Semilattice Polymorphisms on Reflexive Graphs

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Fields Workshop on CSP - 2011

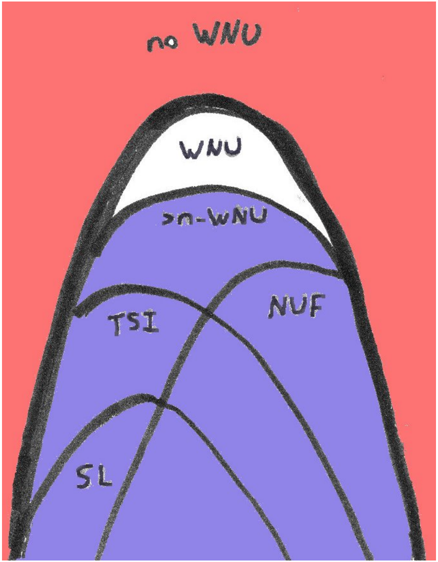
Outline

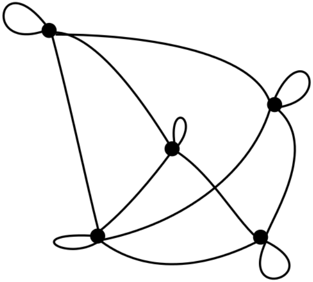
- Motivation
- Definitions and Basics
- A heirarchy of Semilattice Polymorphisms
- Semilattice vs. NUF
- Homotopies of cycles

Recall...

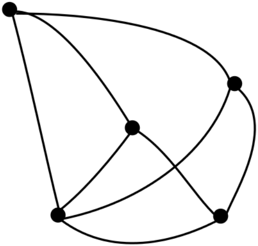
CSP Dichotomoy Classification Conjecture

For a core relational structure $\mathcal{H}$, $\mathrm{CSP}(\mathcal{H})$ is NP-complete if $\mathcal{H}$ omits WNU polymorphisms and is otherwise polynomial time solvable.

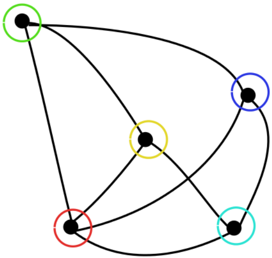

no WNU
WNU
>n-WNU
TSI
NUF
SL



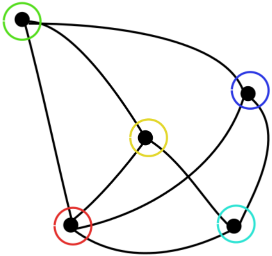
A symmetric graph H is reflexive if every edge has a loop.



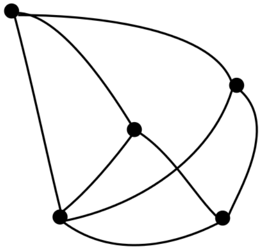
We usually don't draw the loops.



$\mathrm{CSP}(H)$ is trivial for such H, so we assume H also has all singleton unary relations.



- $\mathrm{CSP}(H)$ is H-precolouring extension
- Any H is a core.
- All polymorphisms are idempotent.



We don't draw these singleton relations either.

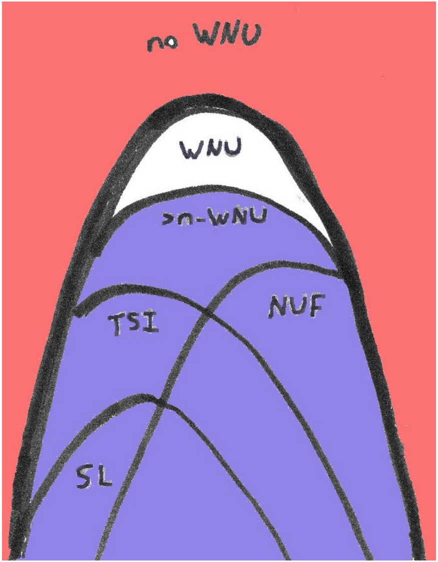
- CSP Dichotomy is equivalent to Dichotomy for symmetric reflexive graphs. [Feder Vardi 98] .

- CSP Dichotomy is equivalent to Dichotomy for symmetric reflexive graphs. [Feder Vardi 98] .
- Dichotomy is done for MinHOM of reflexive graphs. [Gutin Hell Rafiey Yao 07] .

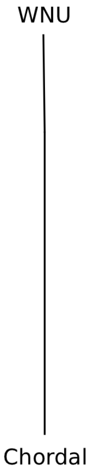
A vague notion: Reflexive Graphs have no Pushing

- One characterization of the set of structures omitting WNU is that one can make edge gadgets for them to encode 3-colouring.
- For this some vertex gadget that will map to one of three spots, and some edge gadget that keeps two vertex gadgets apart.
- To keep them apart the only way we can do it seems to be to *push* them apart, for which we basically need direction, or to *pull* them apart by wrapping the gadget around a 'hole'.
- In mapping to reflexive graphs, there is only pulling, requiring 'holes', which will translate to cycles that we can't move across- induced cycles, but more than just that.

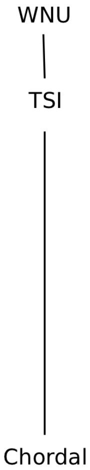
Why Semilattice polymorphisms?



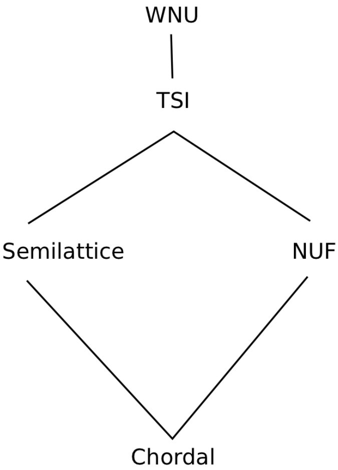
For reflexive graphs that 'hardness' comes from induced cycles that are 'non-contractible'.



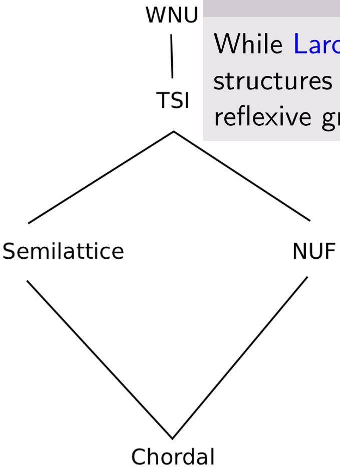
Chordal graphs have no induced cycles, while Larose '04 shows that if a reflexive graph admits WNU then cycles 'contract'.



Graphs admitting TSI 'contract' in a stronger sense.

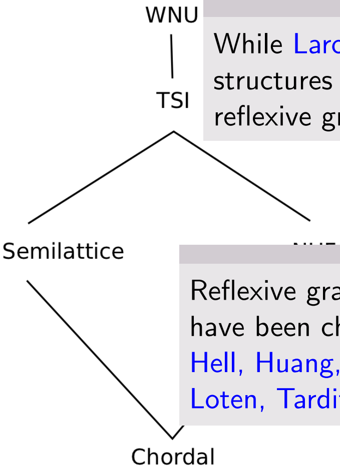


Between this are graphs admitting SL or NUF.



While Larose and Zadori [03] show there are structures in $\mathrm{NUF} \setminus \mathrm{TSI}$, Loten [03] shows for reflexive graphs that $\mathrm{NUF} \subset \mathrm{TSI}$.

Between this are graphs admitting SL or NUF.



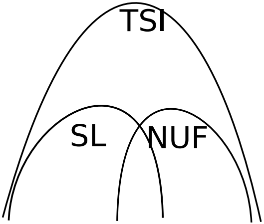
While Larose and Zadori [03] show there are structures in $\mathrm{NUF} \setminus \mathrm{TSI}$, Loten [03] shows for reflexive graphs that $\mathrm{NUF} \subset \mathrm{TSI}$.

Reflexive graphs admitting NU polymorphisms have been characterised by Brewster, Feder, Hell, Huang, MacGillivray [06] and Larose, Loten, Tardif, [06] .

Between this are graphs admitting SL or NUF.

Our goals are to...

- Draw this diagram properly.
- Characterise the various intersections.
- Relate the intersections to the way circles contract.

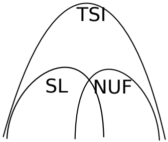


Definition: Polymorphism

A *polymorphism of H* is a homomorphism of H^d to H.

