Analysis of discounted sum of ladder heights within Sparre-Andersen risk model

Hélène Cossette
École d'actuariat, Université Laval

Joint work with:
David Landriault
Etienne Marceau
Khouzeima Moutanabbir

WIM 2012 Toronto
February 3rd 2012

Introduction

- Trufin, Albrecher and Denuit (2010): Properties of a risk measure derived from ruin theory
- Dynamic risk measures motivated by ruin theory
- VaR-Type risk measures based on infinite-time ruin probability
- Link: maximum aggregate loss rv
- Infinite-time ruin probability: survival function of maximum aggregate loss rv
- Analysis of distribution of discounted maximum aggregate loss within Sparre-Andersen risk model

1. Content

- Sparre Andersen risk model
- Moments of the discounted sum of ascending ladder heights over finite and infinite-time horizon
- Special cases:
- Compound Poisson model: general and mixed Erlang claim sizes
- Sparre-Andersen model with exponential claims: general and mixed Erlang interclaim times
- Moment based approximation

2. Classical Sparre-Andersen risk model

- Claim number process (renewal process) : $\underline{N} = \{N(t), t \geq 0\}$
- Interclaim times rvs $\{W_j, j \in \mathbb{N}^+\}$:
  - sequence of iid rvs
  - pdf k, cdf $K(t) = 1 - \overline{K}(t)$, and LT $\widetilde{k}$
- Claim amount rvs $\{X_j, j \in \mathbb{N}^+\}$:
  - sequence of iid positive rvs
  - pdf p, cdf $P(x) = 1 - \overline{P}(x)$, LT $\widetilde{p}$, and mean μ
- For each j, W_j and X_j are independent (time between claims and claim amount independent)
- Aggregate claim amount process $\underline{S} = \{S(t), t \geq 0\}$ where

$$S(t) = \begin{cases} \sum_{i=1}^{N(t)} X_i, & N(t) > 0 \\ 0, & N(t) = 0 \end{cases}$$

3. Surplus Process

- Surplus process associated to portfolio of insurance business: $\underline{U} = \{U(t), t \geq 0\}$

$$U(t) = u + ct - S(t)$$

- $u \geq 0$: initial surplus level
- $c > 0$: premium rate
- Time to ruin rv τ (first passage time below level 0): $\tau = \inf_{t \geq 0} \{t, U(t) < 0\}$ with $\tau = \infty$ if $U(t) \geq 0$ for all $t \geq 0$
- Deficit at ruin: $|U(\tau)|$
- Surplus just prior to ruin: $U(\tau^{-})$

4. Expected discounted penalty function

- Use some results obtained from the analysis of the EDPF to analyze the discounted sum of ascending ladder heights over finite and infinite-time horizon
- Expected discounted penalty function:

$$E\left[e^{-\delta\tau} w\left(U(\tau^-), |U(\tau)|\right) I(\tau < \infty) | U(0) = u\right], \quad u \geq 0$$

  - $\delta \geq 0$: force of interest
  - $w(x, y) : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$: penalty function
  - I: indicator function ($I(A) = 1$ if the event A occurs and 0 otherwise)

5. Net aggregate loss process

- Net aggregate loss process: $\underline{Y} = \{Y(t), t \geq 0\}$ with $Y(t) = S(t) - ct$
- $\{v_i\}_{i=1}^{\infty}$: sequence of ascending ladder epochs associated to $\underline{Y}$
- $v_1 = \inf\{t \geq 0 : Y(t) > 0\}$
- Recursively: $v_i = \inf\{t \geq v_{i-1} : Y(t) - Y(v_{i-1}) > 0\}$
- For convenience: $v_0 = 0$

5. Net aggregate loss process

- Duration of the *ith* ladder epoch: $\gamma_i = \upsilon_i - \upsilon_{i-1}$
- Corresponding (ascending) ladder height (whenever $\upsilon_i < \infty$): $L_i = Y(\upsilon_i) - Y(\upsilon_{i-1})$
- Number of ladder epochs of $\underline{Y}$: $M = \sup\{i \in \mathbb{N} : \upsilon_i < \infty\}$
- $\{(\gamma_i, L_i)\}_{i=1}^{M}$ forms a sequence of iid random pairs, distributed as $(\tau, |U(\tau)|)$ when $u = 0$

