Partitions of graphs with bounded maximum average degree

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Proper k-coloring

Proper k-coloring

Let G be a graph and k ($k \geq 1$) an integer.
A proper k-coloring of G is a mapping $\phi : V(G) \longrightarrow \{1, \cdots, k\}$ such that:

- for every edge xy, $\phi(x) \neq \phi(y)$

In other words, a k-coloring of G is a partition $V_1, V_2, \cdots, V_k$ of $V(G)$ such that V_i is an independent set for every i, i.e., the subgraph induced by V_i has maximum degree zero.

d-improper k-coloring

Burr and Jacobson (1985), Cowen, Cowen, and Woodall (1986), Harary and Jones (1985).

d-improper k-coloring

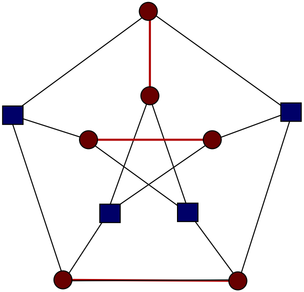
Let G be a graph and k, d $(k, d \geq 1)$ integers.
A d-improper k-coloring of G is a mapping $\psi : V(G) \longrightarrow \{1, \cdots, k\}$ such that :

- $\forall i, 1 \leq i \leq k$, $G[i]$ has a maximum degree at most d
- $G[i]$ is the subgraph induced by color i.

Every vertex v has at most d neighbors receiving the same color as v.

- a d-improper k-coloring : $(d, \cdots, d)$-coloring
- a $(0, 0, 0, 0)$-coloring is a proper 4-coloring.
- a $(2, 2, 2)$-coloring is a 2-improper 3-coloring.

d-improper k-coloring



$(1, 1)$-coloring

Known results

Appel and Haken, 1977

- Every planar graph is $(0, 0, 0, 0)$-colorable.

Cowen, Cowen, and Woodall, 1986

- Every planar graph is 2-improperly 3-colorable, i.e. $(2, 2, 2)$-colorable.

Xu, 2009

Every plane graph with neither adjacent triangles nor 5-cycles is $(1, 1, 1)$-colorable.

Known results-Choosability

Definition

A graph G is d-improper m-choosable, or simply $(m,d)^*$-choosable, if for every list assignment L, where $|L(v)| \geq m$ for every $v \in V(G)$, there exists an L-colouring of G such that each vertex of G has at most d neighbours coloured with the same colour as itself.

Eaton and Hull (1999), Škrekovski (1999)

- Every planar graph is 2-improper 3-choosable: $(3,2)^*$-choosable.

If a graph G is 2-improper 3-choosable then it is $(2,2,2)$-colorable.

Škrekovski proved that for every k, there are planar graphs which are not k-improper 2-colorable.

Known results

Cushing and Kierstead (2009)

Every planar graph is 1-improper 4-choosable ($(4, 1)^*$-choosable).

Dong and Xu 2009

Let G be a plane graph without any cycles of length in $\{4, 8\}$, then G is $(3, 1)^*$-choosable.

Question (Xu and Zhang, 2007)

Is-it true that every planar graph without adjacent triangle is $(3, 1)^*$-choosable.

Every planar graph without adjacent triangle is $(1, 1, 1)$-colorable?

Known results

Definition-Maximum average degree

$$\mathrm{Mad}(G) = max\left\{\frac{2 \cdot |E(H)|}{|V(H)|}, H \subseteq G\right\}.$$

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$$\mathrm{Mad}(G) = max\left\{\frac{2 \cdot |E(H)|}{|V(H)|}, H \subseteq G\right\}.$$

In 1995, Jensen and Toft showed that there is a polynomial algorithm to comput $Mad(G)$ for a given graph G.

T. R. Jensen and B. Toft, Choosability versus chromaticity, Geombinatorics 5(1995), 45-64.

if G is a planar graph with girth g, then $\mathrm{Mad}(G) < \frac{2g}{g-2}$.