Polymorphisms of reflexive graphs are necessarily *idempotent*:

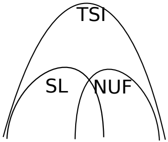
$$\phi(a, a, \ldots, a) = a$$



A polymorphism is TSI if

$$\phi(x_1, \ldots, x_d) = \phi(y_1, \ldots, y_d)$$

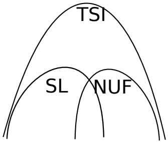
when $\{x_1, \ldots x_d\} = \{y_1, \ldots, y_d\}$ as sets.



A polymorphism is a NUF if

$$
\begin{aligned}
& \phi(x, x, \ldots, x, y) \\
= \;& \phi(x, x, \ldots, y, x) \\
= \;& \vdots \\
= \;& \phi(y, x, \ldots, x, x) = x
\end{aligned}
$$

for all x, y.



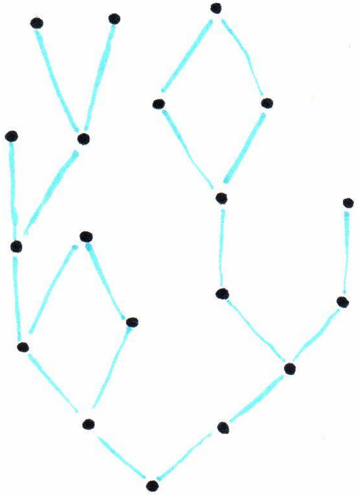
A 2-ary polymorphism is SL (semi-lattice) if it is symmetric and associative.

Recall that

A semilattice operation ϕ on a set of points defines a partial order of the points by

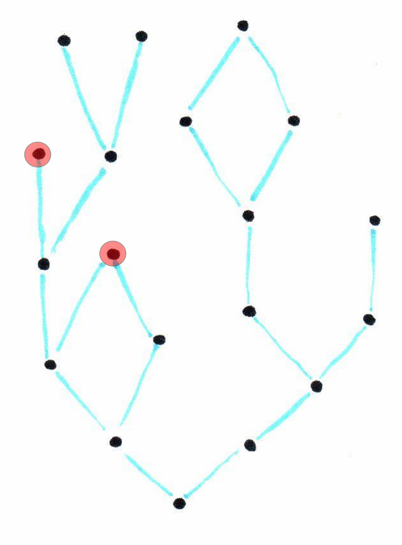
$$u < v \text{ if } \phi(u, v) = u.$$

We represent a semilattice by its Hasse diagram of covers.



Recall that

This partial ordering is a semilattice ordering; that is, every pair of points a, b, has a glb $a \wedge b$.



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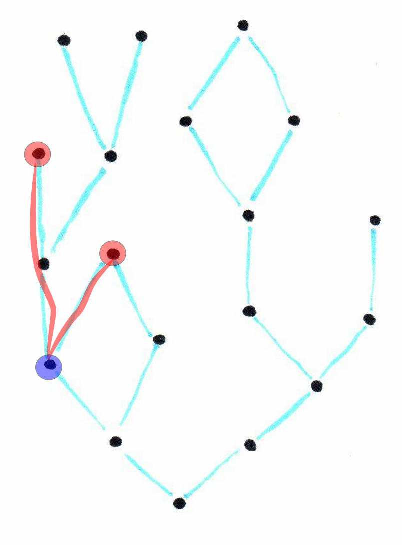
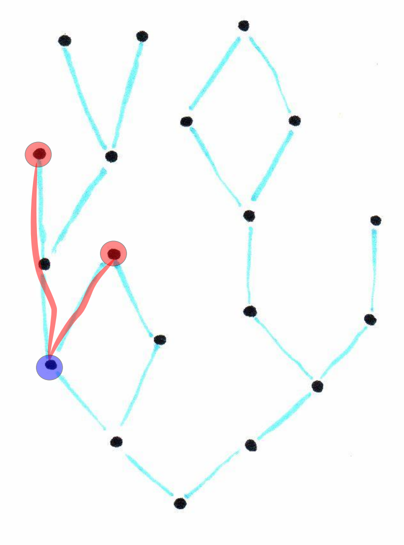
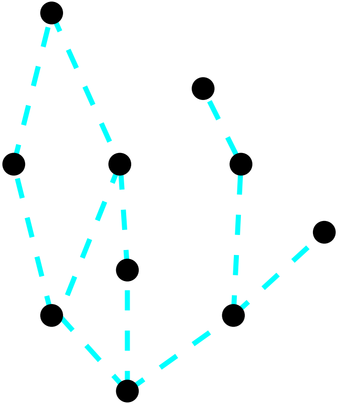


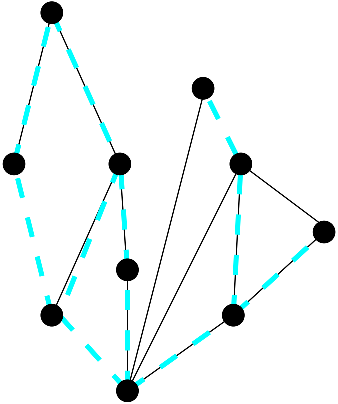
Recall that

The semilattice operation is glb: $\phi(a, b) = a \wedge b$.

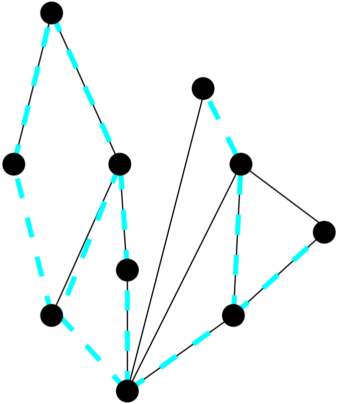




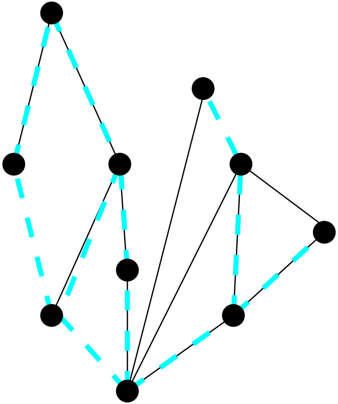
Given a semilattice operation on a set of vertices,



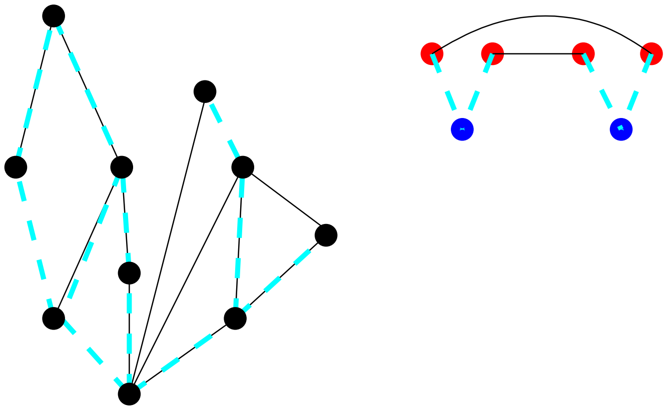
Given a semilattice operation on a set of vertices, and a reflexive graph on the vertices,



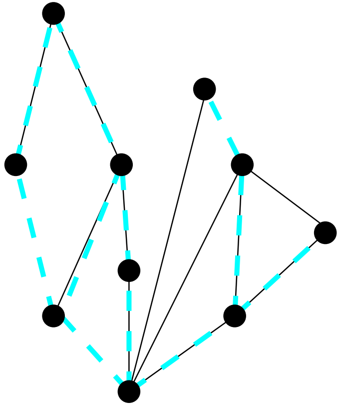
What other edges are needed so that the semilattice operation is a polymorphism?

