Circular colouring of graphs

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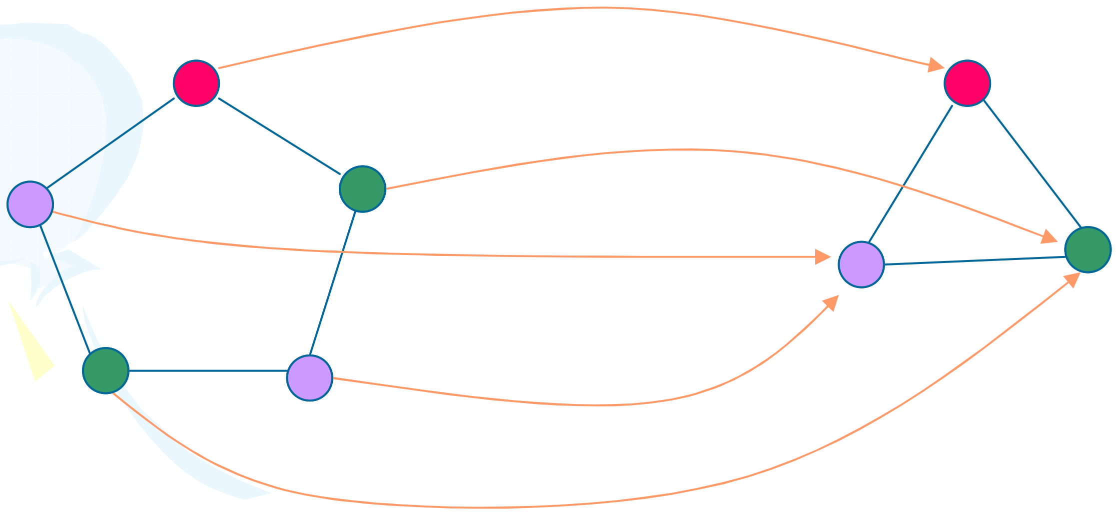
Graph homomorphism = edge preserving map

$G \longrightarrow H$

G $\longrightarrow$ K_n $\Leftrightarrow$ an n-coloring of G.



G, H are homomorphic equivalent means

$G \longrightarrow H$ and $H \longrightarrow G$

$G<H \iff$ G is homomorphic to H.

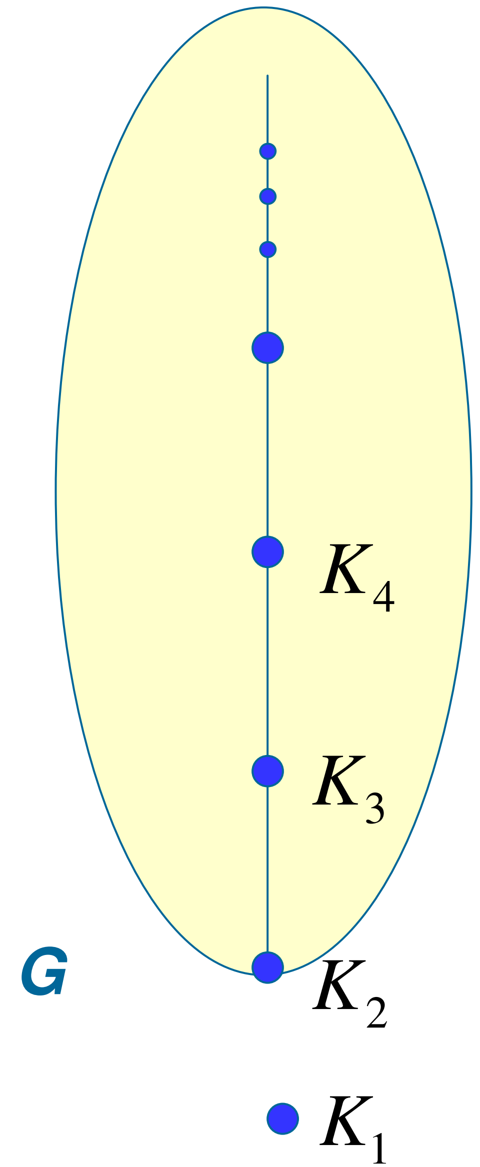
G={finite graphs up to homomorphic equivalence}

(***G***, <) is a partial order.

The set of complete graphs

$$K_1, K_2, K_3, \ldots, K_n, \ldots,$$

form an increasing chain in ***G.***



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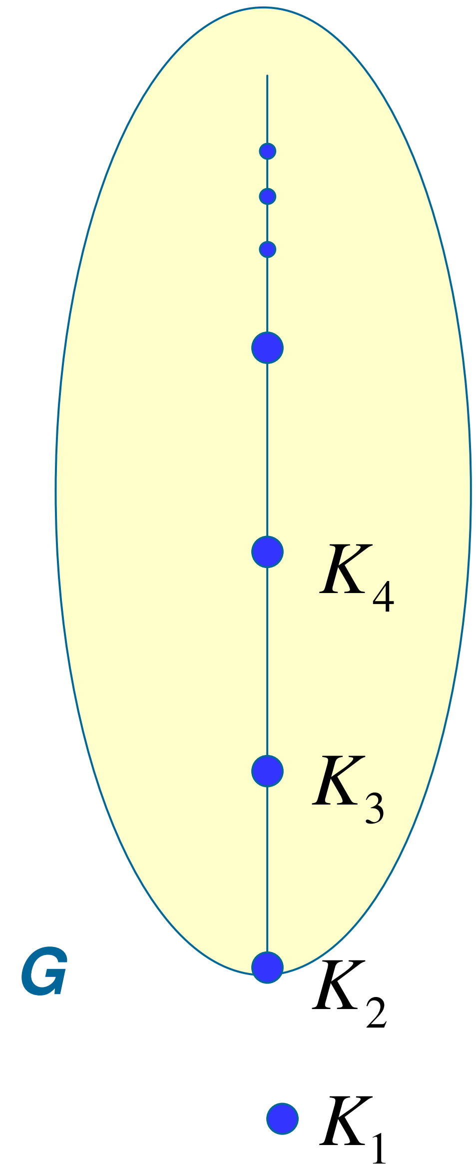
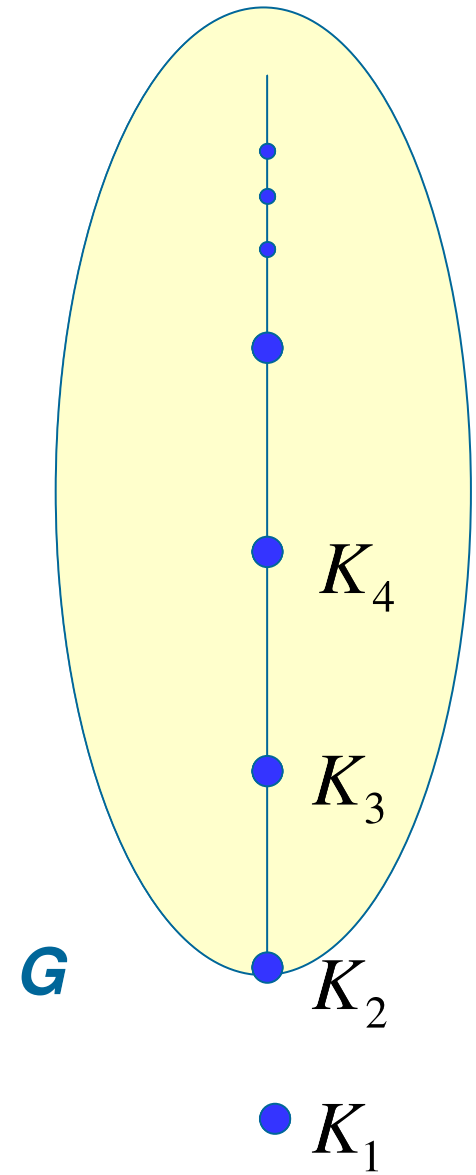
For a G, the chromatic number of G is