Known results

Havet and Sereni, 2006

- For every $k \geq 0$, every graph G with $Mad(G) < \frac{4k+4}{k+2}$ is k-improperly 2-colorable (in fact k-improperly 2-choosable), i.e. (k,k)-colorable

- $k = 1$ $Mad(G) < \frac{8}{3}$: $(1,1)$-colorable (planar, $g = 8$).
- $k = 2$ $Mad(G) < 3$: $(2,2)$-colorable (planar, $g = 6$).

A more general result:

Theorem (Havet and Sereni)

For every $l \geq 2$ and every $k \geq 0$, all graphs of maximum average degree less than $\frac{l(l+2k)}{l+k}$ are k-improper l-choosable.

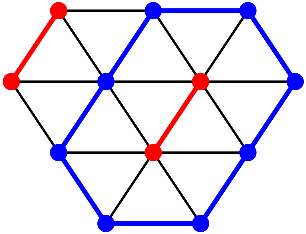
it implies $(k, \cdots, k)$-colorable.

$(d_1, d_2, \cdots, d_k)$-coloring

$(d_1, d_2, \cdots, d_k)$-coloring

A graph G is $(d_1, d_2, \cdots, d_k)$-colorable if and only if:

- it exists a partition of V: $V = V_1 \cup V_2 \cup \cdots \cup V_k$ such that $\forall i \in [1, k]$, $\Delta(G[V_i]) \leq d_i$



$(2, 1)$-coloring

In this talk: $(1, 0)$, $(k, 0)$, $(k, 1)$, (k, j)

$(1, 0)$-colorable

Theorem (Glebov and Zambalaeva, 2007)

Every planar graph is $(1, 0)$-colorable if $g(G) \geq 16$.

Theorem (Borodin and Ivanova, 2009)

Every graph is $(1, 0)$-colorable if $Mad(G) < \frac{7}{3}$

This implies: A planar graph is $(1, 0)$-colorable if $g(G) \geq 14$

Sketch of proof: every graph is $(1,0)$-colorable if $Mad(G) < \frac{7}{3}$

New technique first introduced by Borodin, Ivanova and Kostochka in 2006.

Let $G = (V, E)$ be a minimum counterexample to the theorem.

- $\delta(G) \geq 2$ and G is connected.
- $\frac{\sum_{v \in V} d(v)}{|V|} = \frac{2|E|}{|V|} < \frac{7}{3} \implies 3\sum_{v \in V} d(v) < 7|V|$

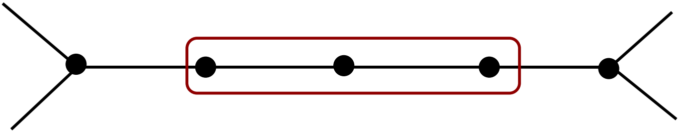
  $\implies \sum_{v \in V}(6d(v) - 14) < 0$
- We give a charge $\forall v \in V \;\; \mu(v) = 6d(v) - 14$
  $d(v) = 2 \implies \mu(v) = -2$, $d(v) = 3 \implies \mu(v) = 4$ etc.
- The total charge is negative, we will redistribute the charges in such a way that the total charge will be non negative. The sum of charges does not change, this is a CONTRADICTION.

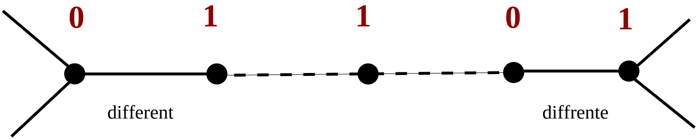
Sketch of proof: every graph is $(1,0)$-colorable if $Mad(G) < \frac{7}{3}$

Let $G = (V, E)$ be a minimum counterexample to the theorem.

Lemme

G does not contain no 2-vertex adjacent to two 2-vertices.