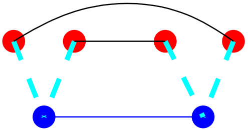


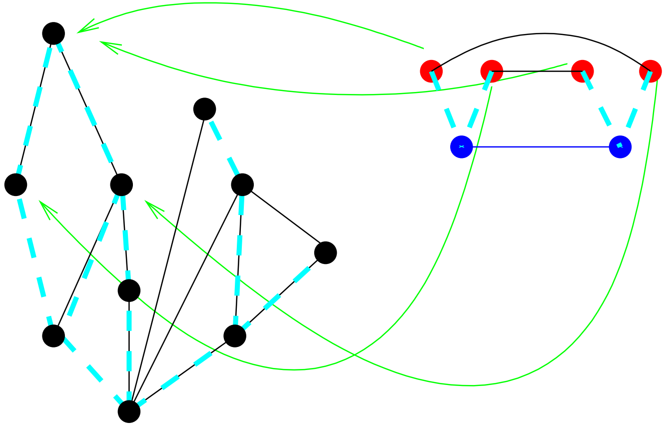
Polymorphism: $u \sim u', v \sim v' \Rightarrow u \wedge v \sim u' \wedge v'$

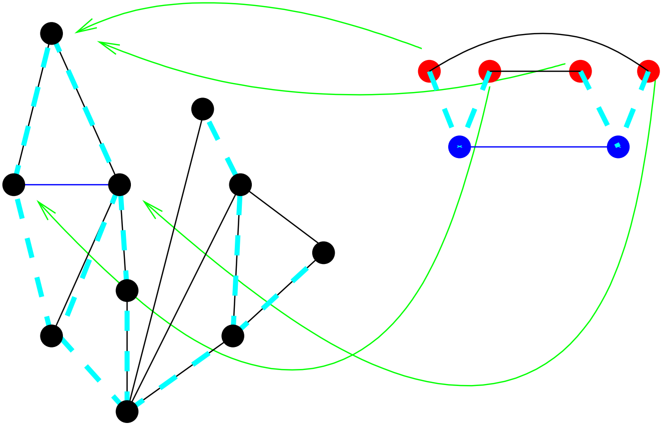


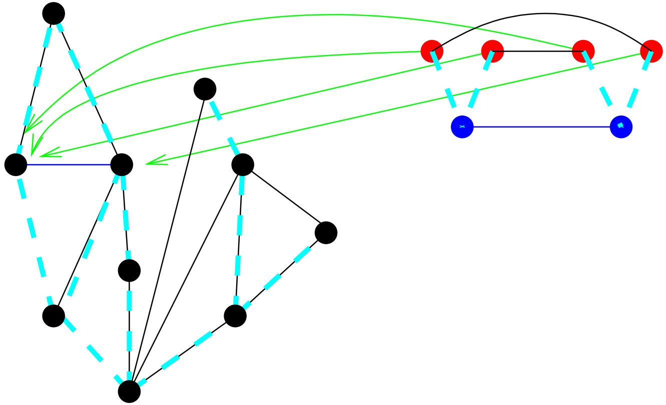
Polymorphism: $u \sim u', v \sim v' \Rightarrow u \wedge v \sim u' \wedge v'$

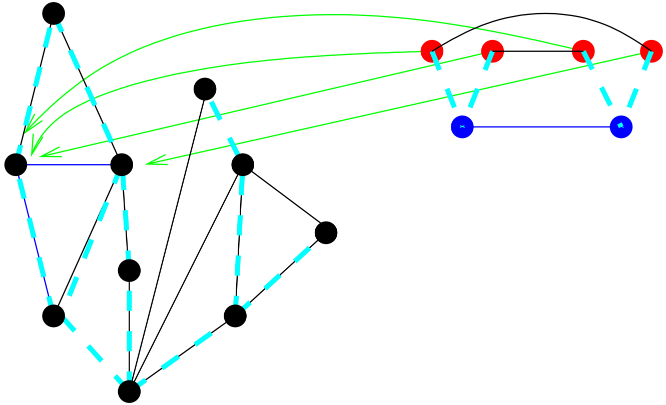


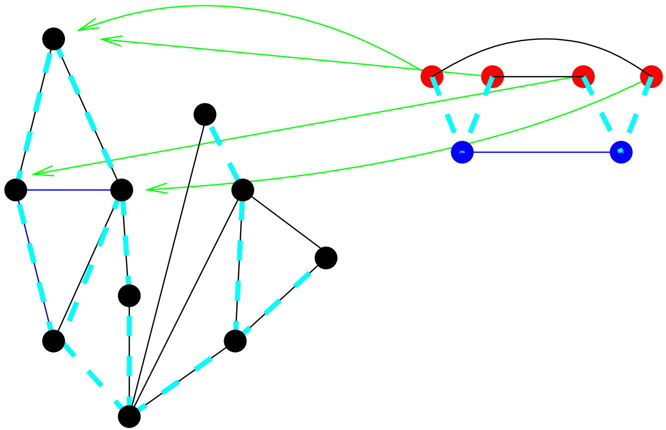


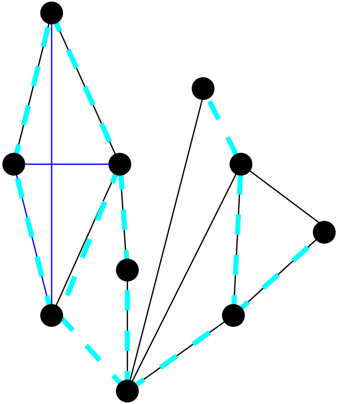


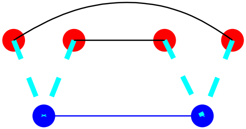


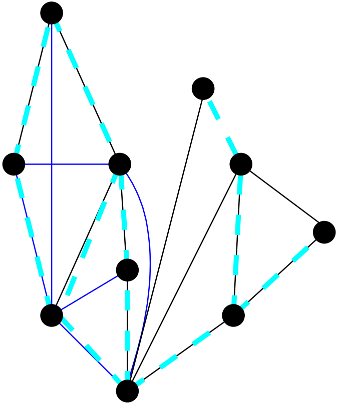


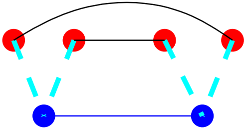


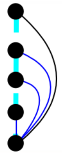




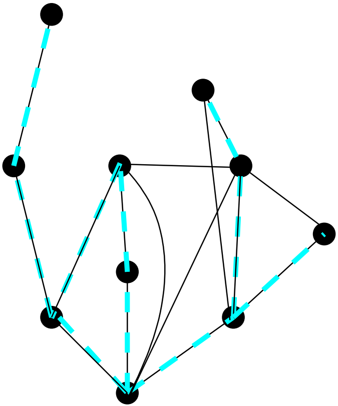






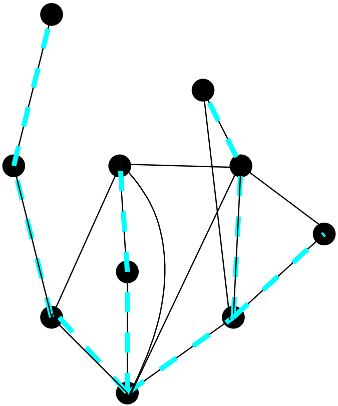


For a chain semilattice, the polymorpihsm property simply becomes the *min*-property, or the X-underbar property, so by Feder Hell 98 , the graphs admitting chain semilattices are exactly interval graphs.



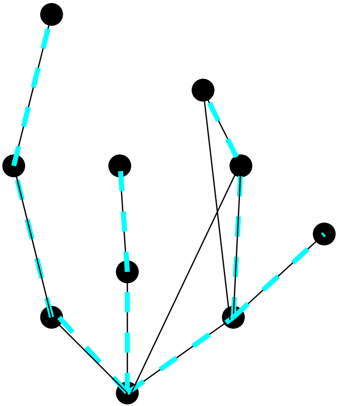
A semilattice polymorphism is ...

- embedded if every *Hasse* edge (blue edge) is a graph edge.



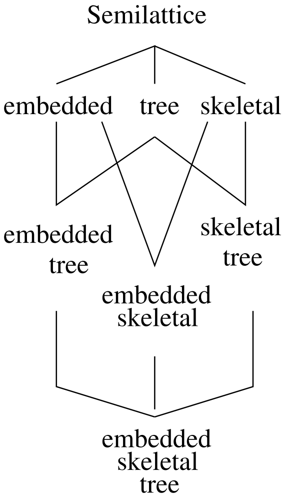
A semilattice polymorphism is ...

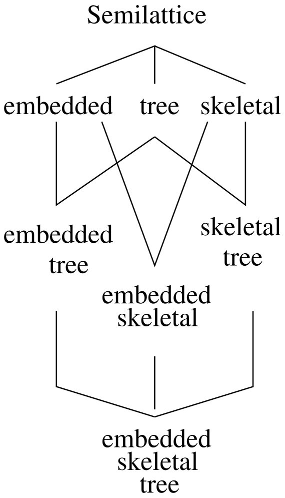
- embedded if every *Hasse* edge (blue edge) is a graph edge.
- tree if the Hasse edges induce a tree.



A semilattice polymorphism is ...

- embedded if every *Hasse* edge (blue edge) is a graph edge.
- tree if the Hasse edges induce a tree.
- skeletal if all graph edges are between comparible vertices.