$$\boxed{\chi(G) = \min\{n : G < K_n\}.}$$



The chain $K_1, K_2, K_3, \ldots, K_n, \ldots,$

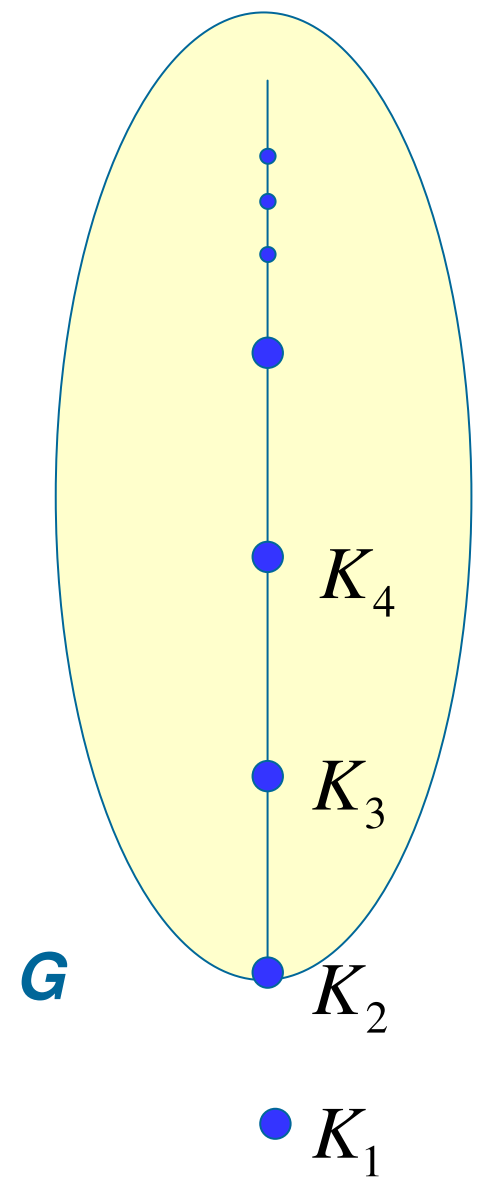
is a scale that measures the chromatic number of graphs.



Theorem [Welzl (1984)]:

Any countable partial order is isomorphic to a suborder of ***G***.

⇨ We can find a dense chain in ***G***.

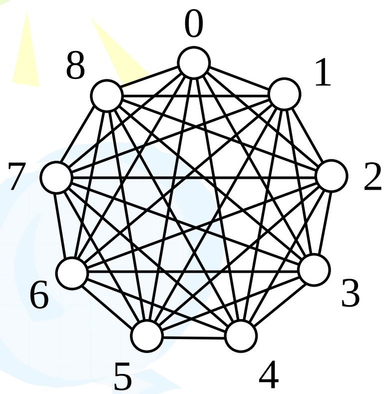


There is a natural dense chain corresponding to rationals.

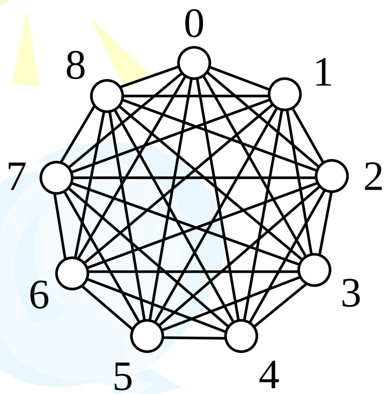
For $\frac{p}{q} \geq 2$, let $K_{\frac{p}{q}}$ be the graph with vertex set

V={0, 1, …, p-1}

i~j $\Longleftrightarrow$ $q \leq |i-j| \leq p-q.$



$K_{9/1}$



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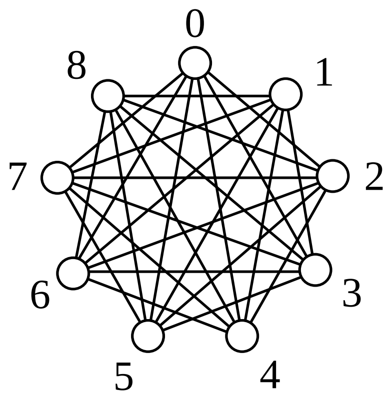
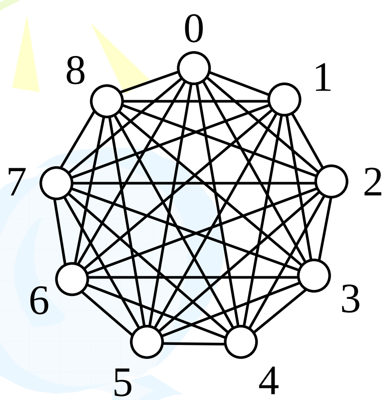


$K_{9/2}$



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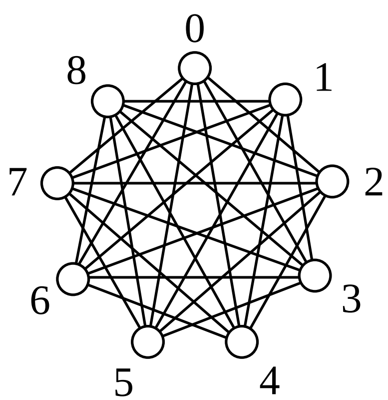
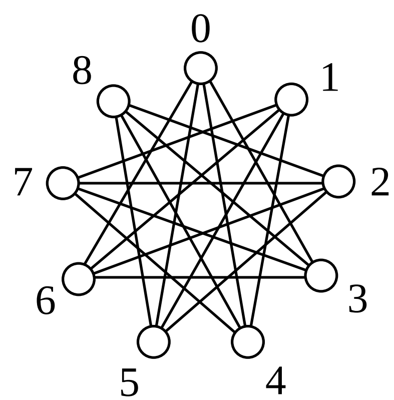
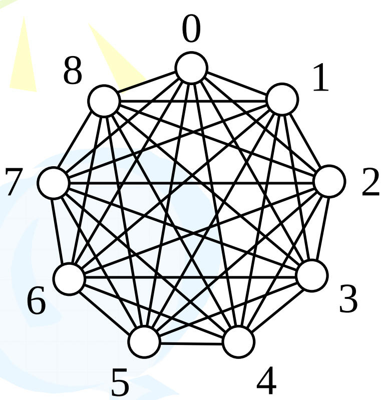