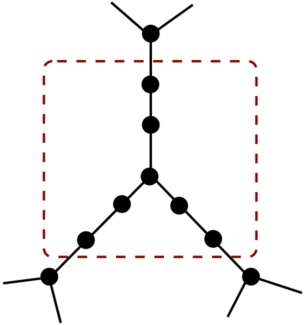


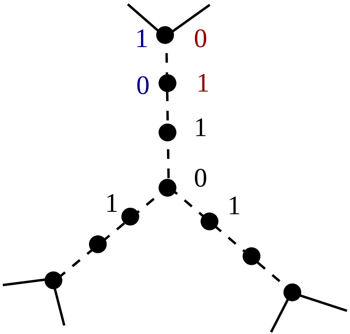


Sketch of proof: every graph is $(1,0)$-colorable if $Mad(G) < \frac{7}{3}$

Lemme

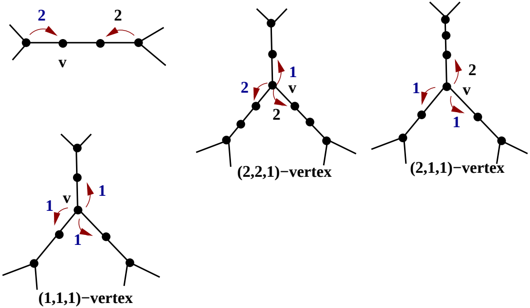
G does not contain a $(2,2,2)$-vertex.





Sketch of proof: every graph is $(1,0)$-colorable if $Mad(G) < \frac{7}{3}$

Recall: $d(v) = 2 \Longrightarrow \mu(v) = -2$, $d(v) = 3 \Longrightarrow \mu(v) = 4$

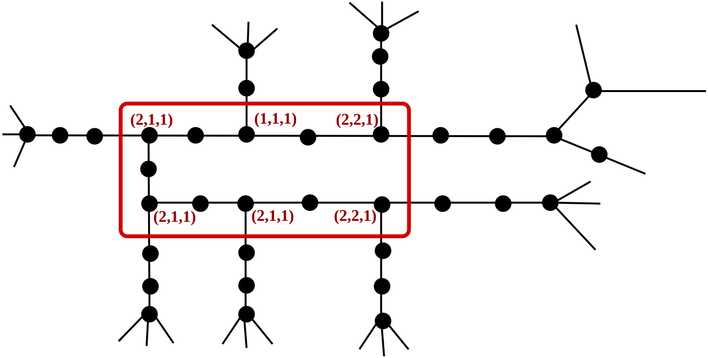


- 2-vertex $\Longrightarrow \mu^*(v) = 0$
- $(2,2,1)$-vertex $\Longrightarrow \mu^*(v) = -1$
- $(2,1,1)$-vertex $\Longrightarrow \mu^*(v) = 0$
- $(1,1,1)$-vertex $\Longrightarrow \mu^*(v) = 1$

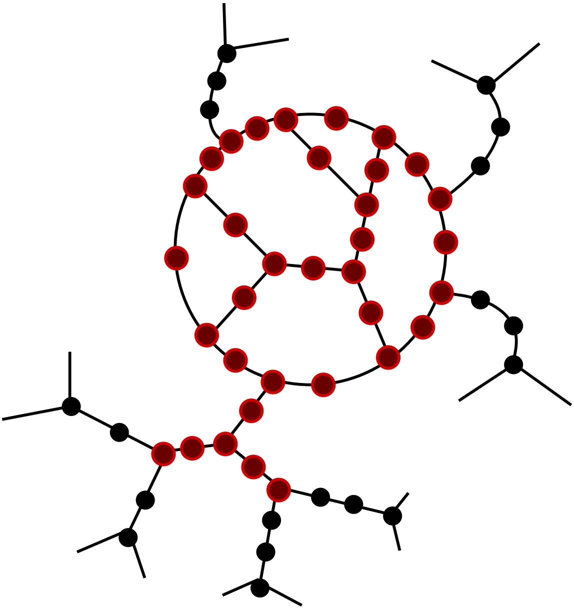
Sketch of proof: every graph is $(1,0)$-colorable if $Mad(G) < \frac{7}{3}$

Feading Area

A maximal subgraph consisting of $(2,2,1)$-,$(2,1,1)$,-$(1,1,1)$-vertices mutually accessible from each other along 1-paths, and of those 2-vertices adjacent to vertices of FA only.