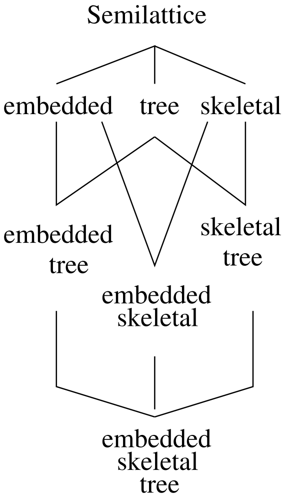

Semilattice
embedded
tree
skeletal
embedded tree
skeletal tree
embedded skeletal
embedded skeletal tree



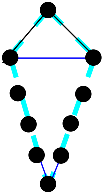
There are SL polymorphisms of graphs that are

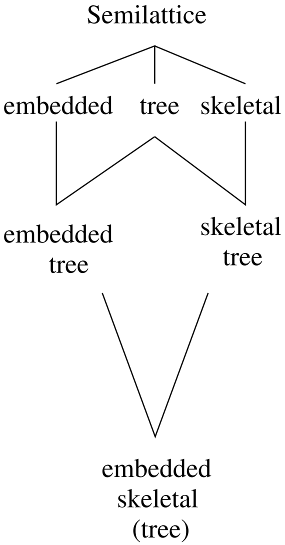
- tree and embedded but not skeletal
- tree and skeletal but not embedded

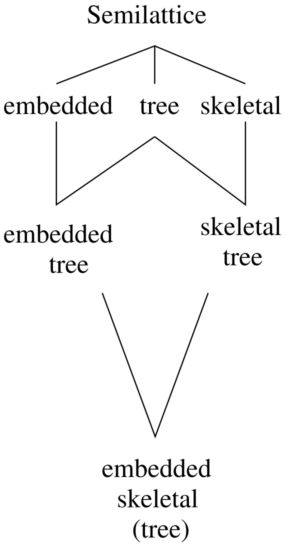
But...



If a SL polymorphism is skeletal and embedded, then it must be tree.




Semilattice
embedded
tree
skeletal
embedded tree
skeletal tree
embedded skeletal (tree)



Proposition

Any graph admitting a tree SL admits an embedded tree SL.

Proposition

Any graph admitting a skeletal SL admits a skeletal embedded tree SL.

Semilattice

embedded

tree

skeletal

TSI

Semilattice

embedded

tree

skeletal

chain = interval graph

On skeletal trees, the polymorphism property again simplifies to the X-underbar property, and so

Easy Proposition

A graph admits a skeletal polymorphism if and only if it is *chordal*.

TSI

Semilattice

embedded

tree

skeletal $=$ chordal

chain $=$ interval graph

Recall that...

a chordal graph can be represented as the intersection graph of a set of subtrees of some tree.

Definition

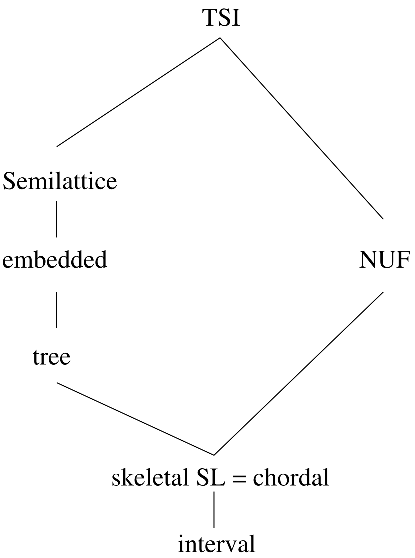
The leafage of a chordal graph H is the minimum number of leaves in a tree that gives an intersection representation of H.

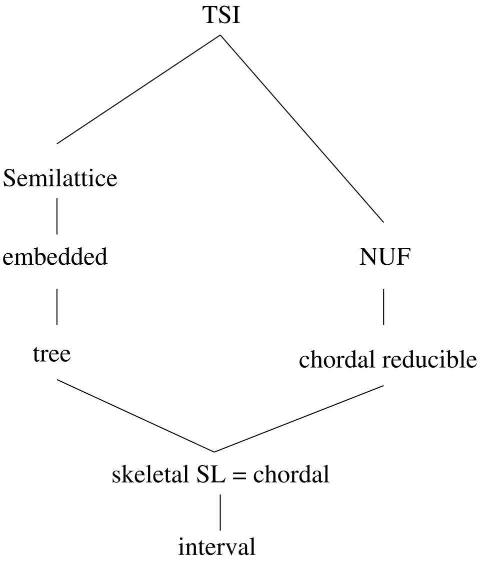
Theorem [BFFHM]

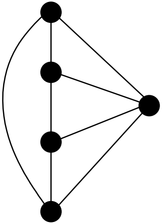
Every chordal graph of leafage k admits a NUF of arity $k+1$.

Proposition

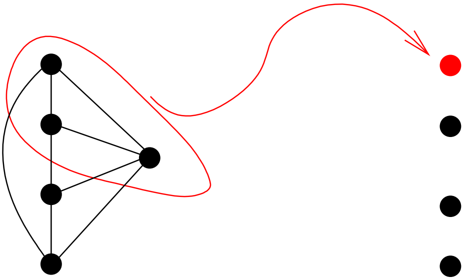
A chordal graph has leafage k if and only if it admits a skeletal SL polymorphism in which the Hasse diagram has k leaves.


TSI
Semilattice
embedded
NUF
tree
skeletal SL = chordal
interval

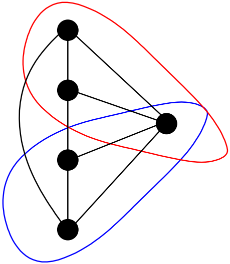

TSI
Semilattice
embedded
tree
NUF
chordal reducible
skeletal SL = chordal
interval

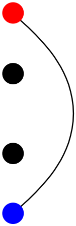


Given a graph H,



Given a graph H, take its clique graph $\mathrm{CL}(H)$,





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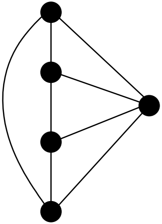

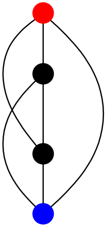

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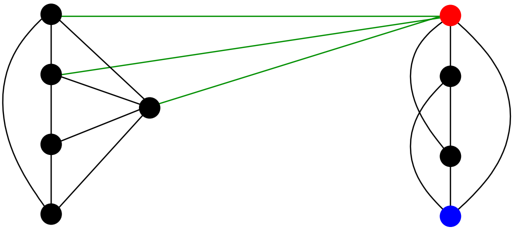
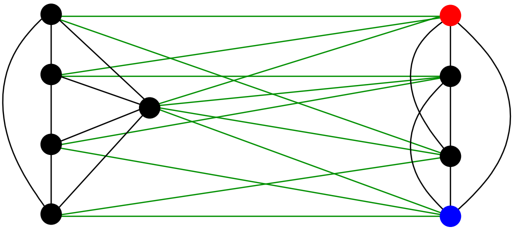


Given a graph H, take its clique graph $\mathrm{CL}(H)$, and add edges between them accoring to incidence: $\mathrm{CR}(H)$.



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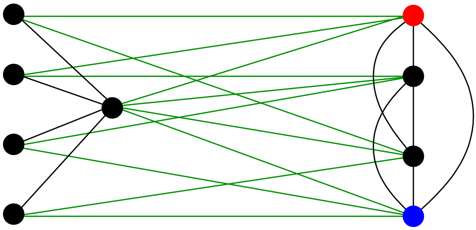
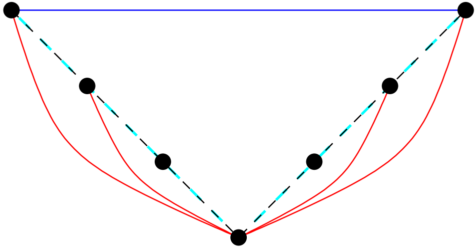


If we can remove edges from H such that it remains connected, and the full graph $\mathrm{CR}^*(H)$ is chordal, then H is chordal reducible .

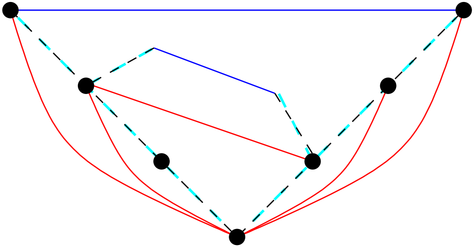
- Chordal graphs are chordal reducible.
- Graphs with a universal vertex are chordal reducible.
- Chordal reducible graphs have NUF of some arity.
- Are all graphs with 4-NU chordal reducible?

V properties of tree SL polymorphisms



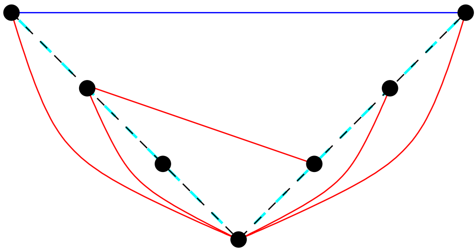
In a graph with an embedded SL polymorphism, and edge between incomparible vertices induces edges in the V below it.

