Minimum cost homomorphism problem for digraphs

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Homomorphism Problem

For graphs (or digraphs) G and H, a mapping $f : V(G) \to V(H)$ is a homomorphism of G to H if $uv \in E(G)$ implies $f(u)f(v) \in E(H)$.

Let H be a fixed graph (digraph). The homomorphism problem for H asks whether an input graph (or digraph) G admits a homomorphism to H.

P.Hell and J. Nešetřil (1990) showed the following dichotomy for graphs.

If H is a bipartite graph or H has a loop then homomorphism problem is polynomial, NP-complete otherwise.

Dichotomy is not known for digraphs.

Theorem(Barto, Kozik, and Niven (2008))
Let H be a smooth digraph (no source and no sink vertex). If each component of the core of H is a directed cycle, HOM(H) is polynomial time solvable. Otherwise HOM(H) is NP-complete.

Minimum Cost Homomorphism

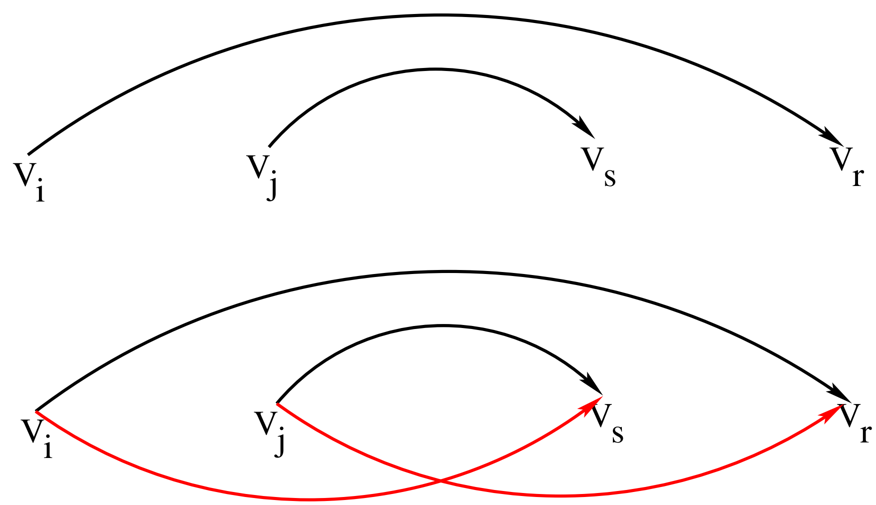
G and H are graphs (digraphs), and $c_i(u)$, for $u \in V(G)$ and $i \in V(H)$, are non-negative numbers. The cost of a homomorphism $f: V(G) \to V(H)$ is

$$\sum_{u \in V(G)} c_{f(u)}(u).$$

If H is fixed, the minimum cost homomorphism problem MinHOM(H) is finding the minimum cost homomorphism from input graph G (with costs) to H.

Min-Max ordering

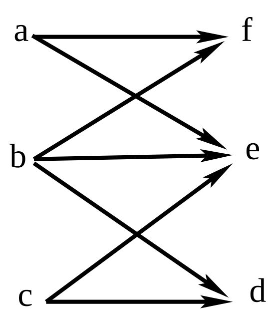
Let H be a digraph. We say that an ordering $v_1, v_2, \ldots, v_p$ of $V(H)$ is a Min-Max ordering of H



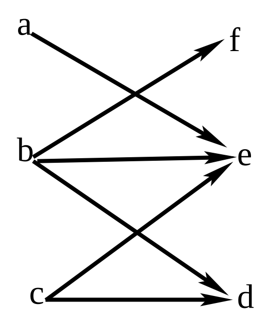
if $v_i v_r, v_j v_s \in E(H)$ implies

$$v_{\min\{i,j\}} v_{\min\{s,r\}} \in E(H), \quad v_{\max\{i,j\}} v_{\max\{s,r\}} \in E(H).$$

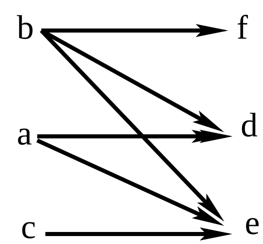
A Min-Max ordering for bipartite graph $G = (V, U)$. Orient all edges from V to U and apply the Min-Max definition.



G Has Min−Max Ordering
c b a d e f



H

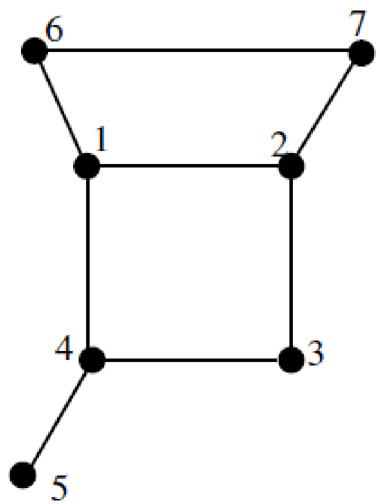


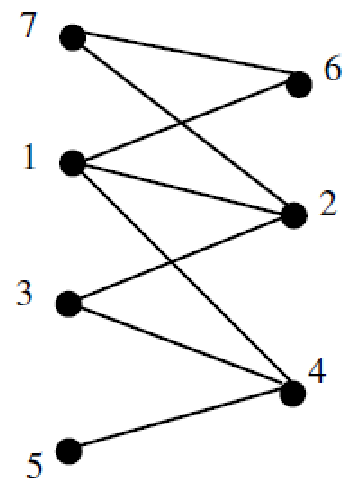
Min−Max ordering of H

Importance of Min-Max ordering for MinHOM(H) is indicated in the following theorem.

Theorem(GRY (2005))

If digraph H has a Min-Max ordering then MinHOM(H) is polynomial-time solvable.