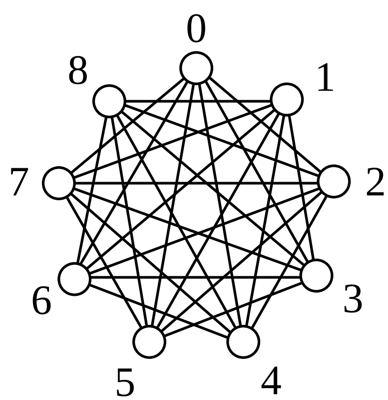
$K_{9/2}$



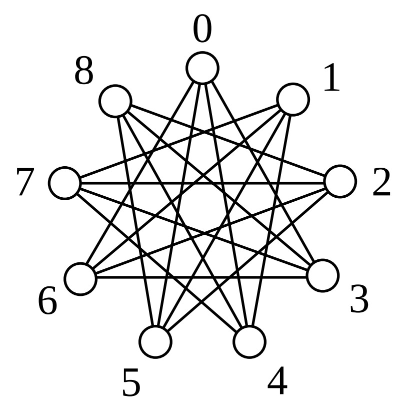
$K_{9/3}$



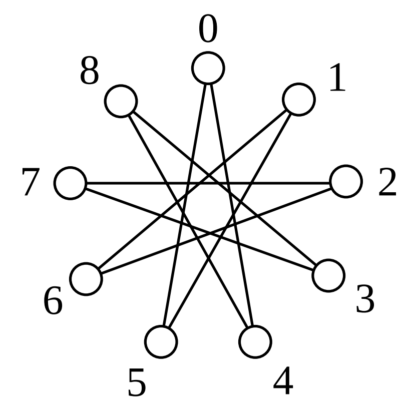
$K_{9/1}$



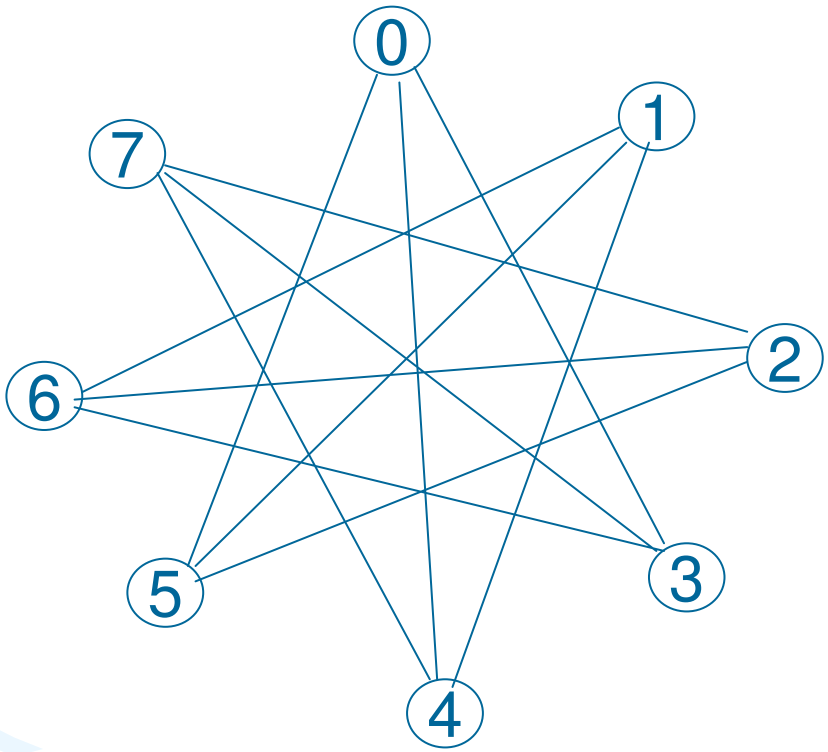
$K_{9/2}$



$K_{9/3}$



$K_{9/4}$



$K_{\frac{8}{3}}$

$$K_{\frac{n}{1}} = K_n$$

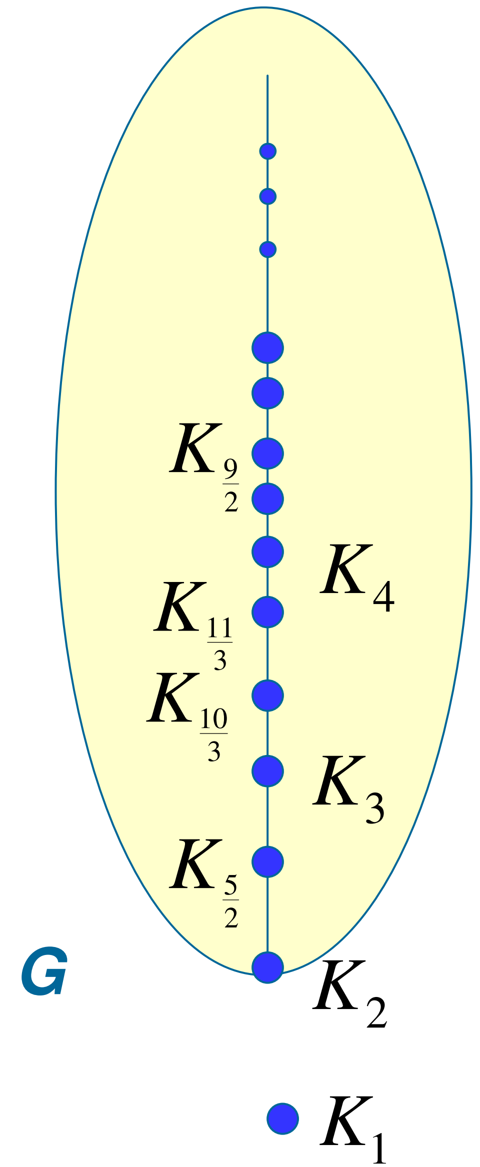
Theorem [Bondy-Hell,1988]

$$K_{\frac{p}{q}} < K_{\frac{p'}{q'}} \Longleftrightarrow \frac{p}{q} \le \frac{p'}{q'}$$

The chain

$$K_1, K_2, K_3, \ldots, K_n, \ldots,$$

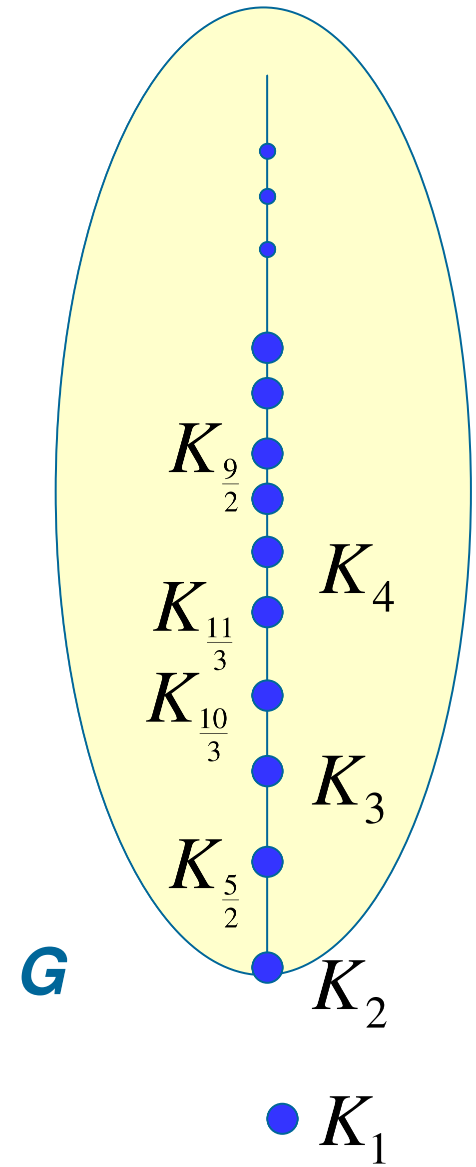
is extended to a dense chain.



Now we use this finer scale
to measure the "chromatic number"
of graphs.