5. Net aggregate loss process

- Graphic: Sample path of $\underline{Y} = \{Y(t), t \geq 0\}$

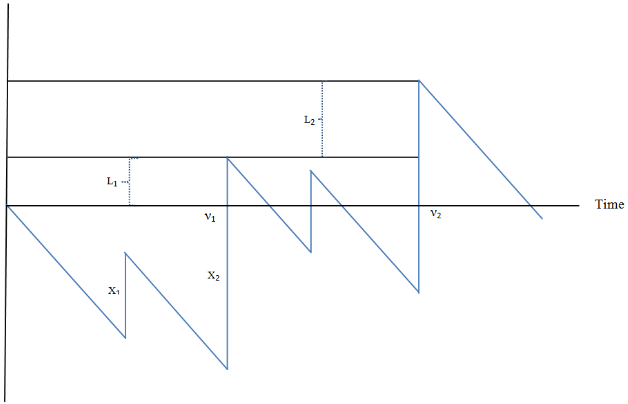

Figure 1. A sample path of $\underline{Y}$.

6. Maximal net aggregate loss process

- Maximal net aggregate loss process: $\underline{Z} = \{Z(t), t \geq 0\}$ where $Z(t) = \sup\{Y(s) : 0 \leq s \leq t\}$
- In the Sparre-Andersen model, $Z \equiv Z(\infty) \sim$ *Compound Geometric* distribution with

$$\Pr(M = m) = (1 - \psi(0))\,(\psi(0))^m\,,\ m = 0, 1, 2, \ldots$$

- Ruin probability:

$$\psi(u) = \Pr(Z > u) = \sum_{m=1}^{\infty} (1 - \psi(0))\,(\psi(0))^m\,\overline{F}_L^{*m}(u)\,,$$

  where $\overline{F}_L^{*m}(x) = 1 - F_L^{*m}(x)$
- Alternative definition for rv $Z(t)$:

$$Z(t) = \begin{cases} \sum_{i=1}^{M} L_i 1(v_i \leq t), & M > 0 \\ 0, & M = 0 \end{cases}$$

  (more relevant in the sequel)

6. Maximal net aggregate loss process

- Discounted maximum aggregate loss process (discounted version of $\underline{Z}$): $\underline{Z}_\delta = \{Z_\delta(t), t \geq 0\}$

$$Z_\delta(t) = \int_0^t e^{-\delta s} dZ(s)$$
$$= \begin{cases} \sum_{i=1}^{M} e^{-\delta \vartheta_i} L_i 1_{\{\vartheta_i \leq t\}}, & M > 0 \\ 0, & M = 0. \end{cases}$$

- Interpretation: assume that at the time of the ith ascending ladder epoch of $\underline{Y}$ (namely, υ_i), a random amount of capital L_i ($i = 1, 2, \ldots$) is injected to the insurance portfolio. Then Z is the sum of all capital amounts injected over time and Z_δ is its discounted version in which the capital injections are discounted at a constant force of interest $\delta \geq 0$.

6. Maximal net aggregate loss process

- In risk theory, $Z_\delta(t)$ can be viewed as the discounted aggregated loss where the interclaim times and claim sizes are distributed as an arbitrary $\gamma_i = v_i - v_{i-1}$ and L_i
- In our context, pairs $\{(\gamma_i, L_i)\}_{i\geq 1}$ are mutually independent, but γ_i and L_i are in general not independent
- Departs from the usual assumption in risk theory for the analysis of the (discounted) aggregate loss