6
7
1
2
4
3
5


7
6
1
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3
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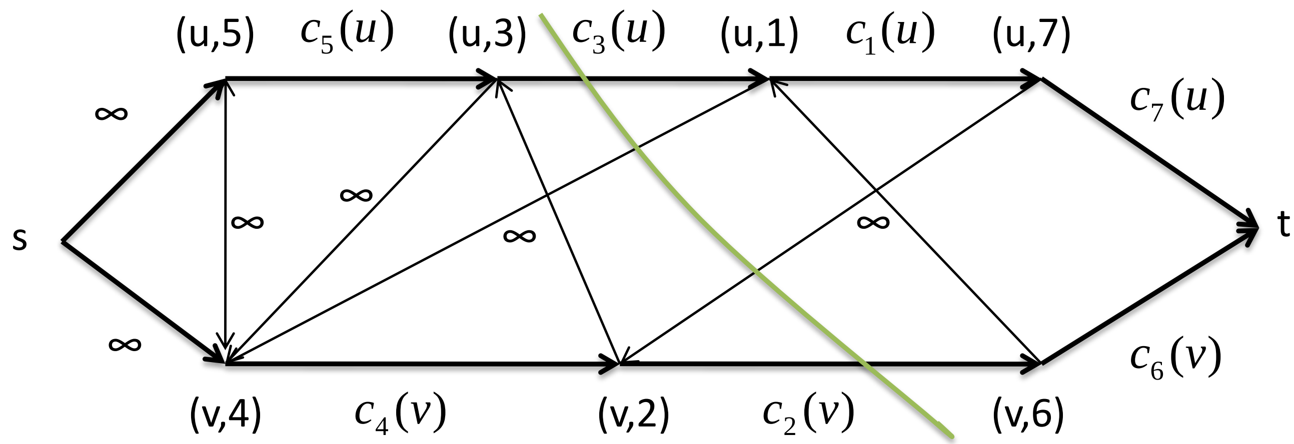
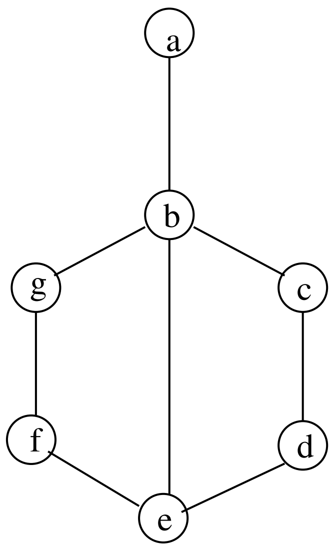
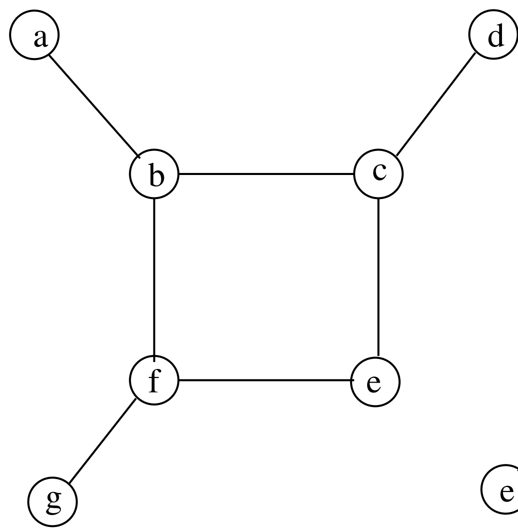

(u,5)
$c_5(u)$
(u,3)
$c_3(u)$
(u,1)
$c_1(u)$
(u,7)
$c_7(u)$
s
t
∞
∞
∞
∞
∞
∞
(v,4)
$c_4(v)$
(v,2)
$c_2(v)$
(v,6)
$c_6(v)$

Theorem (GHRY 2006)

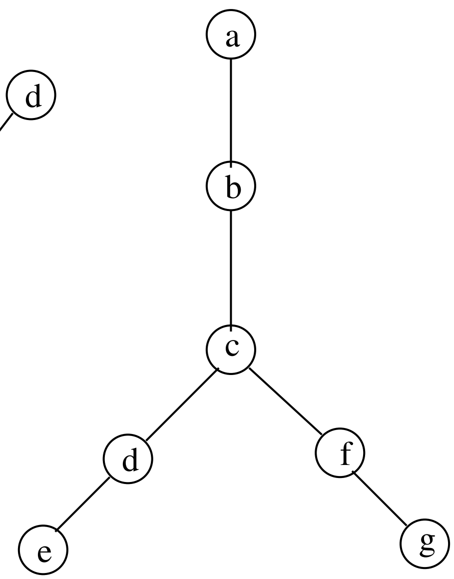
Let H be a bipartite graph. If H contains an induced cycle C_{2k}, $k \geq 3$, or bipartite claw or bipartite net or bipartite tent then MinHOM(H) is NP-complete. Otherwise H has a Min-Max ordering and MinHOM(H) is polynomial.



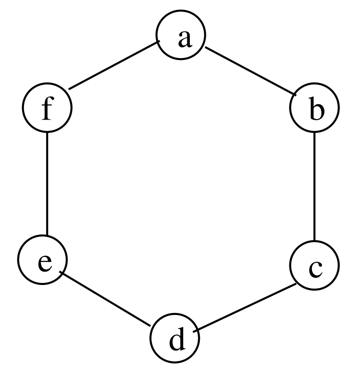
Bipartite Tent



Bipartite Net



Bipartite Claw



Cyle of length 6

By applying the results of [Spinard, Brandstaedt, Stewart (1987)] and [Spinard (2003)] we observe:

Theorem (GHRY 2006)

A bipartite graph H has a Min-Max ordering if and only if it is a proper interval bigraph .

Proper Interval Bigraph

A bipartite graph $G = (V, U)$, which V, U are the set of inclusion-free intervals on the line. A vertex from V is adjacent to a vertex from U if their intervals intersect.

Proper Interval Digraphs

Monotone proper interval digraph H: There exist two families of closed intervals $I_v, v \in V(H)$, and $J_v, v \in V(H)$, and point p_v.

1. if $p_v < p_w$ then left endpoint of I_v precedes left endpoint of I_w and left endpoint of J_v precedes left endpoint of J_w,

2. no I_v contains another I_w with $w \neq v, I_w \neq \emptyset$,

3. no J_v contains another J_w with $w \neq v, J_w \neq \emptyset$, and

4. $uv \in A(H)$ if and only if $I_u \cap J_v \neq \emptyset$.

Theorem HR 2009

A digraph H admits a Min-Max ordering if and only if it is a monotone proper interval digraph.

If H is a monotone proper interval digraph, we define the ordering $<$ by the left-to-right order of the points p_v.

We show that $<$ is a Min-Max ordering of H.