The circular chromatic number

$$\chi_c(G) = \min\left\{\frac{p}{q} : \; G < K_{\frac{p}{q}}\right\}$$



$\chi_c(G)$ is a refinement of $\chi(G)$

$\chi(G)$ is an approximation of $\chi_c(G)$

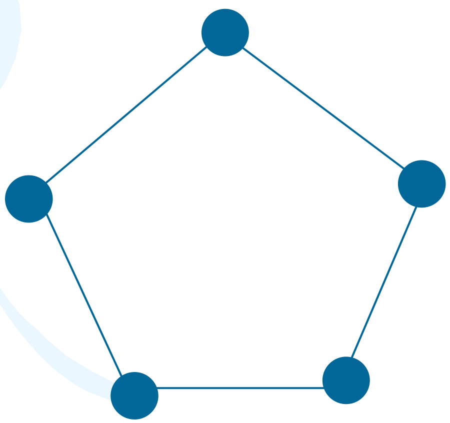
$\chi_c(G)$: the real chromatic number

What’s your age?

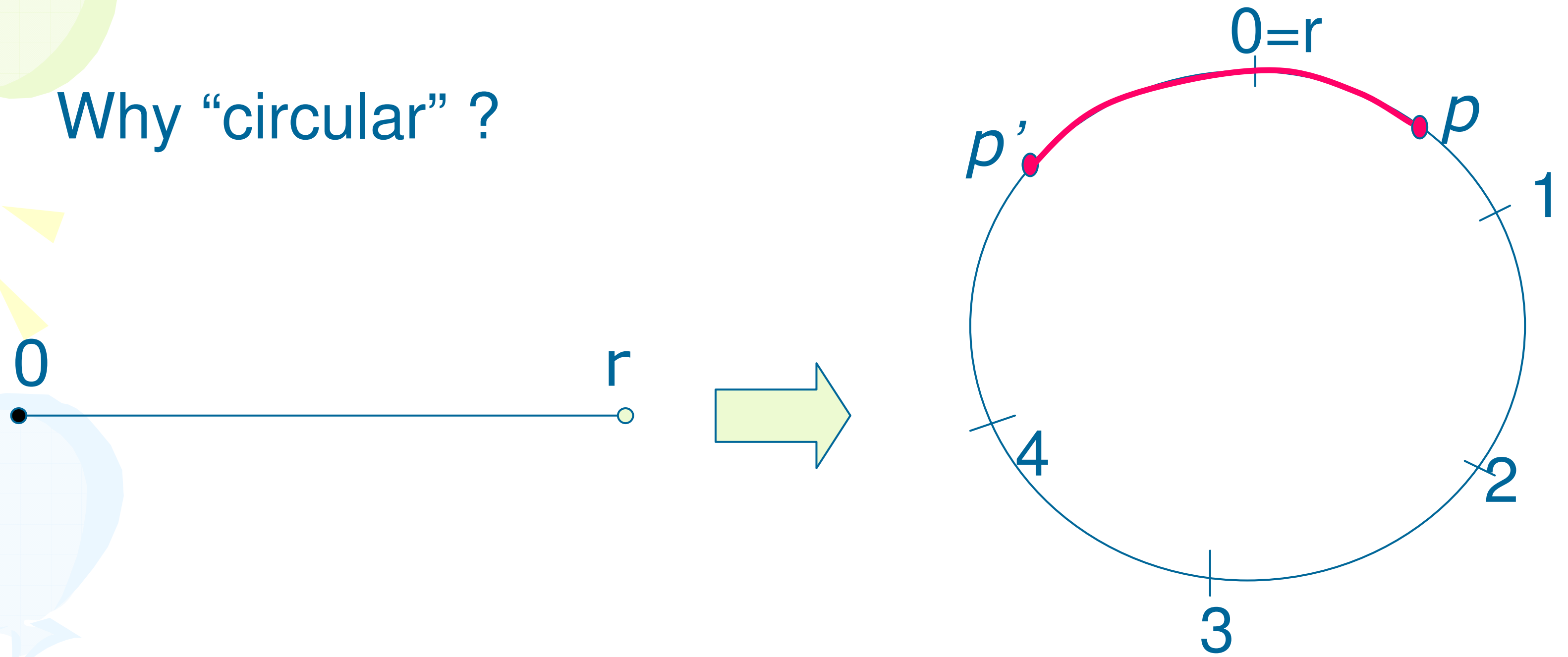
I’m 5.67 years old

What’s you chromatic number ?

My ‘real’ chromatic number is 2.5



Why “circular” ?



The distance between p, p’ in the circle is

$$|p - p'|_r = \min\{\,|p-p'|,\;\; r - |p-p'|\,\}$$

f is a circular r-coloring if

$$x \sim y \quad\Longrightarrow\quad |f(x)-f(y)|_r \geq 1$$

Circular graph coloring is a model for periodic scheduling problem

A. Vince, 1988.
B. star chromatic number $\chi^{*}(G)$

More than 250 papers published, and publications are accelerating.

Quote from Feder, Hell, Mohar (2003):

"The theory of circular colorings of graphs has become an important branch of chromatic graph theory with many exciting results and new techniques."

This is more true today

Bondy-Hell, JGT, 1990
A note on the star chromatic number

I would like to dedicate the talk to Pavol's
65.46 birthday

“The theory of circular colorings of graphs has become an important branch of chromatic graph theory with many exciting results and new techniques.”

It stimulates challenging problems, leads to better understanding of the chromatic graph theory.

$$\chi(C_{2k+1}) = 3$$

$$\chi_c(C_{2k+1}) = 2 + \frac{1}{k}$$

Theorem [Z, 1996] If G is critical n-chromatic and has large girth, then $\chi_c(G) \le (n-1) + \varepsilon.$

Theorem [Nesetril-Zhu,2006]

For any $\varepsilon > 0$ and for any integer k,
there is an integer $n(\varepsilon, k)$ such that for any graph G of treewidth at most k and girth at least $n(\varepsilon, k)$, $\chi_c(G) < 2 + \varepsilon$.

Theorem [Kostochka,Kral,Sereni,Stiebitz, 2010]

For any $\varepsilon > 0$ and for any integer k,
there is an integer $n(\varepsilon, k)$ such that for any graph G of treewidth at most k and odd girth at least $n(\varepsilon, k)$, $\chi_c(G) < 2 + \varepsilon$.

Theorem [Z, 1996] If G is uniquely n-colorable, then $\chi_c(G) = n.$

$$\boxed{\forall g \forall n \exists G, \mathrm{girth}(G) \geq g, \chi_c(G) \geq n.}$$

A classical result of Paul Erdos:

$$\forall g \forall n \exists G, \mathrm{girth}(G) \geq g, \chi(G) \geq n.$$

Theorem [Z, 1996]

$$\forall g \forall r \geq 2 \exists G, \mathrm{girth}(G) \geq g, \chi_c(G) \geq r.$$

Theorem [Nesetril-Z, 2004]

$$\forall g, k, \forall H, \exists G, G \rightarrow H, \mathrm{girth}(G) \geq g,$$

$$\forall H' \text{ with } |V(H')| \leq k,\ G \rightarrow H' \Leftrightarrow H \rightarrow H'$$

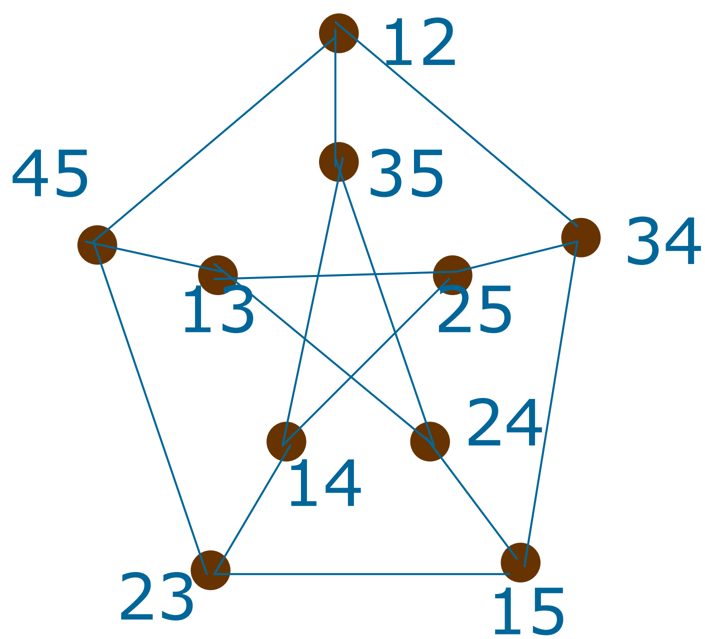
Kneser graph KG(n,k)

Vertex set = all k-subsets of {1,2,…,n}

Two vertices X,Y are adjacent iff $X \bigcap Y = \emptyset$

Petersen graph

= KG(5,2)



There is an easy (n-2k+2)-colouring of KG(n,k):

For i=1,2,..., n-2k+1,

k-subsets with minimum element i
is coloured by colour i.