7. Aggregate discounted ladder heights – General

- First examine LT of $Z_\delta(t)$ =total (discounted) value of all capital injections over the **time interval** $[0, t]$
- Conditioning on time and amount of first capital injection,

$$E\left[e^{-sZ_\delta(t)}\right] = \Pr(\gamma_1 > t) + E\left[e^{-se^{-\delta\gamma_1}\left(L_1 + Z'_\delta(t-\gamma_1)\right)}; \gamma_1 < t\right]$$

- Taking the n-th derivative $(n = 1, 2, ...)$ with respect to s on both sides and letting $s \longrightarrow 0$

$$E\left[(Z_\delta(t))^n\right] = \sum_{l=0}^{n} \binom{n}{l} \int_0^t e^{-n\delta w} f_\gamma(w)\, E\left[(L_1)^{n-l} \,|\gamma_1 = w\right] m_{\delta,l}(t-w)\, dw$$

where $m_{\delta,n}(t) = E\left[(Z_\delta(t))^n\right]$ $(n = 0, 1, ...)$.

7. Aggregate discounted ladder heights – General

- Notation: $\widetilde{m}_{\delta,n}(z) = \int_0^\infty e^{-zt} m_{\delta,n}(t)\,dt$
- Taking LT on both sides yields

$$\begin{aligned}\widetilde{m}_{\delta,n}(z) &= \sum_{l=0}^{n} \binom{n}{l} E\left[e^{-(z+n\delta)\gamma_1}(L_1)^{n-l}\right]\widetilde{m}_{\delta,l}(z) \\ &= \sum_{l=0}^{n-1} \binom{n}{l} \frac{E\left[e^{-(z+n\delta)\gamma_1}(L_1)^{n-l}\right]}{1-E\left[e^{-(z+n\delta)\gamma_1}\right]}\widetilde{m}_{\delta,l}(z) \qquad (1)\end{aligned}$$

- For $\delta > 0$, (1) is also equivalent to

$$\widetilde{m}_{\delta,n}(z) = \sum_{l=0}^{n-1} \binom{n}{l} \frac{E\left[e^{-(z+n\delta)\gamma_1}(L_1)^{n-l};\gamma_1<\infty\right]}{1-E\left[e^{-(z+n\delta)\gamma_1};\gamma_1<\infty\right]}\widetilde{m}_{\delta,l}(z) \qquad (2)$$

7. Aggregate discounted ladder heights – General

- Expressed in terms of discounted moments of deficit at ruin $|U(\tau)|$ for an initial surplus of 0 since $\{(\gamma_i, L_i)\}_{i=1}^{M}$ forms a sequence of iid random pairs, distributed as $(\tau, |U(\tau)|)$ when $u = 0$
- $E\left[e^{-(z+n\delta)\gamma_1}(L_1)^{n-l}; \gamma_1 < \infty\right]$: known in a variety of risk models (thanks to numerous advances on the Gerber-Shiu discounted penalty function)
- See e.g., Lin and Willmot (2000), Tsai and Willmot (2002), and Ren (2007)
- Recursive expression for the moments of $Z_\delta(t)$ can be identified when the inversion wrt z can be performed

7. Aggregate discounted ladder heights – Infinite time

- Moments of $Z_\delta \equiv Z_\delta(\infty)$: **infinite-time** horizon
- Final value theorem (see Feller (1969))

$$\lim_{z \to 0} z \widetilde{m}_{\delta,n}(z) = E\left[(Z_\delta)^n\right]$$

- Multiply by z both sides of

$$\widetilde{m}_{\delta,n}(z) = \sum_{l=0}^{n-1} \binom{n}{l} \frac{E\left[e^{-(z+n\delta)\gamma_1}(L_1)^{n-l}; \gamma_1 < \infty\right]}{1 - E\left[e^{-(z+n\delta)\gamma_1}; \gamma_1 < \infty\right]} \widetilde{m}_{\delta,l}(z) \qquad (3)$$

and take the limit when $z \to 0$

$$E\left[(Z_\delta)^n\right] = \sum_{l=0}^{n-1} \binom{n}{l} \frac{E\left[e^{-n\delta\gamma_1}(L_1)^{n-l}; \gamma_1 < \infty\right]}{1 - E\left[e^{-n\delta\gamma_1}; \gamma_1 < \infty\right]} E\left[(Z_\delta)^l\right] \qquad (4)$$

