Semilattice Polymorphisms on Reflexive Graphs

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Outline

- Motivation
- Definitions and Basics
- A hierarchy of Semilattice Polymorphisms
- Semilattice vs. NUF
- Homotopies of cycles
CSP Dichotomy Classification Conjecture

For a core relational structure $\mathcal{H}$, $\text{CSP}(\mathcal{H})$ is NP-complete if $\mathcal{H}$ omits WNU polymorphisms and is otherwise polynomial time solvable.
Motivation

Reflexive Graphs

Semilattice Polymorphisms

Goals

Semilattice Polymorphisms

SL vs. NUF

Homotopy

End
A symmetric graph $H$ is reflexive if every edge has a loop.
We usually don’t draw the loops.
$\text{CSP}(H)$ is trivial for such $H$, so we assume $H$ also has all singleton unary relations.
- CSP($H$) is $H$-precolouring extension
- Any $H$ is a core.
- All polymorphisms are idempotent.
We don’t draw these singleton relations either.
CSP Dichotomy is equivalent to Dichotomy for symmetric reflexive graphs. [Feder Vardi 98].
- CSP Dichotomy is equivalent to Dichotomy for symmetric reflexive graphs. [Feder Vardi 98].
- Dichotomy is done for MinHOM of reflexive graphs. [Gutin Hell Rafiey Yao 07].
A vague notion: Reflexive Graphs have no Pushing

- One characterization of the set of structures omitting WNU is that one can make edge gadgets for them to encode 3-colouring.
- For this some vertex gadget that will map to one of three spots, and some edge gadget that keeps two vertex gadgets apart.
- To keep them apart the only way we can do it seems to be to *push* them apart, for which we basically need direction, or to *pull* them apart by wrapping the gadget around a 'hole'.
- In mapping to reflexive graphs, there is only pulling, requiring 'holes’, which will translate to cycles that we can’t move across- induced cycles, but more than just that.
Why Semilattice polymorphisms?
For reflexive graphs that 'hardness' comes from induced cycles that are 'non-contractible'.
Chordal graphs have no induced cycles, while Larose '04 shows that if a reflexive graph admits WNU then cycles 'contract'.
Graphs admitting TSI 'contract' in a stronger sense.
Between this are graphs admitting SL or NUF.
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While Larose and Zadori [03] show there are structures in $\text{NUF} \setminus \text{TSI}$, Loten [03] shows for reflexive graphs that $\text{NUF} \subset \text{TSI}$. 
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Reflexive graphs admitting NU polymorphisms have been characterised by Brewster, Feder, Hell, Huang, MacGillivray [06] and Larose, Loten, Tardif, [06].
Our goals are to...

- Draw this diagram properly.
- Characterise the various intersections.
- Relate the intersections to the way circles contract.
**Definition: Polymorphism**

A *polymorphism* of $H$ is a homomorphism of $H^d$ to $H$.

Polymorphisms of reflexive graphs are necessarily *idempotent*:

$$\phi(a, a, \ldots, a) = a$$
A polymorphism is **TSI** if

\[ \phi(x_1, \ldots, x_d) = \phi(y_1, \ldots, y_d) \]

when \( \{x_1, \ldots x_d\} = \{y_1, \ldots, y_d\} \) as sets.
A polymorphism is a **NUF** if

\[
\phi(x, x, \ldots, x, y) = \phi(x, x, \ldots, y, x) = \cdots = \phi(y, x, \ldots, x, x) = x
\]

for all \(x, y\).
A 2-ary polymorphism is **SL** (semi-lattice) if it is symmetric and associative.
Recall that

A semilattice operation \( \phi \) on a set of points defines a partial order of the points by

\[
u < v \text{ if } \phi(u, v) = u.\]

We represent a semilattice by its Hasse diagram of covers.
Recall that

This partial ordering is a semilattice ordering; that is, every pair of points $a, b$, has a $\text{glb } a \land b$. 
Recall that

This partial ordering is a semilattice ordering; that is, every pair of points $a, b$, has a $\text{glb } a \wedge b$. 
Recall that

The semilattice operation is glb: $\phi(a, b) = a \land b$. 
Given a semilattice operation on a set of vertices,
Given a semilattice operation on a set of vertices, and a reflexive graph on the vertices,
What other edges are needed so that the semilattice operation is a polymorphism?
Polymorphism: $u \sim u', v \sim v' \Rightarrow u \land v \sim u' \land v'$
Polymorphism: \( u \sim u', v \sim v' \Rightarrow u \land v \sim u' \land v' \)
Semilattice Polymorphisms
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Motivation

Semilattice Polymorphisms
Picture of a Semilattice
Types of SL Polymorphisms
Chordal Graphs
SL vs. NUF
Homotopy
End
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For a chain semilattice, the polymorphism property simply becomes the $\text{min}$-property, or the $X$-underbar property, so by Feder Hell 98, the graphs admitting chain semilattices are exactly interval graphs.
A semilattice polymorphism is ...

- **embedded** if every Hasse edge (blue edge) is a graph edge.
A semilattice polymorphism is...