V properties of tree SL polymorphisms



Other parts of the graph may induce more edges in the V, so these edges are not enough to ensure we have a polymorphism.

V properties of tree SL polymorphisms



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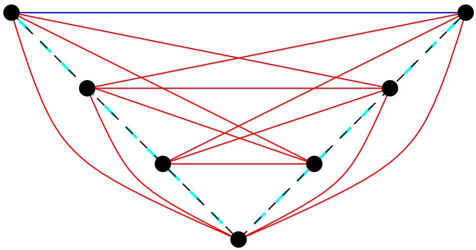
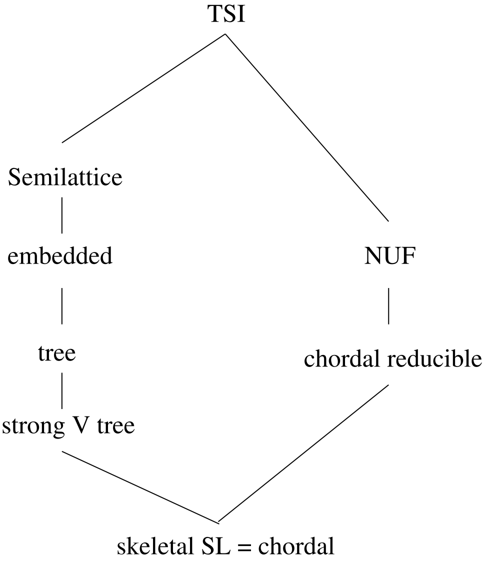
V properties of tree SL polymorphisms



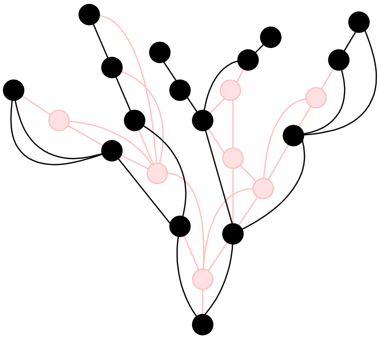
The *strong-V property* more than ensures our SL ordering is a polymorphism.


TSI
Semilattice
NUF
embedded
tree
chordal reducible
strong V tree
skeletal SL = chordal

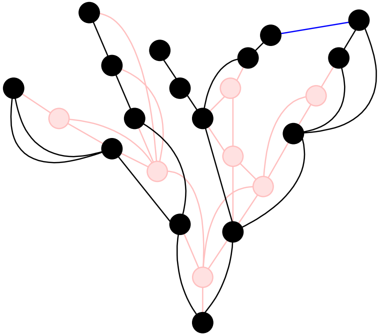
Proposition

Chordal reducible graphs admit strong V tree polymorphisms.

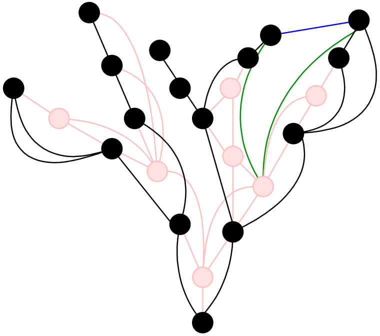
Proof:



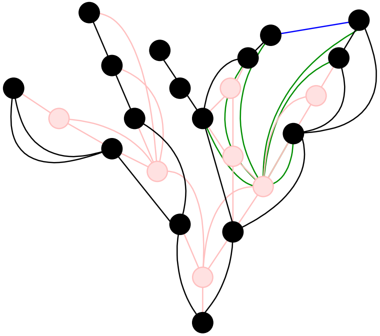
Proof:



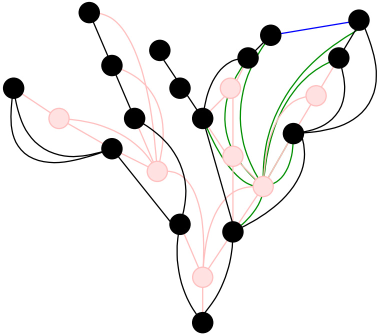
Proof:



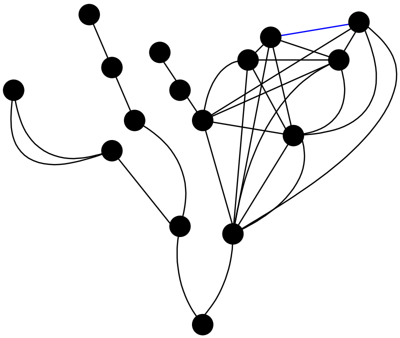
Proof:



Proof:



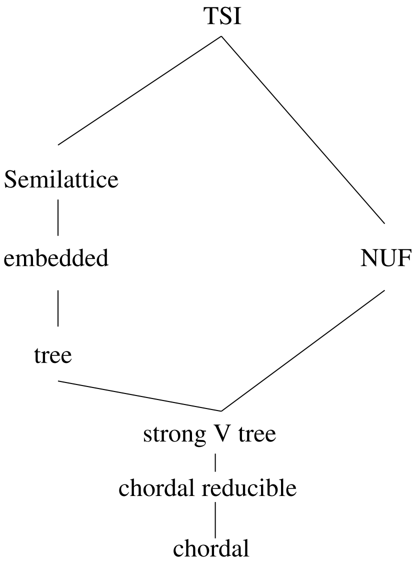
Proof:

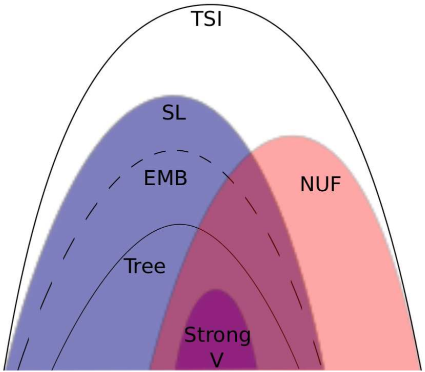


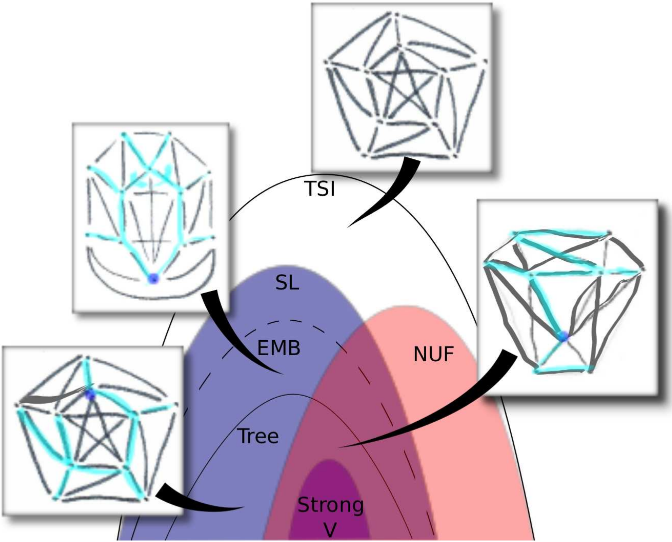
Further...

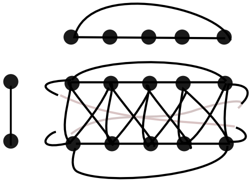
Proposition

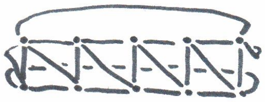
If a reflexive graph admits a strong-V tree SL, then it admits NUF.