Other k-subsets are contained in {n-2k+2,...,n}
and are coloured by colour n-2k+2.

$$\Rightarrow \chi(KG(n,k)) \le n-2k+2.$$

Lovasz Theorem [1978]

$$\chi(KG(n,k)) = n-2k+2.$$

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Chen Theorem [2011]

$$\chi_c(KG(n,k)) = n-2k+2.$$

Lovasz Theorem

For any (n-2k+2)-colouring c of KG(n,k), each colour class is non-empty

Alternative Kneser Colouring Theorem [Chen, 2011]

For any (n-2k+2)-colouring c of KG(n,k), there exists two disjoint (k-1)-subsets S and T, such that the following is true:

S

T

i_1

i_2

$\vdots$

i_{n-2k+2}

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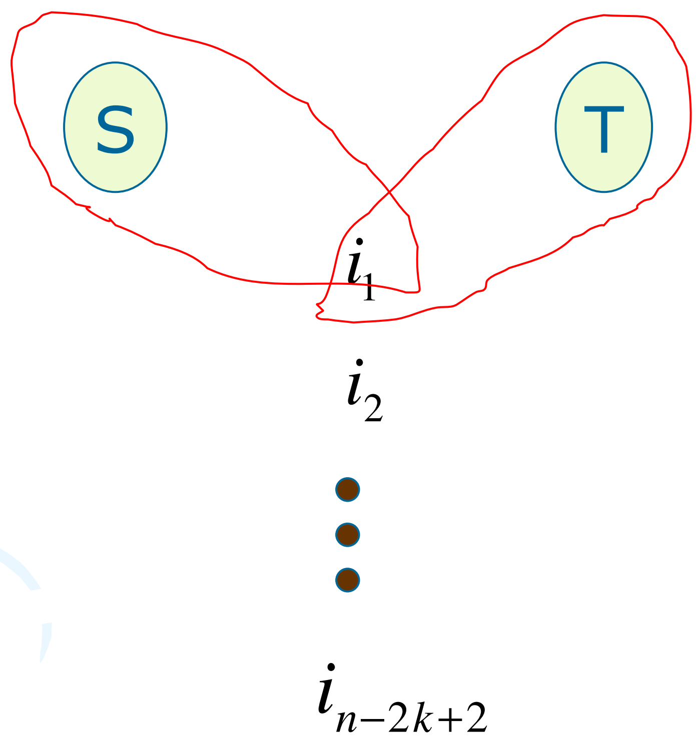

Colour 1

Alternative Kneser Colouring Theorem [Chen, 2011]

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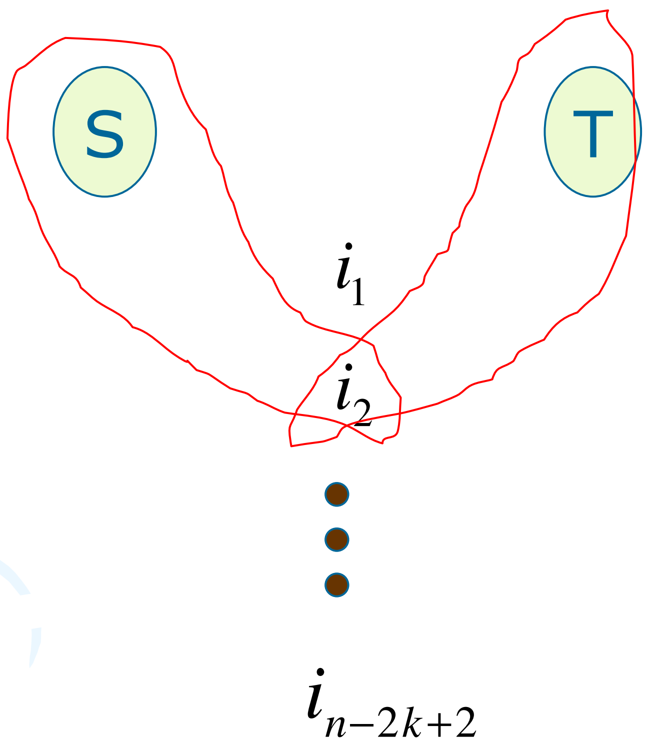
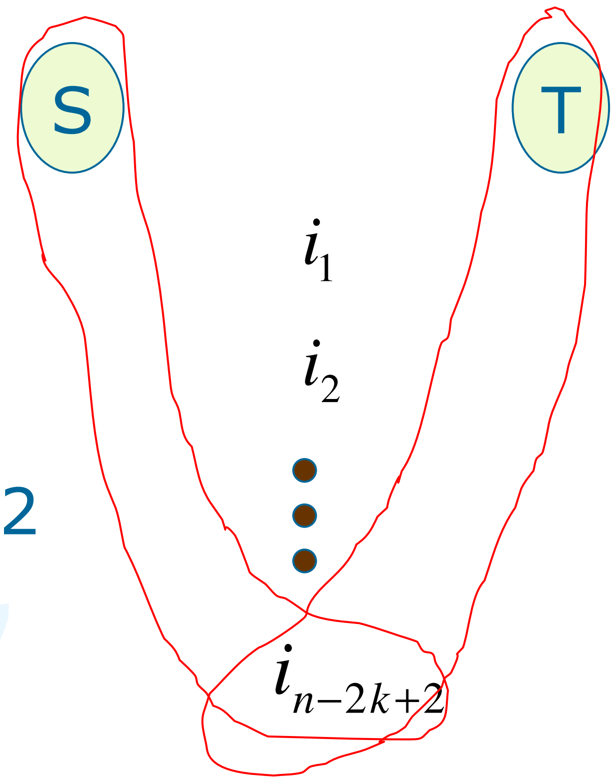


Colour 2

Alternative Kneser Colouring Theorem [Chen, 2011]