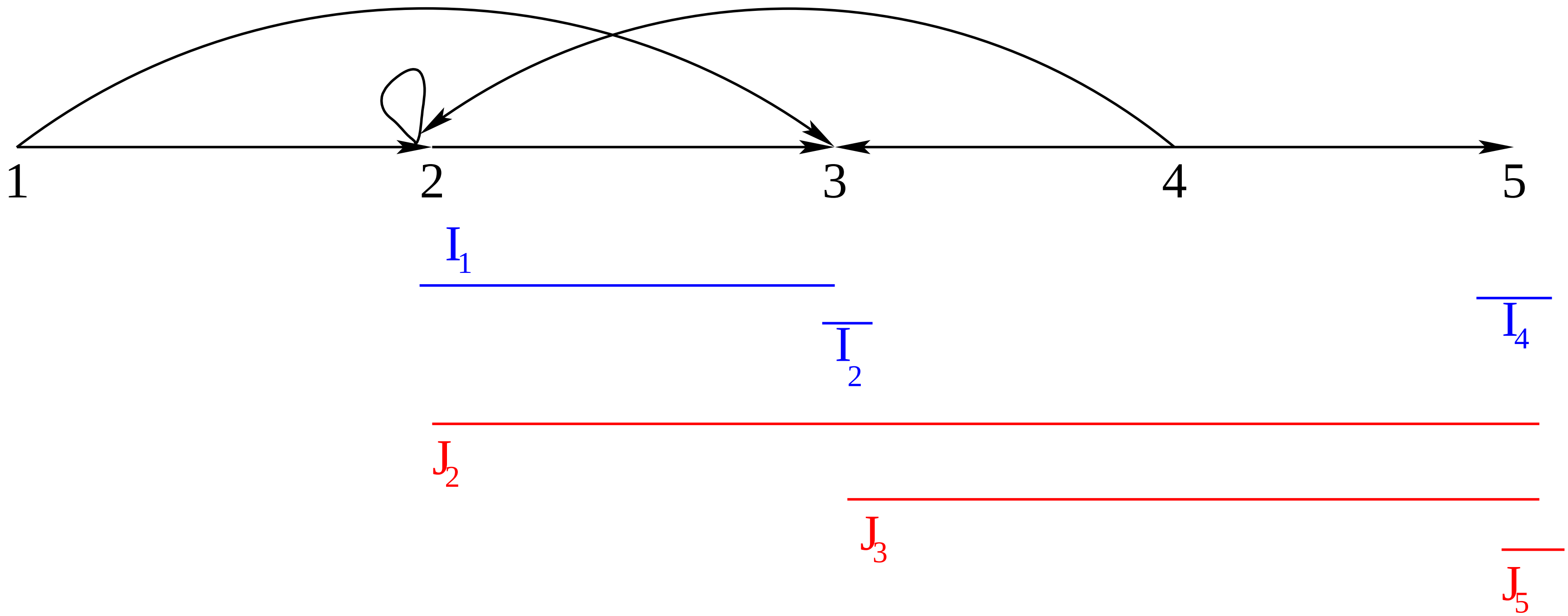
Suppose $u < w, z < v$ and $uv, wz \in A(H)$.

We have $I_u \cap J_v \neq \emptyset$, $I_w \cap J_z \neq \emptyset$.

uv is an arc implies $r(I_u) \geq \ell(J_v)$ and $\ell(J_v) > \ell(J_z)$ since $v > z$; thus $r(I_u) \geq \ell(J_z)$.

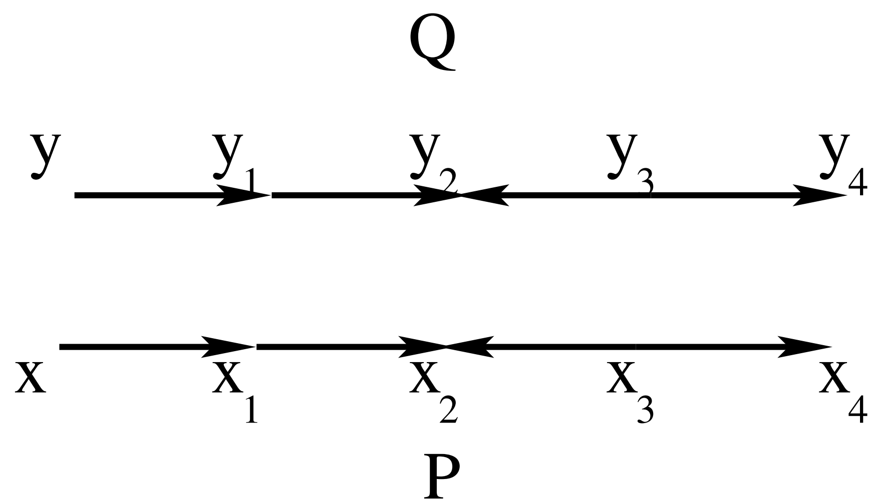
Similarly $r(J_z) \geq \ell(I_w) > \ell(I_u)$, I_u, J_z intersect and $uz \in A(H)$.

By a similar calculation, $wv \in A(H)$.

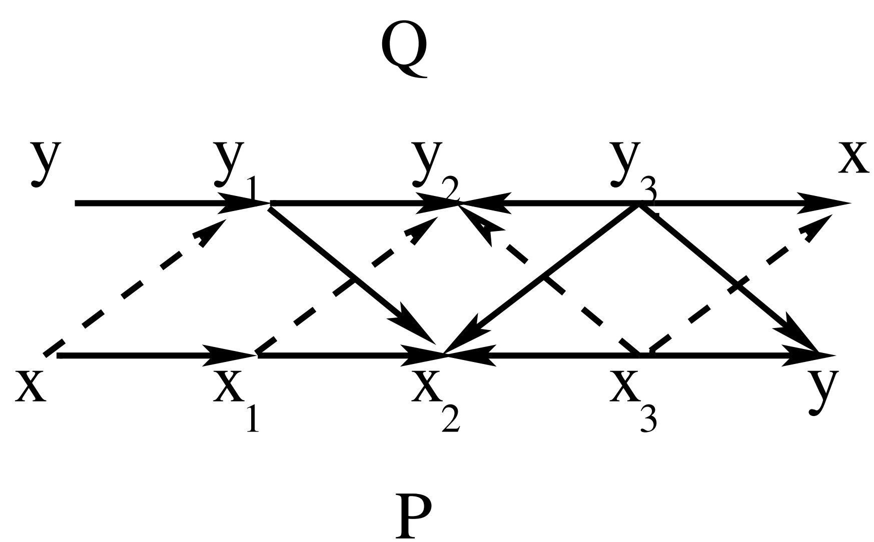

1
2
3
4
5
I_1
I_2
I_4
J_2
J_3
J_5

Digraphs with Min-Max Ordering

Two walks $P = x_0, x_1, \ldots, x_n$ and $Q = y_0, y_1, \ldots, y_n$ in H are *congruent*, if they follow the same pattern of forward and backward arcs.