- **embedded** if every Hasse edge (blue edge) is a graph edge.
- **tree** if the Hasse edges induce a tree.
A semilattice polymorphism is ...

- **embedded** if every Hasse edge (blue edge) is a graph edge.
- **tree** if the Hasse edges induce a tree.
- **skeletal** if all graph edges are between comparable vertices.
There are SL polymorphisms of graphs that are

- tree and embedded but not skeletal
- tree and skeletal but not embedded

But...
If a SL polymorphism is skeletal and embedded, then it must be tree.
Semilattice

Proposition
Any graph admitting a tree SL admits an embedded tree SL.

Proposition
Any graph admitting a skeletal SL admits a skeletal embedded tree SL.
Semilattice

embedded

tree

skeletal
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Picture of a Semilattice

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TSI

Semilattice

embedded

tree

skeletal

chain = interval graph
On skeletal trees, the polymorphism property again simplifies to the $X$-underbar property, and so

<table>
<thead>
<tr>
<th>Easy Proposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A graph admits a skeletal polymorphism if and only if it is <em>chordal</em>.</td>
</tr>
</tbody>
</table>

TSI

Semilattice

embedded

tree

skeletal $=$ chordal

chain $=$ interval graph
Recall that...

A chordal graph can be represented as the intersection graph of a set of subtrees of some tree.

**Definition**

The leafage of a chordal graph $H$ is the minimum number of leaves in a tree that gives an intersection representation of $H$.

**Theorem [BFFHM]**

Every chordal graph of leafage $k$ admits a NUF of arity $k + 1$. 
**Proposition**

A chordal graph has leafage $k$ if and only if it admits a skeletal SL polymorphism in which the Hasse diagram has $k$ leaves.
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Chordal Reducible Graphs and Strong-V

Homotopy

End
Given a graph $H$, 

![Graph Image]
Given a graph $H$, take its clique graph $\text{CL}(H)$,
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Given a graph \( H \), take its clique graph \( \text{CL}(H) \),
Given a graph $H$, take its clique graph $CL(H)$, and add edges between them according to incidence: $CR(H)$. 
Given a graph $H$, take its clique graph $CL(H)$, and add edges between them according to incidence: $CR(H)$. 
If we can remove edges from $H$ such that it remains connected, and the full graph $CR^*(H)$ is chordal, then $H$ is chordal reducible.
- Chordal graphs are chordal reducible.
- Graphs with a universal vertex are chordal reducible.
- Chordal reducible graphs have NUF of some arity.
- Are all graphs with 4-NU chordal reducible?
$V$ properties of tree SL polymorphisms

In a graph with an embedded SL polymorphism, an edge between incomparable vertices induces edges in the $V$ below it.
$V$ properties of tree SL polymorphisms

Other parts of the graph may induce more edges in the $V$, so these edges are not enough to ensure we have a polymorphism.
Other parts of the graph may induce more edges in the $V$, so these edges are not enough to ensure we have a polymorphism.
V properties of tree SL polymorphisms

The *strong-V property* more than ensures our SL ordering is a polymorphism.
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SL

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Semilattice Polymorphisms

TSI

Semilattice

embedded

tree

strong V tree

NUF

chordal reducible

strong V tree

skeletal SL = chordal
Proposition

Chordal reducible graphs admit strong $V$ tree polymorphisms.
Proof:
Proof:
Proof:
Proof:
Proof:
Proof:
### Proposition

If a reflexive graph admits a strong-$V$ tree $\text{SL}$, then it admits $\text{NUF}$.
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strong V tree

chordal reducible

chordal
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The starting point of the theory of homotopy of graphs is the product $I \times C_d$, if $I \times \vec{C}_d$. 
The graph $\text{Hom}(C_d, H)$ for a graph $H$ has as vertices the homomorphisms of $C_d$ to $H$. 
Two are adjacent if they are the restriction of a homomorphism of $I \times C_d$ to the end copies of $C_d$. 

\[ \text{Hom}(C_5, H) \]
Of course, the homomorphisms need not be injective.
We can do the same with a directed cycle $\vec{C_d}$.
As $H$ is reflexive, the constant maps induce a copy of $H$ in $\text{Hom}(C_5, H)$. This is the constant copy of $H$.

A homorphism in $\text{Hom}(C_d, H)$ contracts if it is in the same component as the constant copy of $H$. 
A contraction of a copy $C$ of $C_5$ in $H$ can be viewed as a homomorphism $P_d \times C_5$ to $H$, where the first copy of $C_5$ is $C$ and the last copy is constant.
Consider the graph $H$ above.
In $\text{Hom}(\tilde{C}_5, H)$, the outer cycle is adjacent to the inner $C_5$, so contracts.
However it is isolated in $\text{Hom}(C_5, H)$. 
Theorem [LLT 06, BFHHM 08]

If a reflexive graph admits a NUF then it is dismantlable.

Well known

If a graph $H$ is dismantlable then $\text{Hom}(G, H)$ is connected.