- Equation allows for the recursive calculation of the moments of Z_δ whenever a closed-form expression exists for the discounted moments of the deficit at ruin $|U(\tau)|$ when initial surplus is 0

7. Aggregate discounted ladder heights – Finite time

- Back to finite time case for special cases...
- Key element: analysis of LT of discounted sum of ladder heights

$$\widetilde{m}_{\delta,n}(z) = \sum_{l=0}^{n-1} \binom{n}{l} \frac{E\left[e^{-(z+n\delta)\gamma_1}(L_1)^{n-l};\gamma_1<\infty\right]}{1-E\left[e^{-(z+n\delta)\gamma_1};\gamma_1<\infty\right]} \widetilde{m}_{\delta,l}(z) \qquad (5)$$

- Examine two special classes of models: compound Poisson model and Sparre-Andersen model with exponential claim amounts
  - Class 1: $W \sim Exp(\lambda)$ and $X \sim Dist$ (not specified)
    - Special case: $X \sim Mixed\ Erlang$
  - Class 2: $W \sim Dist$ (not specified) and $X \sim Exp(\beta)$
    - Special case: $W \sim Mixed\ Erlang$

8. Class 1: Compound Poisson risk model – General

- $W \sim Exp(\lambda)$ and $X \sim Dist$ (not specified)
- Discounted moments of the deficit at ruin has been thoroughly studied within classical compound Poisson risk model by Lin and Willmot (2000)
- Capitalizing on their results, aim at analytically inverting (5) to obtain a closed-form expression for $E\left[(Z_\delta)^n\right]$
- Lemma:

$$\frac{E\left[e^{-\delta\gamma_1}(L_1)^l;\gamma_1<\infty\right]}{1-E\left[e^{-\delta\gamma_1};\gamma_1<\infty\right]}=\int_0^\infty e^{-\delta t}g_l(t)\,dt, \qquad (6)$$

where...

8. Class 1: Compound Poisson risk model – General

$$g_l(t) = ce^{-\lambda t} f_l(ct) + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} t^{n-1} e^{-\lambda t} \int_0^{ct} y p^{*n}(ct-y) f_l(y)\, dy$$

$$f_l(t) = \frac{\lambda}{c}\left\{ \mu_{l,t}\overline{P}(t) + \sum_{k=1}^{\infty}\left(\frac{1}{1+\theta}\right)^k \int_0^t \mu_{l,w}\overline{P}(w)\, p_e^{*k}(t-w)\, dw \right\}$$

$\mu_{l,t}$: lth moment of the residual lifetime density $p_t(y) = p(t+y)/\overline{P}$

p_e : equilibrium dist. of $X = \dfrac{\int_x^{\infty} p(y)dy}{\mu}$

p and P : pdf and cdf of the claim amount X

8. Class 1: Compound Poisson risk model – General

- Replacing δ by $z + n\delta$ in (6), one obtains

$$\frac{E\left[e^{-(z+n\delta)\gamma_1}\left(L_1\right)^l; \gamma_1 < \infty\right]}{1 - E\left[e^{-(z+n\delta)\gamma_1}; \gamma_1 < \infty\right]} = \int_0^\infty e^{-zt}\left\{e^{-n\delta t} g_l\left(t\right)\right\} dt \qquad (7)$$

- Substituting (7) into (5) yields

$$\begin{aligned} \widetilde{m}_{\delta,n}\left(z\right) &= \int_0^\infty e^{-zt}\left\{\int_0^t e^{-n\delta w} g_n(w) dw\right\} dt \\ &+ \int_0^\infty e^{-zt}\left\{\sum_{l=1}^{n-1}\binom{n}{l}\int_0^t e^{-n\delta w} g_{n-l}\left(w\right) m_{\delta,l}\left(t-w\right) dw\right\} dt \end{aligned} \qquad (8)$$