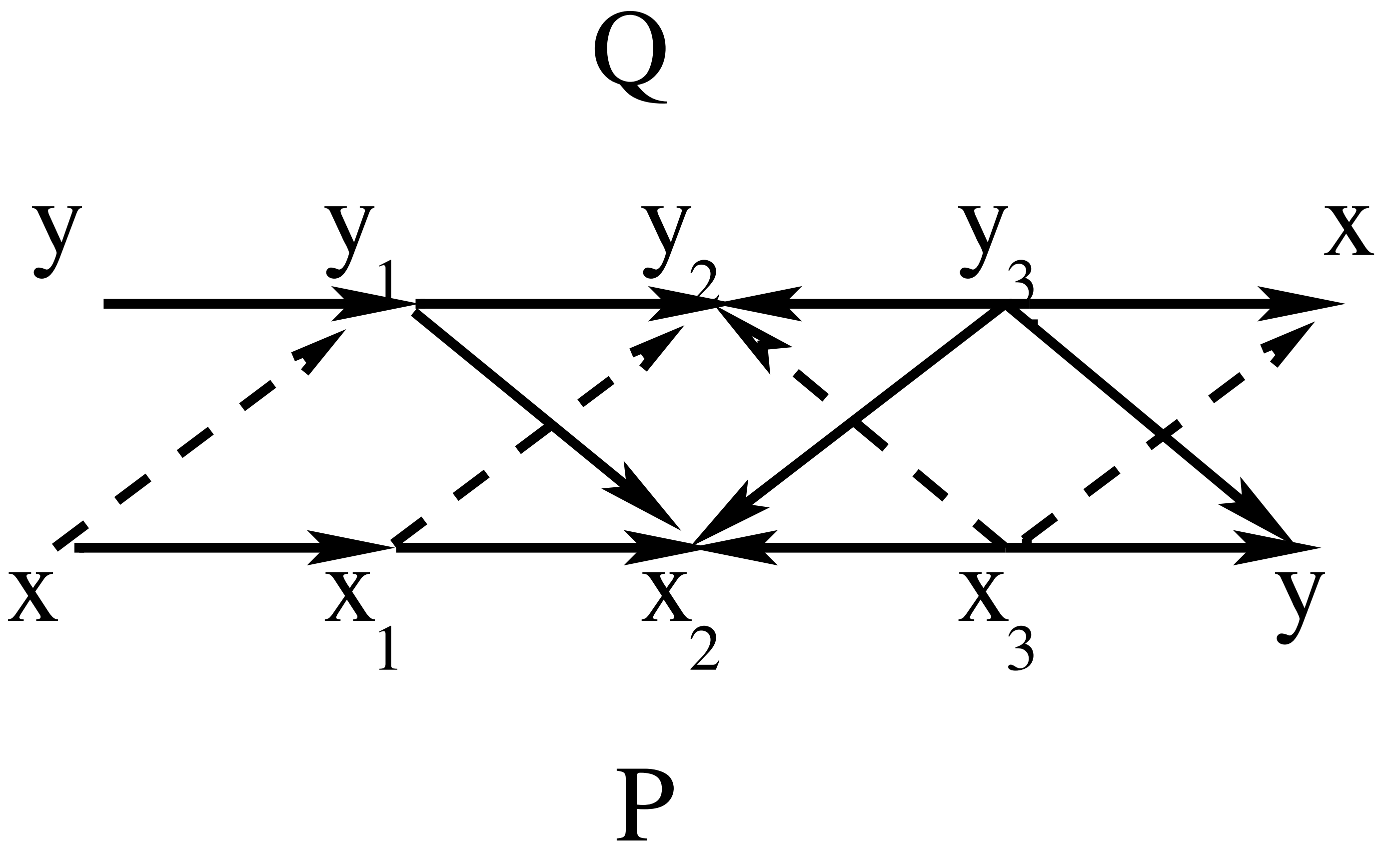


We say an arc $x_i y_{i+1}$ is *a faithful arc from P to Q*, if it is a forward (backward) arc when $x_i x_{i+1}$ is a forward (backward) arc (respectively), and we say an arc $y_i x_{i+1}$ is *a faithful arc from Q to P*, if it is a forward (backward) arc when $x_i x_{i+1}$ is a forward (backward) arc (respectively).



We say that P, Q *avoid each other* if they have no pair of faithful arcs $x_i y_{i+1}$ from P to Q, and $y_i x_{i+1}$ from Q to P.

An *invertible pair* in H is a pair of distinct vertices x, y, such that there exist congruent walks P from x to y and Q from y to x, which avoid each other.


Q
y
y1
y2
y3
x
x
x1
x2
x3
y
P

We define an auxiliary digraph H^* as follows:

1) The vertices of H^* are all ordered pairs (x, y) of distinct vertices of H,

2) There is an arc from (x, y) to (x', y') just if xx', yy' are both forward arcs of H but xy', yx' are not both forward arcs of H.

H admits an invertible pair if and only if there exist u, v so that (u, v) and (v, u) belong to the same weak component of H^*.

How to construct a desired Min-Max ordering $<$?

At each stage of the algorithm, some components of H have already been chosen.

We set $a < b$ if the pair (a, b) belongs to one of the chosen components.

Whenever a component C of H^* is chosen, its dual component C' is discarded.

The objective is to avoid a *circular chain*

$$(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$$

of pairs belonging to the chosen components.

If a component C contains a circular chain then there is no Min-Max ordering.