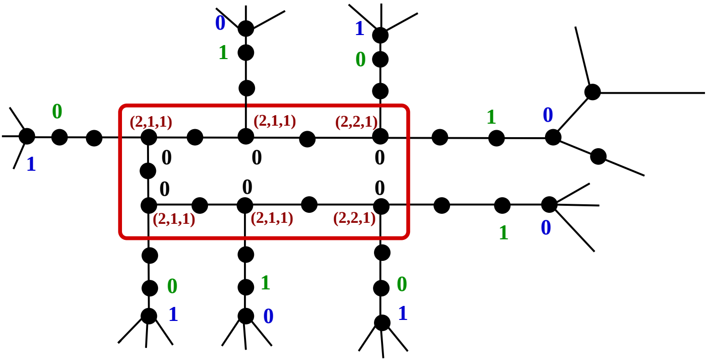
Sketch of proof: every graph is $(1,0)$-colorable if $Mad(G) < \frac{7}{3}$



Sketch of proof: every graph is $(1,0)$-colorable if $Mad(G) < \frac{7}{3}$

Soft component

A feading area FA such that all edges from FA to $G \setminus FA$ belongs to 2-paths



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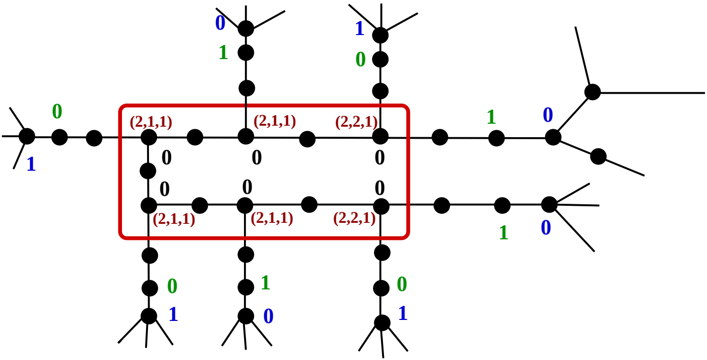


Lemma

G has no soft component.

Sketch of proof: every graph is $(1,0)$-colorable if $Mad(G) < \frac{7}{3}$

A *tough* vertex is a 3-vertex incident with at least one 0-path.
No soft component implies:

Corollary

For each feeding area FA, there exists a xz 1-path such that $x \in FA$ and $z \notin FA$ where z is tough or $d(z) \geq 4$

The key Lemma

Let n_{221} be the number of $(2,2,1)$-vertices in a feeding area FA, n_{111} be the number of $(1,1,1)$-vertices of FA, b be the number of 1-path going from FA to tough or ≥ 4-vertices. Then $n_{221} \leq n_{111} + b$

Rules of discharging

- R1 Every 2-vertex that belong to a 1-path gets 1 from its ends, each 2-vertex that belongs to a 2-path gets charge 2 from the neighbor vertex of degree greater than 2.
- R2 Each $(2,2,1)$-vertex gets 1 from its feeding area, each feeding area gets 1 from each of its $(1,1,1)$-vertex and along each 1-path that goes to a tough vertex or a ≥ 4-vertex.

Sketch of proof: every graph is $(1, 0)$-colorable if $Mad(G) < \frac{7}{3}$

- After appliying R1 and R2, the new charge of a 2-vertex v is $\mu^*(v) = 0$. By the Key lemma the total charge of each feeding area is non negative. Each tough vertex has also a nonnegative charge.

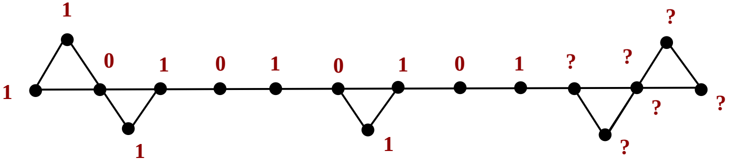
- If $d(v) \geq 4$. v gives at most 2 along each incident edge by R1 and R2, then:
$$\mu^*(v) \geq 6d(v) - 14 - 2d(v) = 4d(v) - 14 > 0$$

The total charge is non negative: a contradiction.

Improvement

Theorem (Borodin and Kostochka, 2010)

Every graph G with $Mad(G) \leq \frac{12}{5}$ is $(1,0)$-colorable and the restriction on $Mad(G)$ is sharp.