So $H$ omits NUF.
Theorem [Loten 03]
The shown graph admits TSI of every arity (so WNU).

Theorem [Larose 04]
If $H$ admits a Taylor term (so WNU) then any cycle $C_d$ in $H$ contracts in $\text{Hom}(C_D, H)$ for large enough $D$. 
So the outer $C_5$, when allowed to expand to a $C_6$ contracts, as it should.
Lemma

Two $C_d$s are homotopic in $\text{Hom}(C_d, H)$ if and only if they are homotopic in $\text{Hom}(C_{d+1}, H)$. 
Proof:
Proof:
Proof:
Proof:
Proof:
Proof:
Proposition

If a reflexive graph $H$ admits a $d$-ary TSI then any copy of $C_d$ contracts in $\text{Hom}(\overrightarrow{C_d}, H)$, (so in $\text{Hom}(C_{d+1}, H)$).
Proof:

For a $C_5$, $a \sim b \sim c \sim d \sim e$ in $H$, the TSI restricted to the following $P_4 \times C_5$ in $H^5$

![Diagram showing a grid with labels from aaaa to edcba, illustrating the contraction]

is a contraction of the $C_5$. 
Larose’s [La04] result that all cycles contract (if allowed to expand) if $H$ has a Taylor term follows

- by the above proof
- from Barto and Kozik’s [BK10] result that such $H$ has a cyclic term of some arity.
So far...

- If $H$ admits WNU then all $C_d$ contract with expansion.
- If $H$ admits SL then all $C_d$ contract by expanding at most one (as $H$ admits TSI of all arity).
- If $H$ admits NUF then all $C_d$ contract without expanding (as $H$ is dismantlable).
Relative Homotopy

**Definition**

For reflexive graphs $G$ and $H$ and vertices $g$ and $h$ of these graphs respectively, let $\text{Hom}(G, g; H, h)$ be the subgraph of $\text{Hom}(G, H)$ induced by homomorphisms taking $g$ to $h$.

Using a result of [LLT 06] on can easily show

**Proposition**

If $G$ admits a NUF, then for all $G, g, H, h$, $\text{Hom}(G, g; H, h)$ is connected.
Relative Homotopy Example

The outer circle contracts relative to the blue vertex.
Relative Homotopy Example

The outer circle contracts relative to the blue vertex.
Relative Homotopy Example

The outer circle contracts relative to the blue vertex.
Relative Homotopy Example

But not relative to the red one. So it omits NUF.
Observation: Not only are reflexive graphs admitting NUF dismantlable, they must have at least two dismantlable vertices.
If $H$ has a semilattice polymorphism $\land$ then applying $\phi(v_1, \ldots, v_5) = v_1 \land \cdots \land v_5$ to
If $H$ has a semilattice polymorphism $\land$ then applying $\phi(v_1, \ldots, v_5) = v_1 \land \cdots \land v_5$ to

gives a homotopy of $a \sim \ldots e \sim a$ to $\land C_5$. 

If $\bigwedge C_5$ is a vertex in $C_5$, then
If $\bigwedge C_5$ is a vertex in $C_5$, then it appears at least twice consecutively in the first step of the homotopy, so the cycle can be viewed as shrinking to a $C_4$. 
And continuing to shrink by at least one vertex per step.
We can view such a homotopy as homomorphism of this to $H$. 
In the case that $\land$ is a (embedded) tree SL
In the case that $\wedge$ is a (embedded) tree $\text{SL}$

$\wedge$ appears at least twice in the first step, whether or not $\wedge C_5$ is in $C_5$, but not necessarily consecutively.
In the case that $\wedge$ is a strong-$\vee$ tree,
In the case that $\wedge$ is a strong-$\forall$ tree,

$\wedge$ appears at least twice consecutively in the first step, whether or not $\wedge C_5$ is in $C_5$. 
In the case that $\land$ is a strong-$\lor$ tree,
In the case that $\land$ is a strong-$V$ tree,

Then the strong $V$-property implies more edges.
In just the first step one can find a 'strong' contraction of the $C_6$ to a $C_3$ in $\text{Hom}(C_6, H)$, and a contraction of $C_9$ to a $C_3$ in $\text{Hom}(C_9, H)$.
So if $H$ has a

- TSI, then cycles contract without expanding.
- NUF, then cycles contract relative to any vertex.
- SL, then cycles contract, shrinking at each step except maybe the first.
- Tree SL, then cycles contract, shrinking at each step.
- Strong-$V$ SL, cycles contract quickly.

( In $\text{Hom}(C_d, H)$. )
So, this...

Omits tree SL as the outer circle is not adjacent in $\text{Hom}(C_5, H)$ to anything with only 4 distinct vertices.
So, this...

This graph has an embedded SL, but omits tree SL by the same reason. ...
So, this...

This graph has a NUF and an tree SL but omits strong-$\forall$ SL.
Also, can any of these classes be characterised in terms of these homotopies of circles?
Also.

Can any of these classes be characterised in terms of these homotopies of circles?
Thank-you