- Examine moments of $Z_\delta(t)$ through functional $\left\{\varphi_{n,\delta}(t), t \geq 0\right\}$ whose LT $\widetilde{\varphi}_{n,\delta}(z) = z\widetilde{m}_{\delta,n}\left(z\right)$.
- Then, $m_{\delta,n}\left(z\right) = \int_0^t \varphi_{n,\delta}(y) dy$

8. Class 1: Compound Poisson risk model – General

- Multiply both sides of (8) by z leading to
-

$$\begin{aligned}\widetilde{\varphi}_{n,\delta}(z) &= z\widetilde{m}_{\delta,n}(z) \\ &= \int_0^\infty e^{-zt}\left\{e^{-n\delta t}g_n(t)\right\}dt \\ &\quad + \int_0^\infty e^{-zt}\left\{\sum_{l=1}^{n-1}\binom{n}{l}\int_0^t e^{-n\delta w}g_{n-l}(w)\,\varphi_{l,\delta}(t-w)dw\right\}dt\end{aligned}$$

- By uniqueness property of LT

$$\varphi_{n,\delta} = e^{-n\delta t}g_n(t) + \sum_{l=1}^{n-1}\binom{n}{l}\int_0^t e^{-n\delta w}g_{n-l}(w)\,\varphi_{l,\delta}(t-w)dw$$

9. Class 1: Compound Poisson risk model – Mixed Erlang claims

- Assume that X follows the mixed Erlang distribution
  - $p(y) = \sum_{i=1}^{\infty} q_i \frac{\beta^i (y)^{i-1} e^{-\beta y}}{(i-1)!}$
  - $\widetilde{p}(s) = Q\left(\frac{\beta}{\beta+s}\right)$
- We show that ...
  - First: $\mu_{l,t}\overline{P}(t)$ is mixed Erlang
  - Second: $f_l(t)$ is mixed Erlang
  - Third: $g_l(t)$ is mixed Erlang
  - Fourth: $\varphi_{n,\delta}(t)$ is mixed Erlang

9. Class 1: Compound Poisson risk model – Mixed Erlang claims

- Expression for $\varphi_{n,\delta}(t)$:

$$\varphi_{n,\delta}(t) = \sum_{j=1}^{\infty} \chi_{n,j} h\left(t; j, \lambda + c\beta + n\delta\right)$$

  where $h\left(t; j, \lambda + c\beta + n\delta\right)$: pdf of an Erlang distribution of order j and scale factor $\lambda + c\beta + n\delta$

- Recursive relation to compute the weights $\chi_{n,j}$
- Closed-form expression for nth moment of $Z_\delta(t)$:

$$m_{n,\delta}(t) = \sum_{j=1}^{\infty} \chi_{n,j} \left(1 - \sum_{i=0}^{j-1} \frac{\left(\left(\lambda + c\beta + n\delta\right)t\right)^i e^{-(\lambda+c\beta+n\delta)t}}{i!}\right)$$

10. Class 2: Sparre-Andersen Risk Model with Exponential claims

- We assume: $X \sim Exp\left(\beta\right)$ and $W \sim Dist\left(\text{not specified}\right)$
- Exponential claims: time of first capital injection γ_1 and its size L_1 are independent
- Result:

$$E\left[e^{-\delta\gamma_1}\left(L_1\right)^n; \gamma_1 < \infty\right] = \frac{n!}{\beta^n} E\left[e^{-\delta\gamma_1}; \gamma_1 < \infty\right]. \tag{9}$$

10. Class 2: Sparre-Andersen Risk Model with Exponential claims

- After some calculations, we obtain
  - for $n = 1, 2, ...$:

$$m_{\delta,n}(t) = \frac{n!}{\beta^n} \left(R_{\delta,n}(t) - R_{\delta,n-1}(t) \right)$$