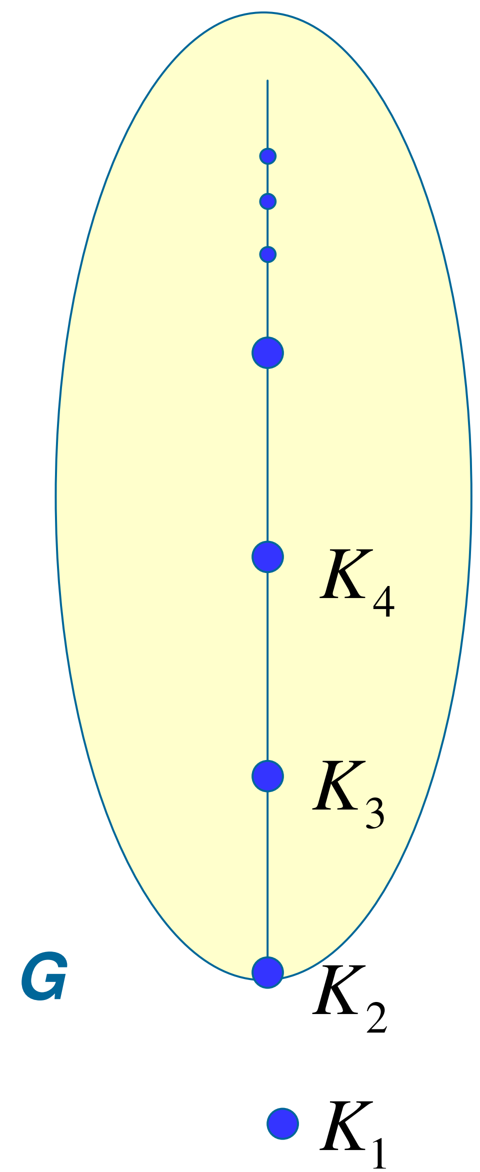
For any (n-2k+2)-colouring c of KG(n,k), there exists two disjoint (k-1)-subsets S and T, such that the following is true:

Colour n-2k+2



For a G, the clique number of G is

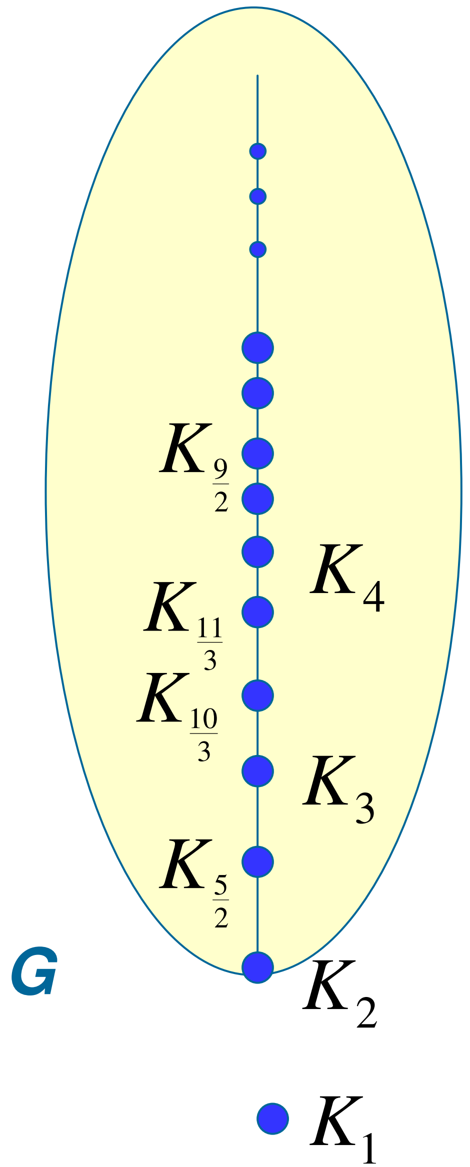
$$\omega(G) = \max\{n : K_n < G\}.$$



Use the finer scale
to measure the “clique number”
of graphs.

The circular clique number

$$\omega_c(G) = \max\left\{ \frac{p}{q} : K_{\frac{p}{q}} < G \right\}$$



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A graph G is circular perfect if for every induced subgraph H of G,

$$\chi_c(H) = \omega_c(H)$$

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A graph G is circular perfect if for every induced subgraph H of G,

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Theorem [Grotschel-Lovasz-Schrijver, 1981]

For perfect graphs, the chromatic number is computable in polynomial time.

A graph G is perfect if for every induced subgraph H of G,

$$\chi(H) = \omega(H)$$

A graph G is circular perfect if for every induced subgraph H of G,

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Theorem [Bochoc Pecher Thiery, 2011]

circular circular

For perfect graphs, the chromatic number is computable in polynomial time.

A key step in the proof is calculating the Lovasz theta number of circular cliques and their complements.

$$\vartheta(G) = \max\left\{ \sum_{(x,y)\in V^2} B(x,y) : \begin{array}{l} B \in R^{V\times V}, B \geq 0 \\ \sum_{x\in V} B(x,x) = 1 \\ B(x,y) = 0 \quad xy \in E \end{array} \right\}$$

$$\forall G, \omega(G) \leq \vartheta(\overline{G}) \leq \chi(G)$$

For perfect graph G, estimating $\vartheta(\overline{G})$ with error less than $\frac{1}{2}$ determines its chromatic number.

Theorem [Bochoc Pecher Thiery, 2011]

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$$\cancel{\forall G, \omega_c(G) \leq \vartheta(\overline{G}) \leq \chi_c(G)}$$

For perfect graph G, estimating $\vartheta(\overline{G})$ with

$$\vartheta(\overline{C}_5) = \sqrt{5}, \quad \omega_c(C_5) = \chi_c(C_5) = \frac{5}{2}$$

hromatic number.

Theorem [Bochoc Pecher Thiery, 2011]]

circular circular

For perfect graphs, the chromatic number is computable in polynomial time.

A key step in the proof is calculating the Lovasz theta number of circular cliques and their complements.

G circular perfect with $\chi_c(G) = k/d$

$$\Rightarrow \quad \vartheta(\overline{G}) = \vartheta\left(\overline{K}_{k/d}\right)$$

$$2d \le k \le |V(G)|$$

For all (k,d), with $(k,d) = 1$, $2d \le k \le n$,

$\vartheta\left(\overline{K}_{k/d}\right)$ are all distinct and separated by at least ε for some ε with polynomial space encoding.

There are very few families of graphs for which the theta number is known.

Theorem [Bochoc Pecher Thiery, 2011]

circular circular

For perfect graphs, the chromatic number is computable in polynomial time.

A powerful tool in the study of list colouring graphs is

Combinatorial Nullstellensatz

Assume $V(G) = \{v_1, v_2, \cdots, v_n\}$

Give G an arbitrary orientation.

$$Q_G(x_1, \cdots, x_n) = \prod_{(v_i, v_j) \in \vec{E}} (x_i - x_j)$$

c is a proper colouring of G

$\Leftrightarrow Q_G(c(v_1), \cdots, c(v_n)) \neq 0$

Find a proper colouring=
find a nonzero assignment
to a polynomial

Combinatorial Nullstellensatz:

Let F be a field, $f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$

Suppose the degree of f is $\sum_{j=1}^{n} t_j$.

If the coefficient of $\prod_{j=1}^{n} x^{t_j}$ in f is nonzero,

then for any subsets $S_1, \ldots, S_n$ of F

with $|S_j| = t_j + 1$,

there exist $s_1 \in S_1, \ldots, s_n \in S_n$ such that

$f(s_1, \ldots, s_n) \neq 0$

What is the polynomial for circular colouring?