Rules of choosing the next component :

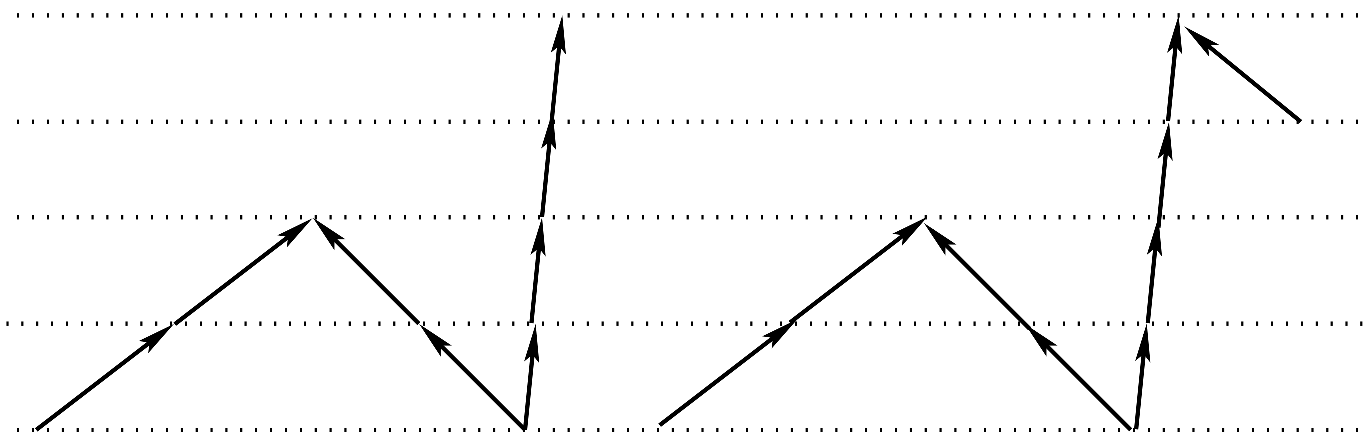
From yet un-chosen and un-discarded components:

- Choose a component C of maximum height.

- If C creates a circular chain then choose the dual component C'.

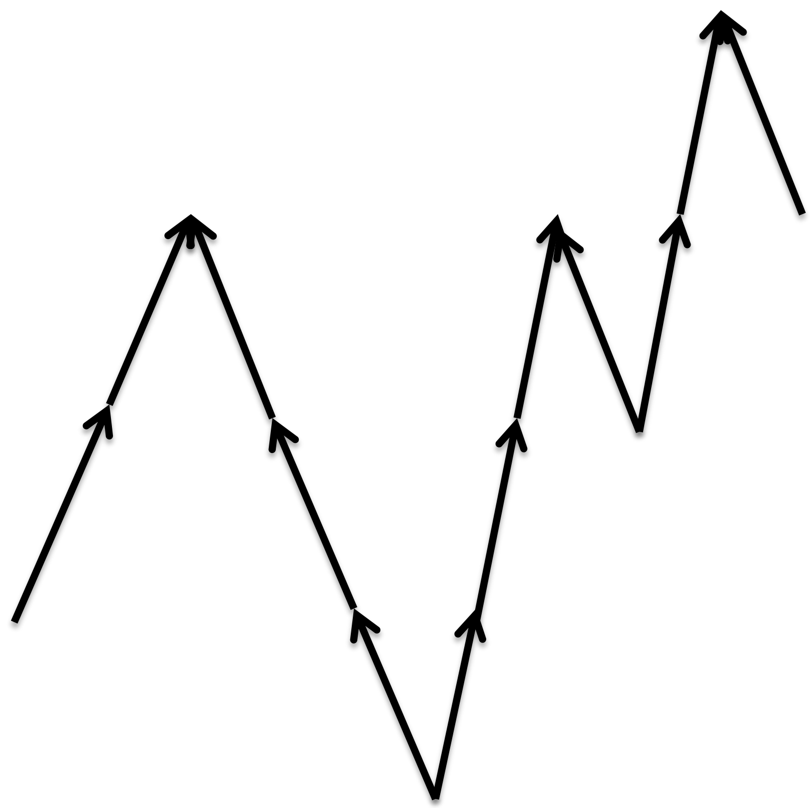
We say a walk is *constricted* if the level of the starting vertex is the lowest level and the level of the ending vertex is the highest level of all vertices of the walk

The *net length* of a walk is the number of forward arcs minus the number of backward arcs.



P Q

The height is 4 and the net-length of the path is 2



Lemma

(Hell (2004)) Let P_1 and P_2 be two constricted walks of net length r. Then there is a constricted walk P of net length r that admits a homomorphism f_1 to P_1 and a homomorphism f_2 to P_2, such that each f_i takes the starting vertex of P to the starting vertex of P_i and the ending vertex of P to the ending vertex of P_i.

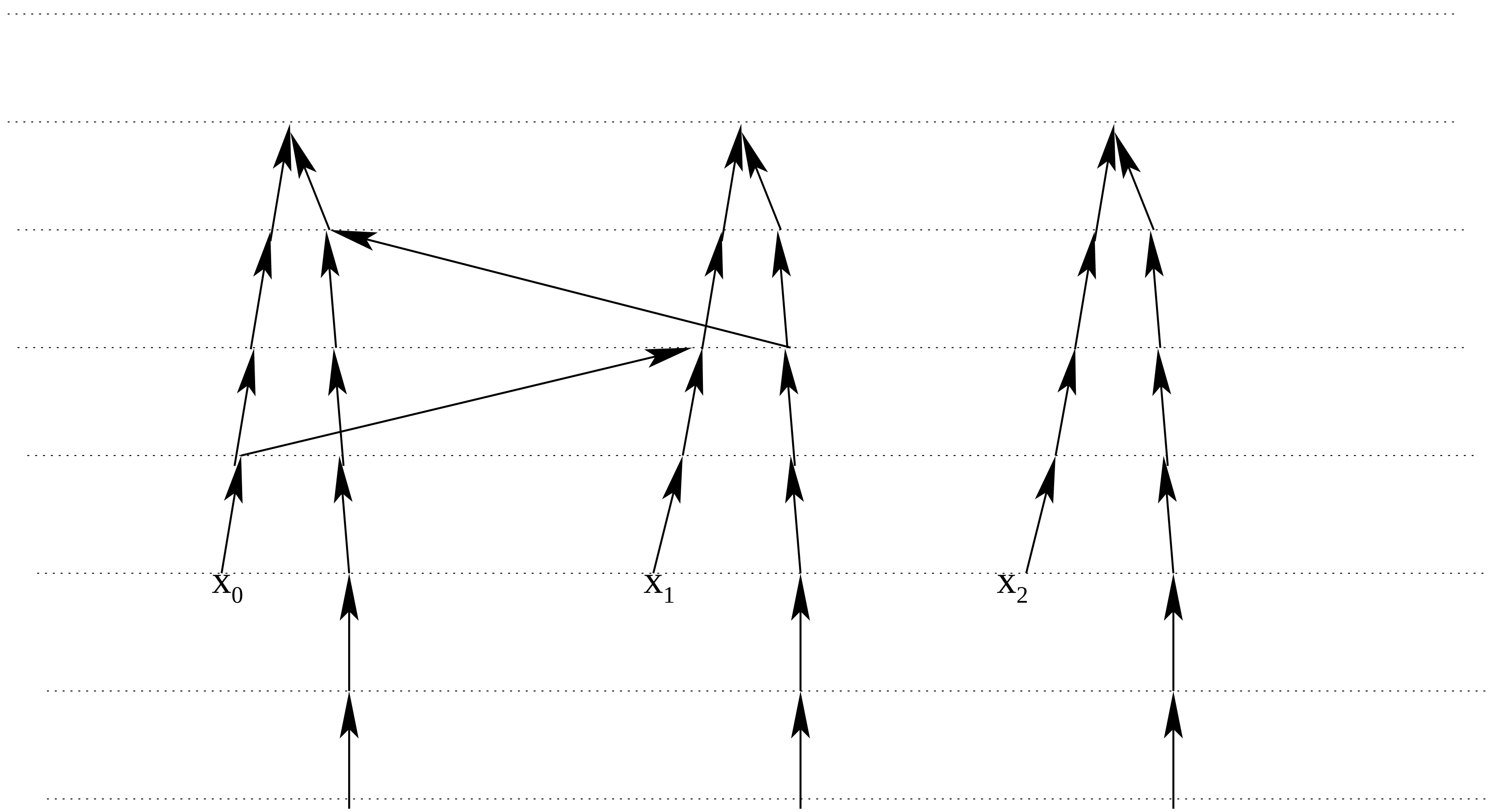
We say P is a *common pre-image* of P_1 and P_2.