  - $R_{\delta,0}(t) = 0$
  - $R_{\delta,n}(t) = \int_0^t r_{\delta,n}(x)\, dx$
  - $r_{\delta,1}(y) = h_\delta(y)$
  - $r_{\delta,n}(y) = r_{\delta,n-1}(y) + h_{n\delta}(y) + \int_0^y h_{n\delta}(x)\, r_{\delta,n-1}(y-x)\, dx$, $n = 2, 3, ...$
  - $h_{j\delta}(y) = e^{-j\delta y} \sum_{n=1}^{\infty} k^{*n}(y) \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right) \frac{(c\beta y)^j e^{-c\beta y}}{j!}$
  - k is the pdf of W

11. Class 2: Sparre-Andersen Risk Model – Exponential claims & Mixed Erlang interclaim times

- We assume that W follows a mixed Erlang distribution
- It implies that
  - $h_\delta(y)$ is mixed Erlang
  - $r_{\delta,l}(y)$ is mixed Erlang with weights obtained recursively:

$$r_{\delta,l}(y) = \sum_{i=1}^{\infty} \pi_{n,i} \tau_{\lambda+c\beta+l\delta,i}(y)$$

  - $m_{\delta,n}(t)$ is mixed Erlang
- Expression for $m_{\delta,n}(t)$:

$$m_{\delta,n}(t) = \frac{n!}{\beta^n} \sum_{i=1}^{\infty} \left(\pi_{n,i} - \eta_{n-1,i}\right) \left(1 - \sum_{j=0}^{i-1} \frac{((\lambda + c\beta + n\delta)t)^j e^{-(\lambda+c\beta+n\delta)t}}{j!}\right)$$

12. Moment-based approximation

- Further analyze distribution of $Z_\delta(t)$ which can not be derived
- Precise knowledge of this distribution is crucial to compute popular risk measures VaR and TVaR
- Note that rv $Z_\delta(t)$ has a mass at zero and its cdf is of the form

$$F_{Z_\delta(t)}(x) = 1 - \psi(0,t) + \psi(0,t)\, F_{Y_\delta(t)}(x)$$

- $\psi(0,t) =$ prob.of ruin within $[0,t]$ with an initial surplus of 0
- $F_{Y_\delta(t)} =$ cdf of a strictly positive rv $Y_\delta(t)$ whose moments are given by

$$E\left[Y_\delta(t)^n\right] = \frac{E\left[Z_\delta(t)^n\right]}{\psi(0,t)} = \frac{m_{\delta,n}(t)}{\psi(0,t)} \tag{10}$$

- Objective: provide a moment-matching method to approximate the distribution of $Y_\delta(t)$ given (10) and elements previously presented
- Moment-matching techniques: well known research topic
- Approach: approximate unknown dist. with mixture of known distributions

12. Moment-based approximation

- Johnson and Taffe (1989): based on finite mixture of Erlang dist. with different scale parameters and identical shape parameters
- Feldmann and Whitt (1998) - based on finite mixture of exponentials
- Technique proposed to approximate dist.of $Y_{\delta}(t)$:
    - based on densiness of mixed Erlang class (see Tijms (1994))
    - finite mixture of Erlang dist. with identical scale parameters but different shape parameters

12. Moment-based approximation

- Let the rv $Y_\delta(t)$ be approx. by the mixed Erlang rv $\widetilde{Y}_\delta(t)$ where

$$\widetilde{Y}_\delta(t) = \sum_{k=1}^{K} C_k$$

  where $C_k \sim Exp(\beta)$ and K is discrete rv defined on $\mathbb{N}^+$ with $\Pr(K = k) = \zeta_k$.
- Cdf of $\widetilde{Y}_\delta(t)$:

$$F_{\widetilde{Y}_\delta(t)}(x) = \sum_{k=1}^{\infty} \zeta_k H(x; k, \beta)$$

- Finding dist. of $\widetilde{Y}_\delta(t)$ that approximates dist. of $Y_\delta(t)$: finding ζ_k $(k = 1, 2, ...)$

12. Moment-based approximation

- Assume $K \in \{1, 2, ..., n\}$ and defined such that $E\left[(Y_\delta(t))^j\right] = E\left[\left(\widetilde{Y}_\delta(t)\right)^j\right], j = 1, ..., m_0$
- Since $K \in \{1, 2, ..., n\}$: approx. **finite** mixture of Erlang dist.