Assume $V(G) = \{v_1, v_2, \cdots, v_n\}$

Give G an arbitrary orientation.

$$Q_G(x_1, \cdots, x_n) = \prod_{(v_i, v_j) \in \vec{E}} (x_i - x_j)$$

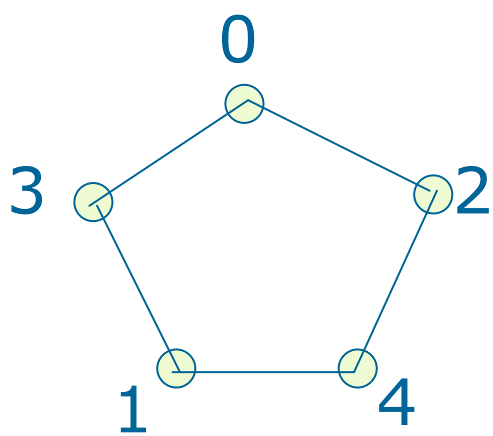
Assume $p \geq q$ are positive integers

Let $Z_p = \{0, 1, \ldots, p-1\}$

A (p, q)-colouring of G is

$c : V \to Z_p$ such that

$$x \sim y \Longrightarrow q \leq |c(x) - c(y)| \leq p - q$$

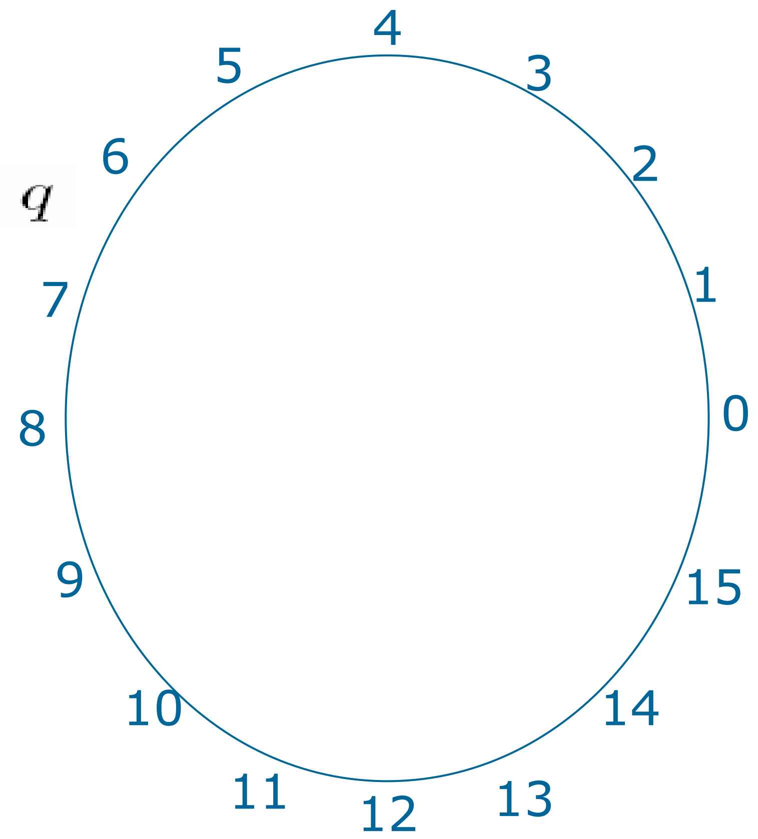


A $(5, 2)$-colouring of C_5

A $(p, 1)$-colouring is a p-colouring

Color set $Z_p = \{0, 1, \ldots, p-1\}$

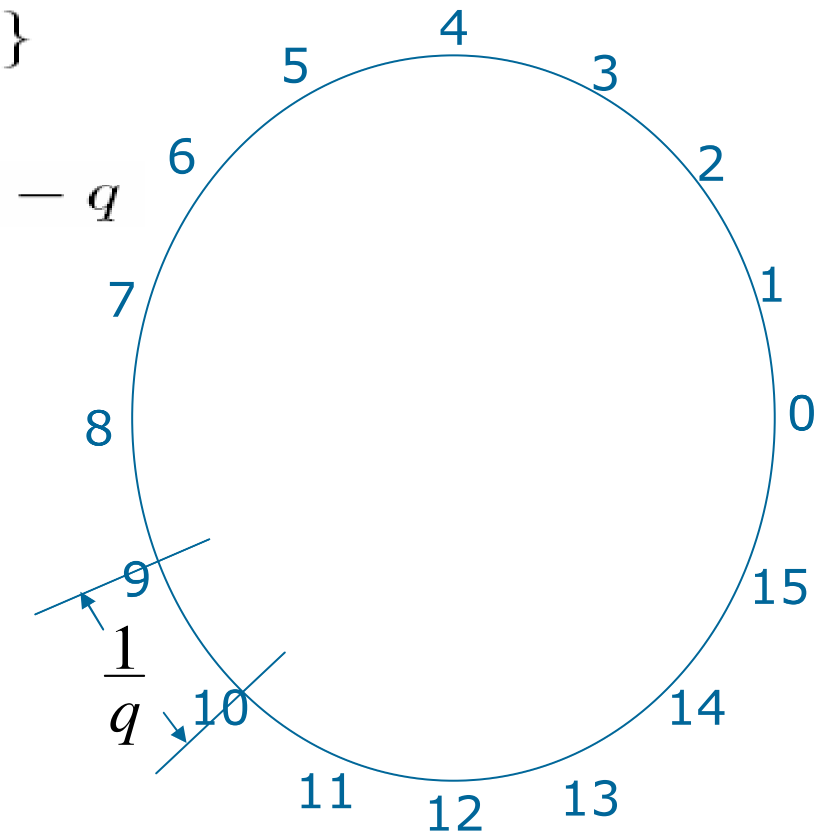
$$x \sim y \Longrightarrow q \leq |c(x) - c(y)| \leq p - q$$



Colors assigned to adjacent vertices have circular distance at least q

Color set $Z_p = \{0, 1, \ldots, p-1\}$

$$x \sim y \Longrightarrow q \leq |c(x) - c(y)| \leq p - q$$



Colors assigned to adjacent vertices have circular distance at least 1

The circle has perimeter $\frac{p}{q}$

The circular chromatic number of G is

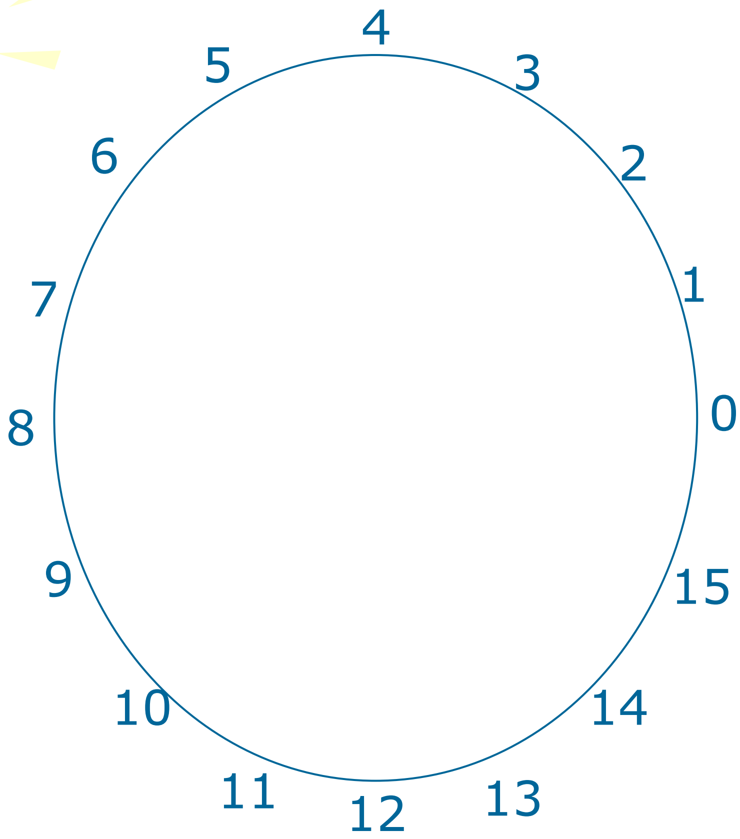
$$\chi_c(G) = \min\{p/q : G \text{ has a } (p,q)\text{-colouring } \}$$

What is the polynomial for circular colouring?