Lemma

Let x_0, x_1, x_2 be three vertices of H such that the component C of H^*, containing (x_0, x_1), contains neither of $(x_0, x_2), (x_2, x_1)$.

Let P, Q, T be congruent walks starting at x_0, x_1, x_2 respectively.

If P and Q avoid each other, then Q and T also avoid each other (and P and T also avoid each other).

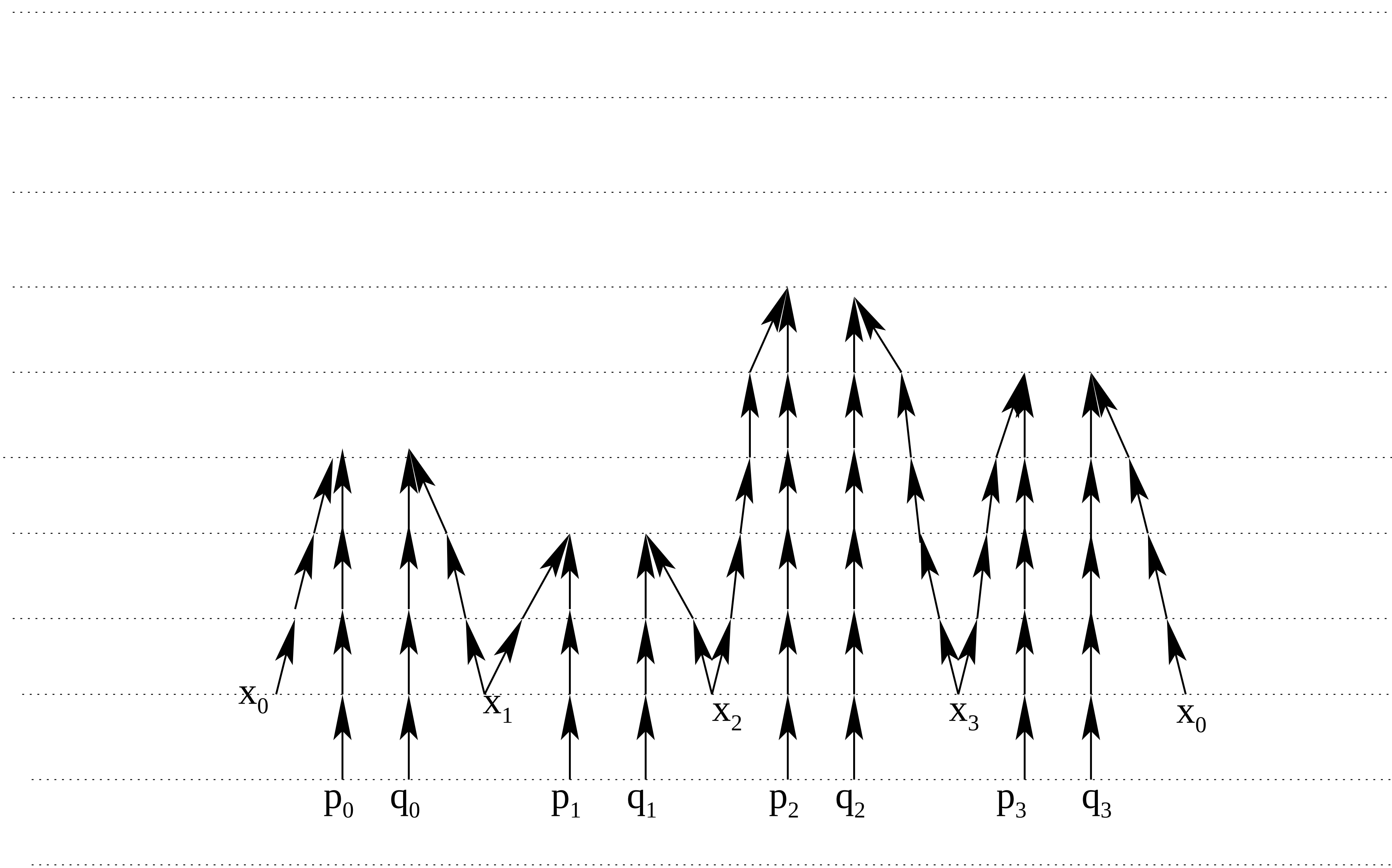

x_0
x_1
x_2

We assume that the circular chain has minimum length.

Lemma

If there is a circular chain $(x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n), (x_n, x_0)$ and there are nearly constricted walks from below; from (p_i, q_i) to (x_i, x_{i+1})

then there is a another circular chain in D using some of the p_i's and q_i's.



x_0
x_1
x_2
x_3
x_0
p_0
q_0
p_1
q_1
p_2
q_2
p_3
q_3

Lemma

If $(x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n), (x_n, x_0)$ is a circular chain and (x_n, x_0), (x_i, x_{i+1}) are in the same component, and there is a walk constricted from below with net-length zero then there is a shorter circular chain.