TSI
Semilattice
NUF
embedded
tree
strong V tree
chordal reducible
chordal

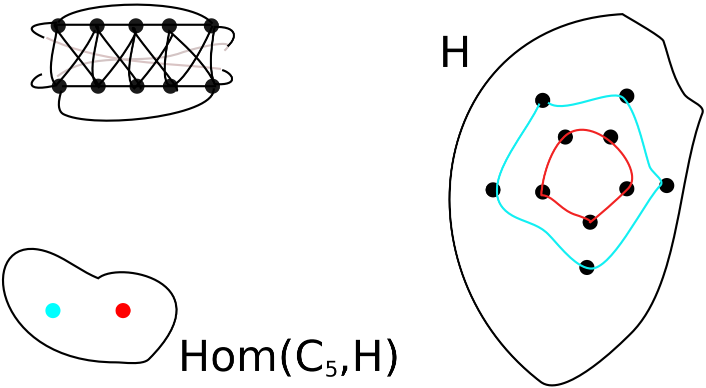

TSI
SL
EMB
NUF
Tree
Strong
V


TSI
SL
EMB
NUF
Tree
Strong
V

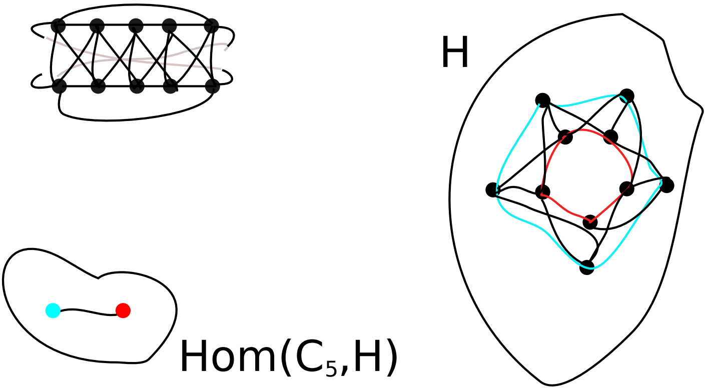




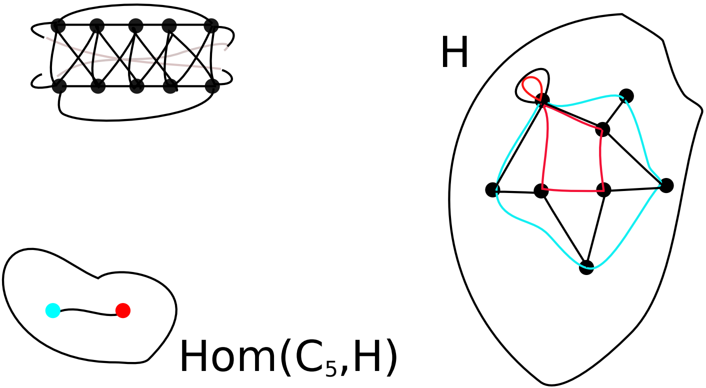
The starting point of the theory of homotopy of graphs it the product $I \times C_d$, if $I \times \vec{C_d}$.



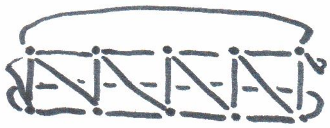
The graph $Hom(C_d, H)$ for a graph H has as vertices the homomorphisms of C_d to H.



Two are adjacent if the are the restriction of a homomorphism of $I \times C_d$ to the end copies of C_d.

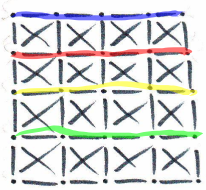


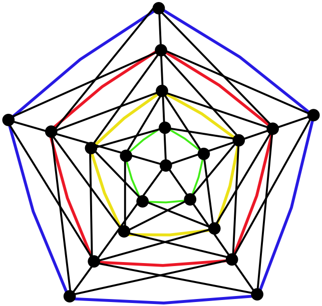
Of course, the homomorphisms need not be injective.



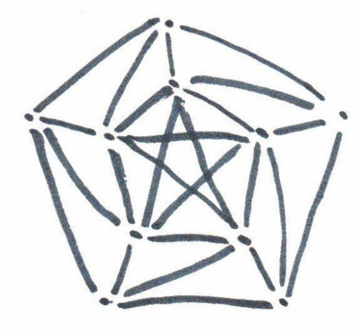
We can do the same with a directed cycle $\vec{C}_d$.

- As H is reflexive, the constant maps induce a copy of H in $\mathrm{Hom}(C_5, H)$. This is the *constant copy of* H.
- A homorphism in $\mathrm{Hom}(C_d, H)$ *contracts* if it is in the same component as the constant copy of H.

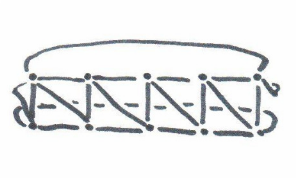
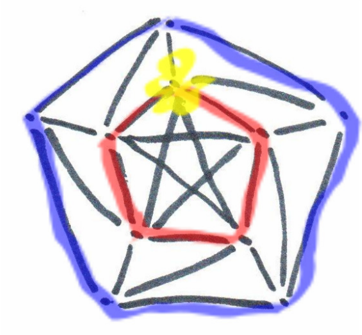




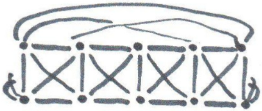
A contraction of a copy C of C_5 in H can be viewed as a homomorphism $P_d \times C_5$ to H, where the first copy of C_5 is C and the last copy is constant.

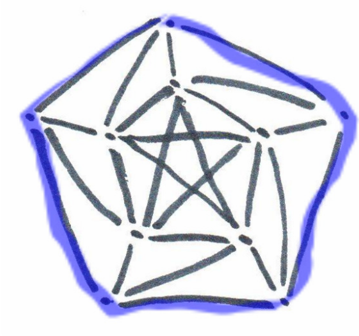


Consider the graph H above.

In $\mathrm{Hom}(\vec{C_5}, H)$, the outer cycle is adjacent to the inner C_5, so contracts.





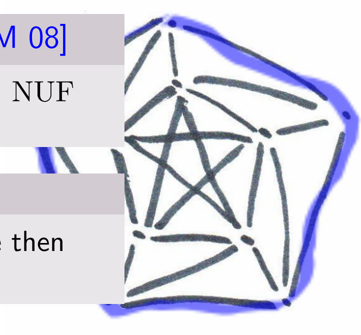
However it is isolated in $\mathrm{Hom}(C_5, H)$.

Theorem [LLT 06, BFHHM 08]

If a reflexive graph admits a NUF then it is dismantlable.

Well known

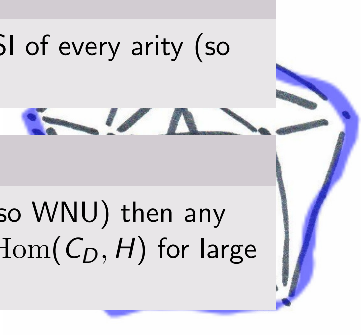
If a graph H is dismantlable then $\mathrm{Hom}(G, H)$ is connected.



So H omits NUF.

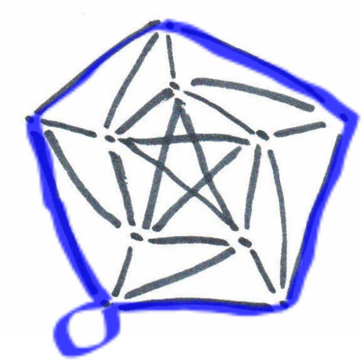
Theorem [Loten 03]

The shown graph admits TSI of every arity (so WNU).

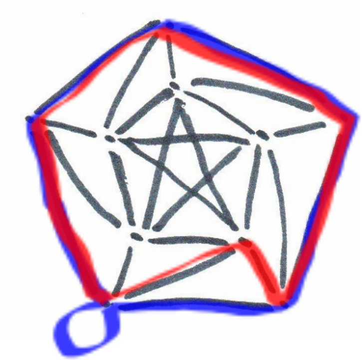


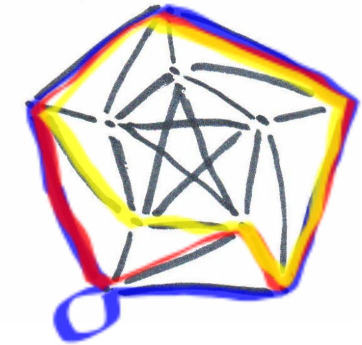
Theorem [Larose 04]

If H admits a Taylor term (so WNU) then any cycle C_d in H contracts in $\mathrm{Hom}(C_D, H)$ for large enough D.



So the outer C_5, when allowed to expand to a C_6 contracts, as it should.

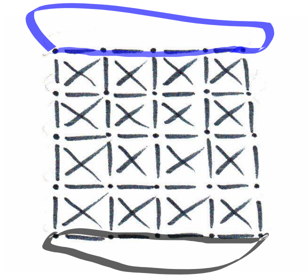


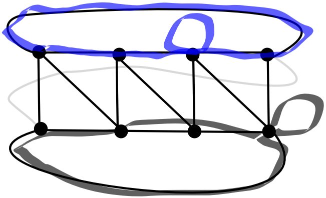


Lemma

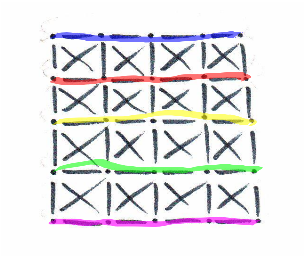
Two C_ds are homotopic in $\mathrm{Hom}(\vec{C_d}, H)$ if and only if they are homotopic in $\mathrm{Hom}(C_{d+1}, H)$.