$$\begin{aligned}\sum_{k=1}^{n} \zeta_k &= 1 \\ \mu_l &= E\left[(Y_\delta(t))\right] = \sum_{k=1}^{n} \zeta_k \frac{\prod_{i=0}^{l-1}(k+i)}{\beta^{m_0}}, \text{ for } l = 1, 2, ..., m_0.\end{aligned}$$

12. Moment-based approximation

- Example $m_0 = 2$:

$$\begin{aligned}
(1)\ \sum_{k=1}^{n} \zeta_k &= 1 \\
(2)\ \sum_{k=1}^{n} \zeta_k k &= E\left[Y_\delta(t)\right]\beta \\
(3)\ \sum_{k=1}^{n} \zeta_k k(k+1) &= E\left[Y_\delta^2(t)\right]\beta^2
\end{aligned}$$

- To this system of equations exist an infinite number of solutions
- However, exist a finite number of distr. with (m_0+1) atoms that satisfy such a system of (m_0+1) equations

12. Moment-based approximation

- Denote by $A\left(\underline{\mu} = (\mu_1, \mu_2, ..., \mu_{m_0}), n, \beta\right)$: space of discrete distributions for K such that moment constraints are verified
- For β and m_0 fixed, n must be large enough to ensure a solution exists in $A\left(\underline{\mu} = (\mu_1, \mu_2, ..., \mu_{m_0}), n, \beta\right)$
- Rule of thumb: select approximation that minimizes difference between their $(m_0 + 1)$-th moment among set of eligible approximations

13. Numerical illustration

- Interclaim times and claim sizes: exponentially distributed with mean 1
- Premium rate: $c = 1.2$ and force of interest: $\delta = 4\%$
- First 6 moments of $Z_\delta(t)$ for $t = 10$, 50, 100 and 200:

m_0 / t	10	50	100	200
1	2.0904	2.7911	2.8291	2.8309
2	10.3922	16.5036	16.8041	16.8173
3	71.3258	132.4999	135.4742	135.5999
4	612.7045	1315.6872	1350.1879	1351.605563
5	6254.4329	15381.4273	15837.2994	15855.6367
6	73472.8075	205368.4629	212098.8478	212364.9329

- Moments of $Z_\delta(t)$ converge as t becomes large, since $Z_\delta(t)$ converges to $Z_\delta(\infty)$ (strictly positive solvency margin)

13. Numerical illustration

- Recall that $E\left[Y_\delta(t)^n\right] = \frac{E[Z_\delta(t)^n]}{\psi(0,t)} = \frac{m_{\delta,n}(t)}{\psi(0,t)}$
- Prob.that no ascending ladder heights occurs in $(0, t]$:

t	10	50	100	200
$1-\psi(0,t)$	0.25519	0.18129	0.16996	0.16738

- Values of $\psi(0,t)$ obtained with 100000 Monte Carlo simulations of $Z_\delta(t)$
- For approximation, consider a time horizon of $t = 200$ and assume $K \in \{1, 2, ..., 10\}$
- For convenience, assume that K has pmf ρ $(\rho_k = \Pr(K = k))$

13. Numerical illustration

- Mixed Erlang models obtained to approximate $Y_\delta(t)$ when first m_0 moments matched:

m_0	β	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
2	0.75	0.2404	0.4962	–	–	0.2635	–
3	0.75	0.3885	–	0.4987	–	0.0115	0.1013
4	0.75	0.3568	0.1217	0.3288	0.0965	–	0.0962

- Figure:compare cdf of mixed Erlang dist. with empirical cdf (10 million Monte Carlo simulations of $Z_\delta(t)$)

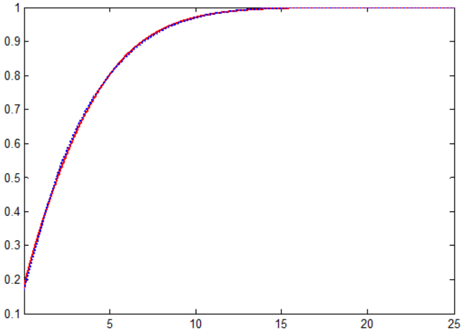

Figure 2. Cdf of the a Mixed Erlang distributions
(2 moments; 3 atoms) with the empirical cdf
that resulted from 10 millions MC simulations of $Z_\delta(t)$.