Finally we show that by adding the last component (containing (x_0, x_n)) we do not create another circular chain

$$(b_0, b_1), (b_1, b_2), ..., (b_{m-1}, b_m), (b_m, b_0).$$

Equivalent Statements

1. H is a monotone proper interval digraph

2. H admits a Min-Max ordering

3. H has no invertible pair and no induced oriented cycle of net length greater than one

4. no weak component of H^* contains a circular chain

Extended Min-Max ordering

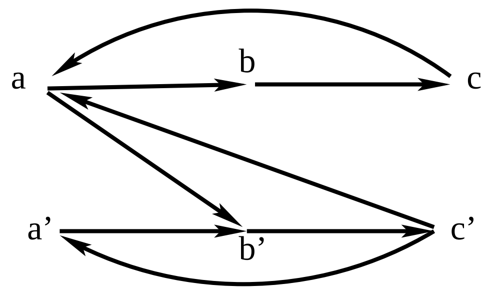
Fix a homomorphism $\ell : H \to \vec{C}_k$.

Call the vertices of $\vec{C}_k$ by $0, 1, \ldots, k-1$,

Set $V_i = \ell^{-1}(i)$. A *k-Min-Max ordering* of H is a linear ordering $<$ of each V_i, so that the Min-Max property is satisfied for vertices in any two circularly consecutive sets V_i and V_{i+1}

Note that a Min-Max ordering is also a k-Min-Max ordering for any k and ℓ;

There are digraphs with a k-Min-Max but no Min-Max ordering.



3 Min−Max Ordering

Theorem (GRY (2006))

If H has k-Min-Max ordering then MinHOM(H) is polynomial.

Consider a subgraph of H^* defined as follows. The digraph $H^{(k)}$ is the subgraph of H^* induced by all ordered pairs (x, y) of with $\ell(x) = \ell(y)$.

We say that (u, v) is a *k-invertible pair* if there is in $H^{(k)}$ an oriented walk joining (u, v) and (v, u).

Theorem HR

The following statements are equivalent for a digraph H with a homomorphism ℓ to $\vec{C}_k$.

- H admits a k-Min-Max ordering
- H does not contain an induced unbalanced oriented cycle of net length other than k, and does not contain a k-invertible pair.

NP-completeness

Theorem

Let H' be an induced subdigraph of digraph H. If MinHOM(H') is NP-complete then MinHOM(H) is NP-complete.

Theorem (Alekseev and Lozin (2003))

The problem of finding a maximum independent set in a 3-partite graph G (even given the three partite sets) is NP-complete.

Lemma

H is a digraph with two vertices x, y; S is a digraph with two vertices s, t.

Suppose we have costs $c_j(i)$ of mapping vertices i of S to vertices j of H where $c_x(s) = c_x(t) = 1, c_y(s) = c_y(t) = 0$, and such that there exists

- a homomorphism $f : S \to H$ mapping s to x and t to y of total cost 1 (i.e., in which all other vertices of S, different from s, t, map to vertices of H with costs zero)

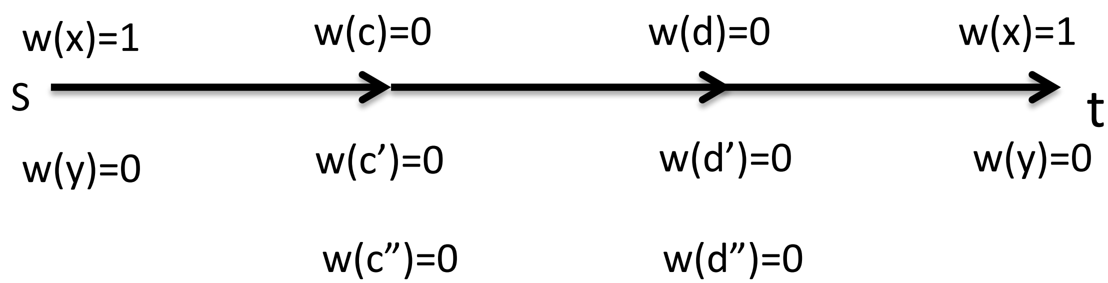
- a homomorphism $g : S \to H$ mapping s to x and t to x of total cost 2 (other vertices map with costs zero)

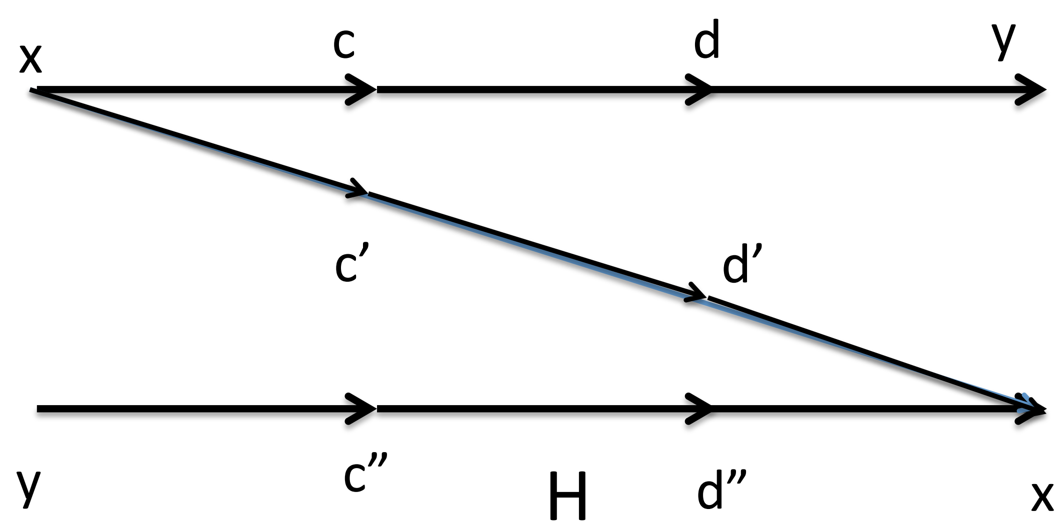
- a homomorphism $h : S \to H$ mapping s to y and t to x, of total cost 1 (other vertices map with costs zero)

- but no homomorphism $S \to H$ mapping s to y and t to y of cost at most $|V(S)|$.