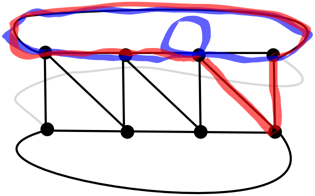
Proof:



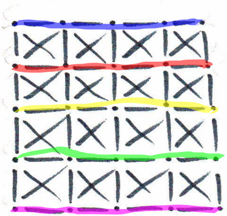


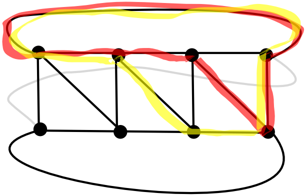
Proof:



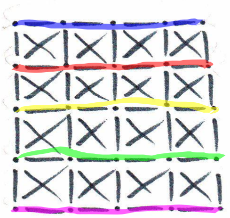


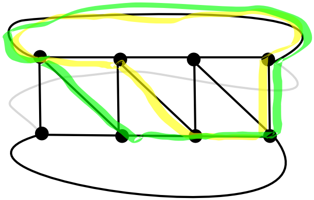
Proof:



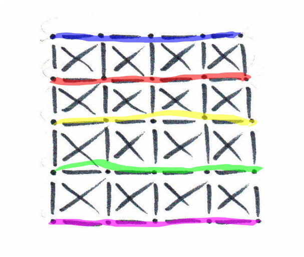


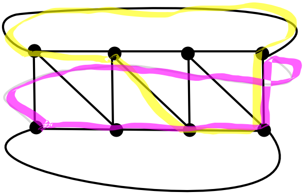
Proof:



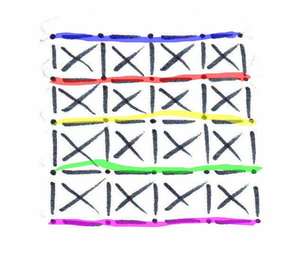
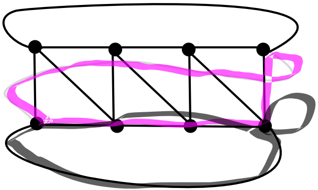


Proof:





Proof:

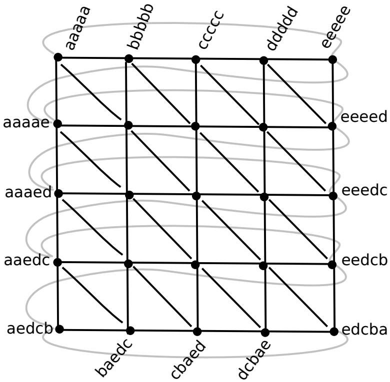
 

Proposition

If a reflexive graph H admits a d-ary TSI then any copy of C_d contracts in $\mathrm{Hom}(\vec{C_d}, H)$, (so in $\mathrm{Hom}(C_{d+1}, H)$).

Proof:

For a C_5, $a \sim b \sim c \sim d \sim e$ in H, the TSI restricted to the following $P_4 \times C_5$ in H^5



is a contraction of the C_5.

An aside

Larose's [La04] result that all cycles contract (if allowed to expand) if H has a Taylor term follows

- by the above proof
- from Barto and Kozik's [BK10] result that such H has a cyclic term of some arity.

So far...

- If H admits WNU then all C_d contract with expansion.
- If H admits SL then all C_d contract by expanding at most one (as H admits TSI of all arity).
- If H admits NUF then all C_d contract without expanding (as H is dismantlable).

Relative Homotopy

Definition

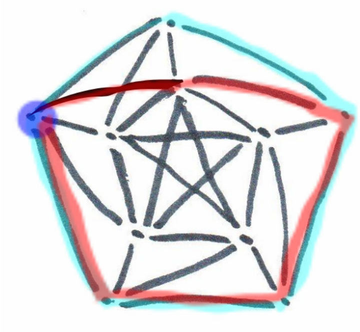
For reflexive graphs G and H and vertices g and h of these graphs respectively, let $\mathrm{Hom}(G, g; H, h)$ be the subgraph of $\mathrm{Hom}(G, H)$ induced by homomorphisms taking g to h.

Using a result of [LLT 06] on can easily show

Proposition

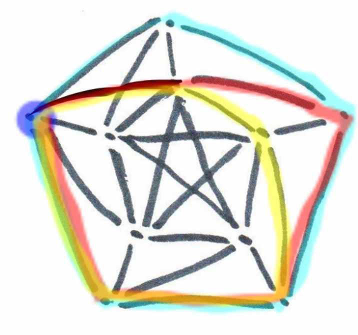
If G admits a NUF, then for all G, g, H, and h, $\mathrm{Hom}(G, g; H, h)$ is connected.

Relative Homotopy Example



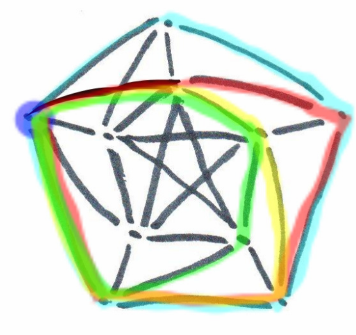
The outer circle contracts relative to the blue vertex.

Relative Homotopy Example



The outer circle contracts relative to the blue vertex.

Relative Homotopy Example



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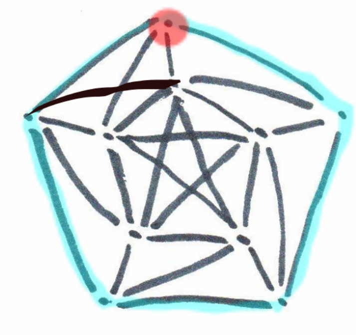
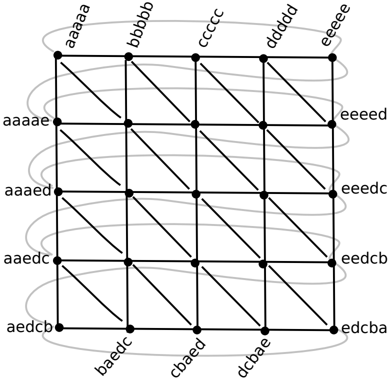
Relative Homotopy Example



But not relative to the red one. So it omits NUF.

Observation: Not only are reflexive graphs admitting NUF dismantlable, they must have at least two dismantlable vertices.

If H has a semilattice polymorphism $\wedge$ then applying $\phi(v_1, \ldots, v_5) = v_1 \wedge \cdots \wedge v_5$ to



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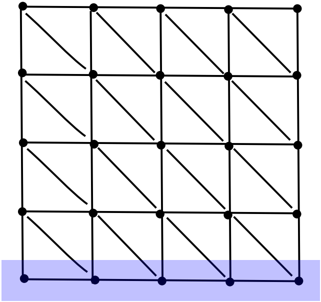
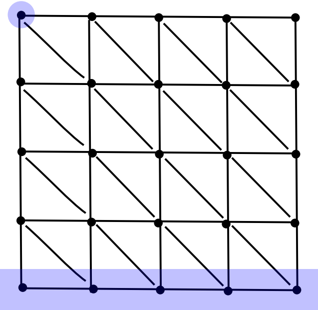


gives a homotopy of $a \sim \ldots e \sim a$ to $\bigwedge C_5$.

If $\bigwedge C_5$ is a vertex in C_5, then



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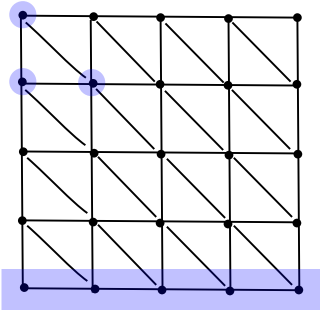
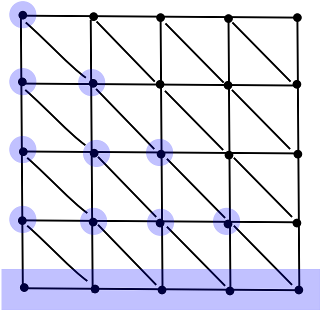
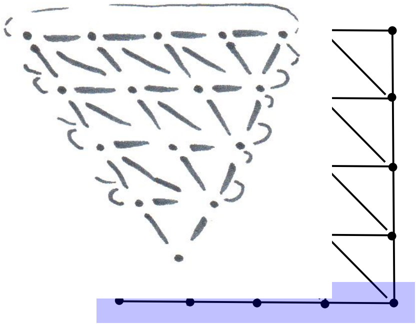


it appears at least twice consecutively in the first step of the homotopy, so the cycle can be viewed as shrinking to a C_4.

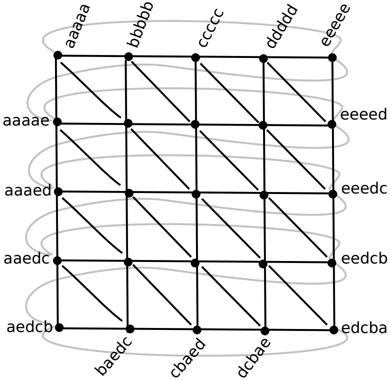


And continuing to shrink by at least one vertex per step.