Assume $V(G)=\{v_1, v_2, \cdots, v_n\}$

Give G an arbitrary orientation.

$$f(x_1, x_2, \ldots, x_n) = \prod_{(v_j, v_{j'}) \in D} \prod_{k=-q+1}^{q-1} (x_j - e^{2\pi i k/p} x_{j'}).$$



$$j \leftrightarrow e^{(2\pi j / p)i}$$

An idea of Norine

$\gamma : Z_p \to \mathbb{C}$ be defined as $\gamma(l) = e^{2\pi i l/p}$

$$\boxed{f(x_1, x_2, \ldots, x_n) = \prod_{(v_j, v_{j'}) \in D} \prod_{k=-q+1}^{q-1} (x_j - e^{2\pi i k/p} x_{j'}).}$$

$h : V \to Z_p$ is a (p, q)-colouring of G

$$\Updownarrow$$

$$f(\gamma(h(v_1)), \gamma(h(v_2)), \ldots, \gamma(h(v_n))) \neq 0.$$

Combinatorial Nullstellensatz:

Let F be a field, $f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$

Suppose the degree of f is $\sum_{j=1}^{n} t_j$.

If the coefficient of $\prod_{j=1}^{n} x^{t_j}$ in f is nonzero,

then for any subsets $S_1, \ldots, S_n$ of F

with $|S_j| = t_j + 1$,

there exist $s_1 \in S_1, \ldots, s_n \in S_n$ such that

$$f(s_1, \ldots, s_n) \neq 0$$

Theorem [Alon-Tarsi]

Suppose D is an orientation of G with $|EE(D)| \neq |OE(D)|$. Then G is $(d_D^+ + 1)-$choosable

$$\phi : E(D) \to \{0, 1, \ldots, 2q-1\} \ \textit{eulerian} \iff$$

for each vertex v, $\displaystyle\sum_{e \in E_D^+(v)} \phi(e) = \sum_{e \in E_D^-(v)} \phi(e).$

$$a_l = \sum_{J \subseteq \{-q+1,\dots,q-1\}, |J|=l} \prod_{j \in J} (-e^{2\pi i j/p})$$

the coefficient of the monomial $\prod_{j=1}^{n} x_j^{t_j}$ in f is

$$\sum_{\phi \text{ is eulerian}} \prod_{e \in D} (-1)^{\phi(e)} a_{\phi(e)}$$

An eulerian mapping φ is even if $\sum_{e \in E(D)} \varphi(e)$ is even

$$\boxed{w_{p,q}(\varphi) = \prod_{e \in E(D)} a_{\varphi(e)}(p,q)}$$

$$\boxed{w_{p,1}(\varphi) = 1}$$

$$\boxed{|EE(D)| \neq |OE(D)| \Leftrightarrow \sum_{\varphi \text{ is even}} w_{p,1}(\varphi) \neq \sum_{\varphi \text{ is odd}} w_{p,1}(\varphi)}$$

Theorem [Norine-Wong-Z, 2008]

Suppose D is an orientation of G with

$$\sum_{\varphi \text{ is even}} w_{p,q}(\varphi) \neq \sum_{\varphi \text{ is odd}} w_{p,q}(\varphi).$$

If L is a *p-list* with $|L(v)| = d_D^+(v)(2q-1)+1$, then G is L-(p,q)-colourable.

Theorem [Norine-Wong-Z, (JGT 2008)]

Suppose G is bipartite, l is a list-size assignment such that for any subgraph H of G,

$$\sum_{x \in V(H)} (l(x) - 1) \geq |E(H)|(2q - 1)$$

Then G is l-(p,q)-choosable.

q=1 case was proved by Alon-Tarsi in 1992.

Corollary[Norine]:
Even cycle are circular 2-choosable.

The only known proof uses combinatorial nullstellensatz

$$\chi_{c,l}(G) = \min\left\{ t: \begin{array}{l} \forall p, q, \text{if } l(x) \geq tq, \text{ then} \\ G \text{ is } l-(p,q)-\text{choosable} \end{array} \right\}$$

$$\chi_{c,l}(G): \text{circular choosability of G}$$

Corollary: Even cycle are circular 2-choosable.

$$\chi_{c,l}(C_{2k}) = 2$$

Theorem [Z, 2005]: For any positive integer k, for any $\varepsilon > 0$, there is a k-degenerate graph G with

$$\chi_{c,l}(G) > 2k - \varepsilon$$

$$\boxed{\chi_l(G) \leq k+1}$$

Conjecture: Every 2-choosable graph is circular 2-choosable.

Equivalent formulation:

$\theta_{2,2,2k}$ is circular 2-choosable.

Theorem [Norine-Wong-Z, 2008]

If G is bipartitie then $\chi_{c,l}(G) \leq 2\left\lceil \frac{\operatorname{mad}(G)}{2} \right\rceil$

$$\chi_{c,l}(\theta_{2,2,2k}) \leq \frac{4k+8}{2k+3}$$

Theorem [Liu-Norine-Pan-Z,2010]

Every 2-choosable graph is consecutive circular 2-choosable.

Circular chromatic index

Theorem [Vizing] For any simple graph G,

$$\Delta(G) \le \chi'(G) \le \Delta(G) + 1$$

Corollary For any simple graph G,

$$\Delta(G) \le \chi_c'(G) \le \Delta(G) + 1$$

If $\Delta(G) = 2$, then

$$\chi_c'(G) \in \left\{2, 3, 2+\frac{1}{2}, 2+\frac{1}{3}, \ldots, 2+\frac{1}{k}, \ldots\right\}$$

What are the possible values of circular chromatic Indices?

Theorem [Afshani-Ghandehari-Ghandehari-Hatami-Tusserkani-Z, 2005]

If $\Delta(G) = 3$, then $\chi_c'(G) \in [3, \frac{11}{3}] \cup \{4\}$

$(\frac{11}{3}, 4)$ is a gap

Are there other gaps?

Petersen graph is the only known graph with circular chromatic index 11/3.

No graphs are known to have $\chi_c'(G) \in (\frac{7}{2}, \frac{11}{3})$

$(\frac{7}{2}, \frac{11}{3})$ is a possible gap.

Theorem [Kral-Macajova-Mazak-Sereni, 2011]

$$\Delta(G)=3, \mathrm{girth}(G)\geq 6 \Rightarrow \chi_c'(G)\leq \frac{7}{2}.$$

If $\Delta(G)=2$, then

$$\chi_c'(G)\in\left\{2,3,2+\frac{1}{2},2+\frac{1}{3},\ldots,2+\frac{1}{k},\ldots\right\}$$

Theorem [Lukot'ka-Mazak, 2011]

For any rational $r\in[3,10/3]$, there is a cubic G with $\chi_c'(G)=r.$

Theorem [Kaiser-Kral-Skrekovki-Z,2007]

$$\forall \varepsilon > 0, k \geq 2, \exists g, \text{girth}(G) \geq g \Rightarrow \chi_c'(G) < k + \varepsilon.$$

Theorem [Lin-Wong-Z,2011]

For any rational $r \in [2k+1, 2k+1+\frac{1}{4}]$, there is a (2k+1)-regular graph G with

$\chi_c'(G) = r.$

Question: Are there other gaps ?

Question: Are there other intervals where circular chromatic indices are dense ?


Thank you!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