G_3

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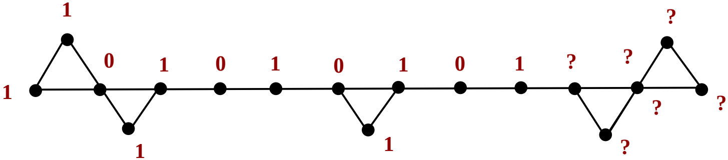


G_3

$$Mad(G_p) = \frac{2|E(G_p)|}{|V(G_p)|} = \frac{12p+6}{5p+2} = \frac{12}{5} + \frac{6}{5(5p+2)}$$

$(k, 0)$-coloring

$(k, 0)$-coloring

- Bipartition V_1,V_2 of $V(G)$
- $\Delta(G[V_1]) \leq k$ and $G[V_2]$ is a stable set.

Color k: vertices of V_1
Color 0: vertices of V_2.

$k = 1$

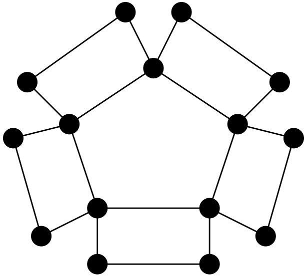
Theorem (Borodin and Kostochka, 2010)

Every graph is $(1, 0)$*-colorable if* $Mad(G) \leq \frac{12}{5}$

Corollary

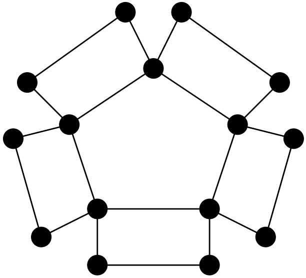
A planar graph is $(1, 0)$*-colorable if* $g(G) \geq 12$

Outerplanar graphs



Smallest value of k such that G admits a $(k, 0)$-coloring?

Outerplanar graphs



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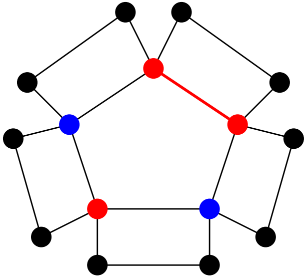

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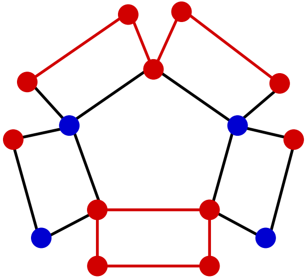


$(2, 0)$-coloring

Outerplanar graphs

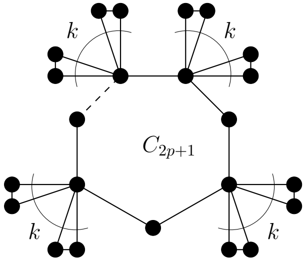
Claim

- Outerplanar graphs with girth 4 are (2,0)-colorable.
- Outerplanar graphs with girth 5 are (1,0)-colorable.

(the girth of a graph G is the length of a shortest cycle of G.)

Outerplanar graphs

For outerplanar graphs with girth 3, k is unbounded.



Non $(k, 0)$-colorable outerplanar graph with girth 3

Outerplanar graphs

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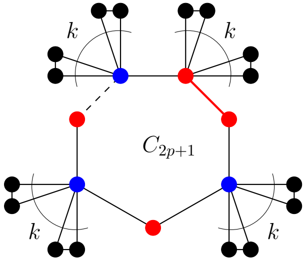

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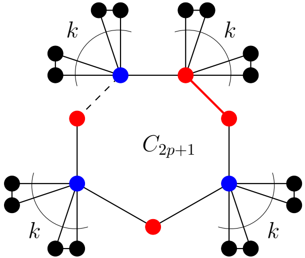

Non $(k, 0)$-colorable outerplanar graph with girth 3

$$Mad(G_{p,k}) = \frac{2|E(G_{p,k})|}{|V(G_{p,k})|} = \frac{2((3k+2)(p+1)-1)}{(2k+2)(p+1)-1} \xrightarrow[p\to\infty]{} \frac{3k+2}{k+1}$$

Sparse graphs

Key concepts:

soft components, feading area

Theorem (Borodin, Ivanova, Montassier, Ochem, R., 2009)

Let $k \geq 0$ be a integer. Every graph with maximum average degree smaller than $\frac{3k+4}{k+2}$ is $(k, 0)$-colorable.