We can view such a homotopy as homomorphism of this to H.

In the case that $\wedge$ is a (embedded) tree SL



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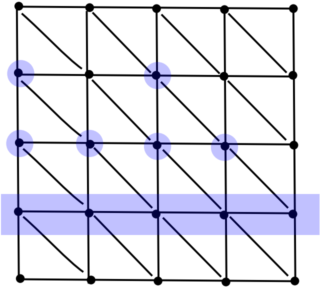
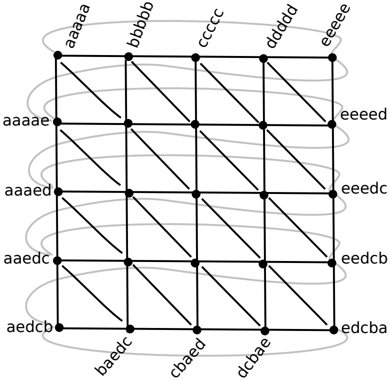


$\bigwedge$ appears at least twice in the first step, whether or not $\bigwedge C_5$ is in C_5, but not necessarily consecutively.

In the case that $\wedge$ is a strong-V tree,



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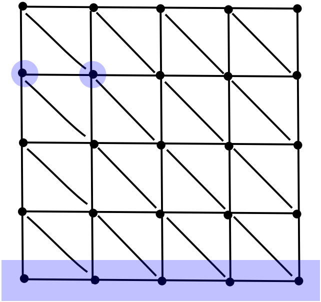
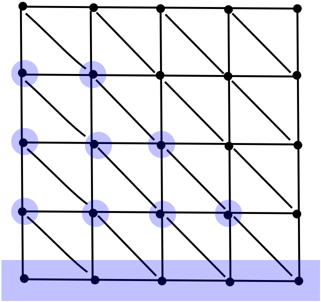
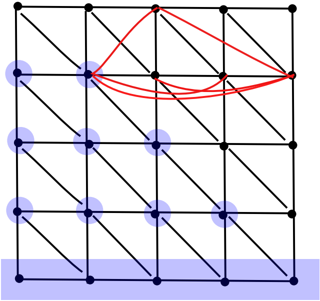


$\bigwedge$ appears at least twice consecutively in the first step, whether or not $\bigwedge C_5$ is in C_5.

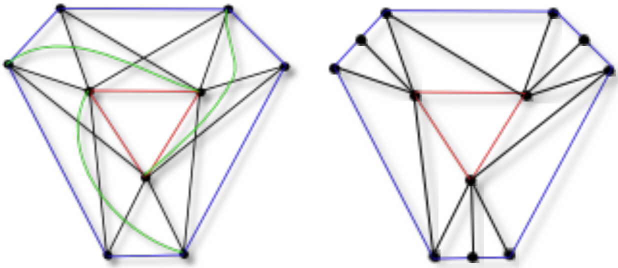
In the case that $\wedge$ is a strong-V tree,



In the case that $\wedge$ is a strong-V tree,



Then the strong V-property implies more edges.



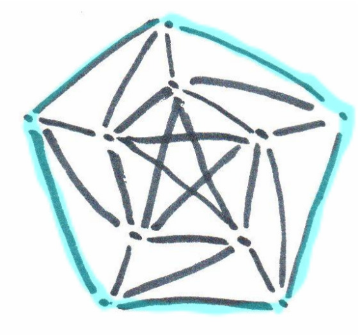
In just the first step one can find a 'strong' contraction of the C_6 to a C_3 in $\mathrm{Hom}(C_6, H)$, and a contraction of C_9 to a C_3 in $\mathrm{Hom}(C_9, H)$.

So if H has a

- TSI, then cycles contract without expanding.
- NUF, then cycles contract relative to any vertex.
- SL, then cycles contract, shrinking at each step except maybe the first.
- Tree SL, then cycles contract, shrinking at each step.
- Strong-V SL, cycles contract quickly.

(In $\mathrm{Hom}(\vec{C_d}, H)$.)

So, this...

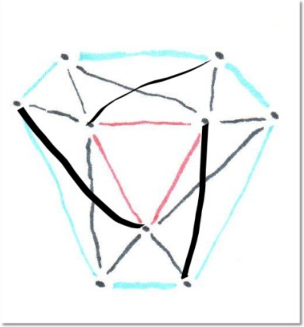


Omits tree SL as the outer circle is not adjacent in $\mathrm{Hom}(\vec{C_5}, H)$ to anything with only 4 distinct vertices.

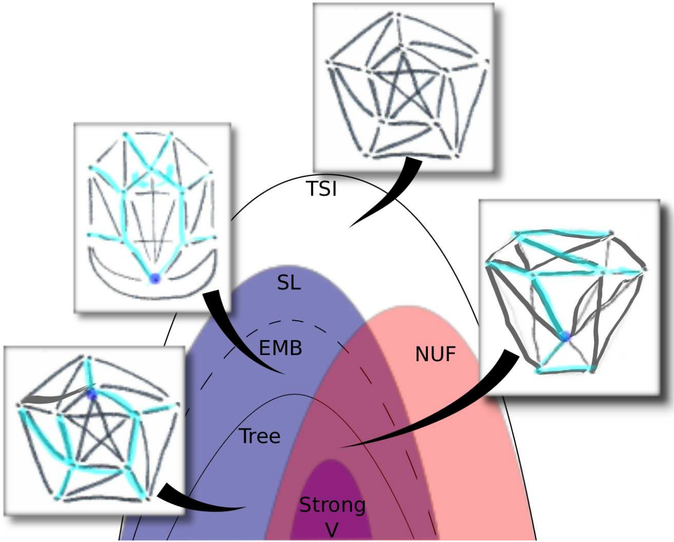
So, this...

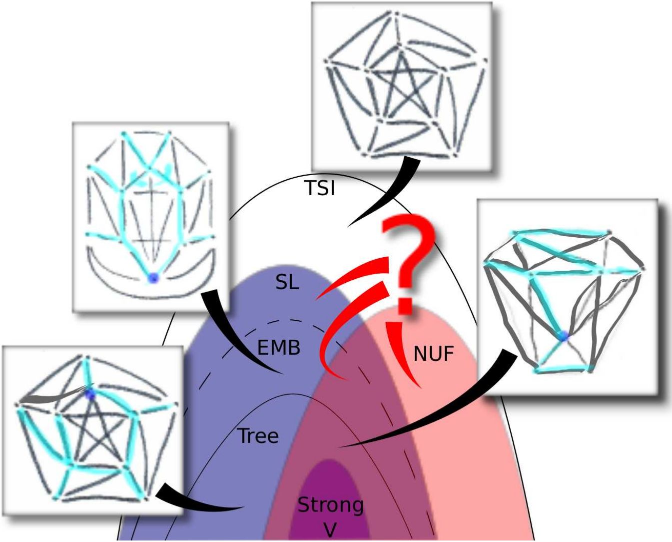
This graph has an embedded SL, but omits tree SL by the same reason. ...

So, this...



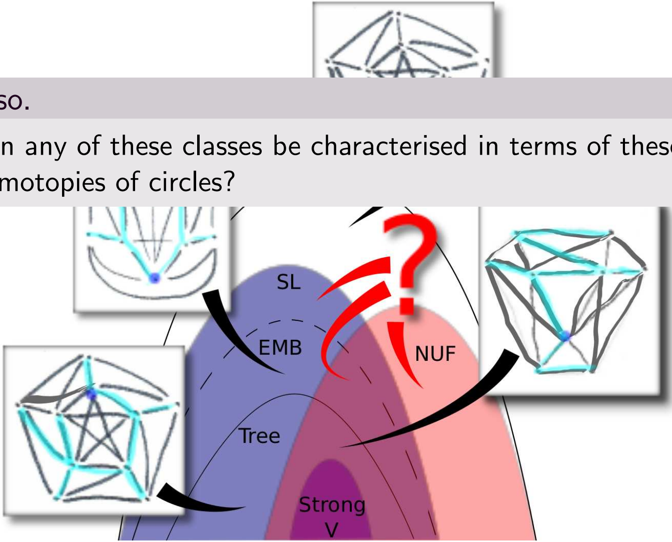
This graph has a NUF and an tree SL but omits strong-V SL.


TSI
SL
EMB
NUF
Tree
Strong
V


TSI
SL
EMB
NUF
Tree
Strong
V

Also.

Can any of these classes be characterised in terms of these homotopies of circles?



Thank-you