13. Numerical illustration

- Figure:compare cdf of mixed Erlang dist. with empirical cdf (10 million Monte Carlo simulations of $Z_\delta(t)$)

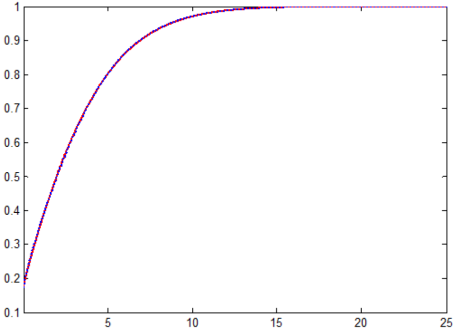

Figure 3. Cdf of the a Mixed Erlang distributions (3 moments; 4 atoms) with the empirical cdf that resulted from 10 millions MC simulations of $Z_\delta(t)$.

13. Numerical illustration

- Figure:compare cdf of mixed Erlang dist. with empirical cdf (10 million Monte Carlo simulations of $Z_\delta(t)$)

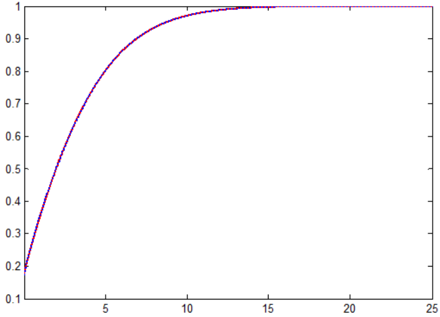

Figure 4. Cdf of the a Mixed Erlang distributions (4 moments; 5 atoms) with the empirical cdf that resulted from 10 millions MC simulations of $Z_\delta(t)$.

13. Numerical illustration

- Approximations obtained with 3 and more moments are very good
- Comparison of $VaR_\kappa\left(Z_\delta\left(200\right)\right)$ and $TVaR_\kappa\left(Z_\delta\left(200\right)\right)$ with simulation

$m_0 \backslash \kappa$	0.95	0.99	0.995
2	8.8586800	12.4862927	13.9031533
3	8.7480563	12.5766479	14.0930303
4	8.7625088	12.5599779	14.0645152
simulation	8.7625	12.5484	14.0566

Table 3: Values of $VaR_\kappa\left(Z_\delta\left(200\right)\right)$

13. Numerical illustration

$m_0 \backslash \kappa$	0.95	0.99	0.995
2	11.0969683	14.4676157	15.8164438
3	11.1064767	14.6950350	16.1364655
4	11.1028197	14.6636129	16.0959678
simulation	11.0959	14.6619	16.1005

Table 4: Values of $TVaR_\kappa\left(Z_\delta\left(200\right)\right)$

- With first 4 moments matched, VaR and TVaR compare very well with simulated counterparts

14. Conclusion

- We insist on the importance to study the behavior of Z and $Z(t)$ for capital assessment
- We generalize Z and $Z(t)$ to Z_δ and $Z_\delta(t)$
- We consider computation of moments of Z_δ and $Z_\delta(t)$
- We examine in more detail two particular classes of Sparre Andersen risk models:
  - Class 1: $W \sim Exp(\lambda)$ and $X \sim Dist$ (not specified)
  - Class 2: $X \sim Exp(\beta)$ and $W \sim Dist$ (not specified)
- We use a moment-matching approximation to evaluate VaR and TVaR

si le montant de sinistres obéit à une loi exp, alors le surplus à la ruine obéit à une expon et indép du temps à la ruine