- $Mad(G) < \frac{5}{2} \rightarrow (2, 0)$
- $Mad(G) < \frac{13}{5} \rightarrow (3, 0)$

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Optimality:

$$\lim_{p \to \infty} Mad(G_{p,k}) = \frac{3k+2}{k+1} < \frac{3k+4}{k+2} + \frac{1}{k+3}.$$

Corollary

Every planar graph G is:

- $(1,0)$-colorable if $g(G) \geq 14$,
- $(2,0)$-colorable if $g(G) \geq 10$,
- $(3,0)$-colorable if $g(G) \geq 9$,
- $(4,0)$-colorable if $g(G) \geq 8$,
- $(8,0)$-colorable if $g(G) \geq 7$.

Planar graphs

Corollary

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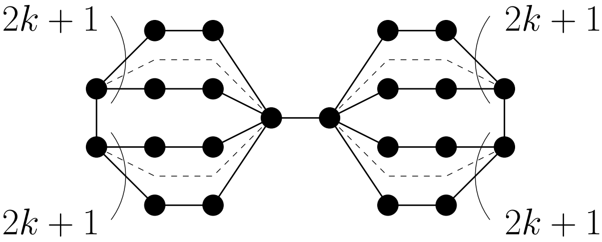
- $(3, 0)$-colorable if $g(G) \geq 9$,
- $(4, 0)$-colorable if $g(G) \geq 8$,
- $(8, 0)$-colorable if $g(G) \geq 7$.

For planar graphs with girth 6, k is unbounded.



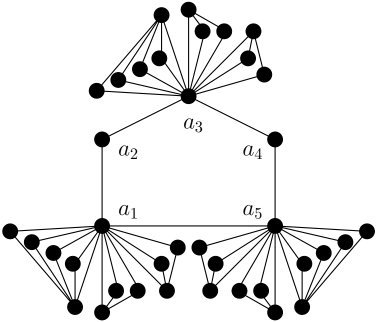
Non $(k, 0)$-colorable planar graph with girth 6

$(k, 1)$-coloring

Theorem (Borodin, Ivanova, Montassier, R., 2010)

Let $k \geq 2$ be a integer. Every graph with maximum average degree smaller than $\frac{10k+22}{3k+9}$ is $(k, 1)$-colorable.

$(k,1)$-coloring



An example of $G_{n,k}$ with $n = 3$ and $k = 3$.

Non $(k,1)$-colorable graph.

$$Mad(G_{n,k}) = \frac{2|E(G_{n,k})|}{|V(G_{n,k})|} = \frac{2(2n-1+5(k-1)n+n(2k+3))}{2n-1+3(k-1)n+n(k+2)} = \frac{2(7nk-1)}{n(4k+1)-1}$$

$$\lim_{n\to\infty} Mad(G_{n,k}) = \frac{14k}{4k+1}$$

(k, j)-coloring

For planar graphs:

$g(G)$	$(k, 0)$	$(k, 1)$	$(k, 2)$	$(k, 3)$	$(k, 4)$
3,4	$\times$	$\times$	$\times$	$\times$	$\times$
5	$\times$	?	$(13, 2)$	$(7, 3)$	$(4, 4)$ [HS06]
6	$\times$	$(5, 1)$	$(2, 2)$ [HS06]		
7	$(8, 0)$	$(2, 1)$			
8	$(4, 0)$	$(1, 1)$ [HS06]			
9	$(3, 0)$				
10	$(2, 0)$				
14	$(1, 0)$				

(k,j)-coloring: very last result

Let $F(j,k)$ denote the supremum of x such that every graph G with $Mad(G) < x$ is (k,j)-colorable. It is easy to see that $F(0,0) = 2$.

Theorem (Borodin and Kostochka, 2011)

Let $j \geq 0$ *and* $k \geq 2j+2$. *Then* $F(j,k) = 2\left(2 - \frac{k+2}{(j+2)(k+1)}\right)$.

- If $\mathsf{Mad(G)} < \frac{3k+2}{k+1}$ $(k \geq 2)$ then G is $(k,0)$-colorable.
- If $\mathsf{Mad}(G) < \frac{10k+8}{3k+3}$ $(k \geq 4)$ then G is $(k,1)$-colorable.

Steinberg Conjecture

Conjecture (Steinberg, 1976)

Every planar graph without 4 *and* 5*-cycles is* 3*-colorable (*$(0,0,0)$*-colorable)*

Erdős' relaxation '91: Determine the smallest value of k, if it exists, such that every planar graph without cycles of length from 4 to k is 3-colorable.

- $k \leq 11$ Abbott and Zhou ('91)
- $k \leq 10$ Borodin ('96)
- $k \leq 9$ Borodin ('96) and Sanders and Zhao ('95)
- $k \leq 8$ Salavatipour (2002)
- $k \leq 7$ Borodin et al. (2005)

Steinberg Conjecture

Let $\mathcal{F}$ be the family of planar graphs without cycles of length 4 and 5.
Can we prove that every graph in $\mathcal{F}$ is:

- $(1, 0, 0)$-colorable?

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Can we prove that every graph in $\mathcal{F}$ is:

- $(1, 0, 0)$-colorable?
- $(1, 1, 0)$-colorable?
- $(1, 1, 1)$-colorable (Xu, 2009)

Steinberg Conjecture

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Can we prove that every graph in $\mathcal{F}$ is:

- $(1, 0, 0)$-colorable?
- $(1, 1, 0)$-colorable?
- $(1, 1, 1)$-colorable (Xu, 2009)
- $(2, 0, 0)$-colorable?

Theorem (Chang, Havet, Montassier, R. 2011)

Every graph of $\mathcal{F}$ *is* $(2,1,0)$*-colorable.*

$\mathbf{(2,1,0)}$-colorability of $\mathcal{F}$

$$\omega(v) = 2d(v) - 6 \text{ and } \omega(f) = r(f) - 6.$$

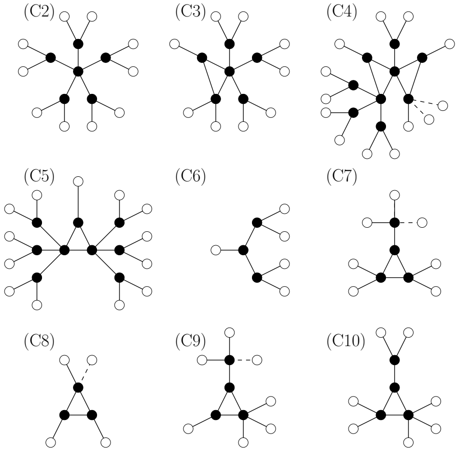
By Euler's Formula $|V| - |E| + |F| = 2$ and $\sum_{v \in V} d(v) = 2|E| = \sum_{f \in F} r(f)$, we have:

$$\sum_{v \in V} \omega(v) + \sum_{f \in F} \omega(f) = -12 < 0.$$

$$\text{If } r(f) = 3 \text{ then } \omega(f) = -3$$

$\mathbf{(2,1,0)}$-colorability of $\mathcal{F}$

Reducible configurations for $\mathbf{(2,1,0)}$-coloring.
(C1) *G contains no 2^{-}-vertices.*



$(2,1,0)$-colorability of $\mathcal{F}$

The discharging rules are as follows:

R1. Every 4-vertex gives $\frac{1}{2}$ to each pendent 3-face.
R2. Every 5^+-vertex gives 1 to each pendent 3-face.
R3. Every 4-vertex gives 1 to each incident 3-face.
R4. Every non-light 5-vertex gives 2 to each incident poor $(3,5,5)$-face.
R5. Every 5-vertex gives $\frac{3}{2}$ to each incident non-poor $(3,5,5)$-face or $(3,4,5)$-face.
R6. Every 5-vertex gives 1 to each other incident 3-face.
R7. Every 6^+-vertex gives 2 to each incident 3-face.

Thank you for your attention!