Then MinHOM(H) is NP-complete.

S





H

Theorem HR

If there are two induced unbalanced oriented cycles with different net-length in H then MinHOM(H) is NP-complete.

Dichotomy HR

If H admits a k-Min-Max ordering (for some $k \geq 1$) then MinHOM(H) is polynomial time solvable. Otherwise MinHOM(H) is NP-complete.

List Homomorphism Problem

The list homomorphism problem for H, LHOM(H), asks whether an input graph (or digraph) G with lists (sets) $L_u \subseteq V(H), u \in V(G)$, admits a homomorphism f to H in which all $f(u) \in L_u, u \in V(G)$.

Feder, Hell and Huang (1999) found a dichotomy classification for list homomorphism of graphs.

In a special case of bipartite graphs:

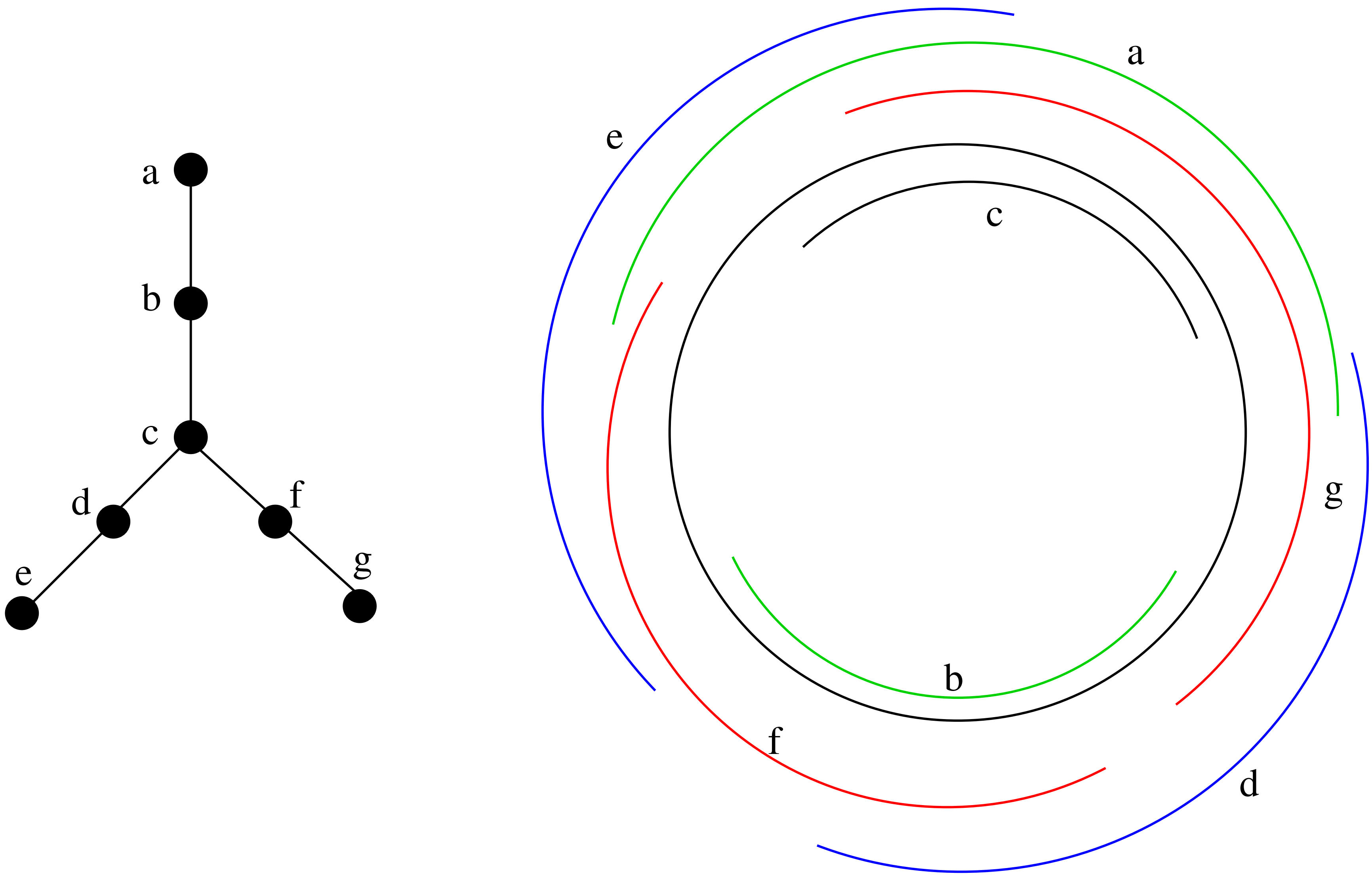
If complement of bipartite graph H is a circular arc graph then LHOM(H) is polynomial and NP-complete otherwise.

Circular Arc Graphs

We say complement of bipartite graph H is a circular arc graph if :

1) There is a family of circular arc A_v for each vertex $v \in V(H)$.

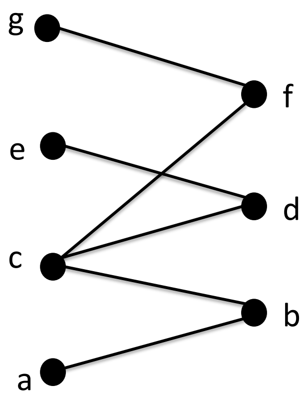
2) uv is an edge of H if A_v and A_u do not intersect.

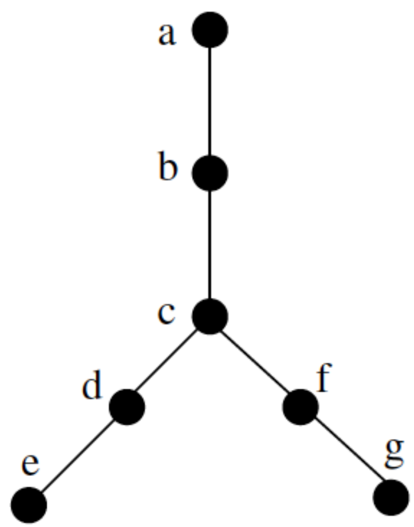


a
b
c
d
e
f
g
a
e
c
g
b
f
d

Theorem HR 2011

Bipartite graph H admits a min ordering iff its complement is a circular arc graphs.





Theorem(Bulatov(2003))

Let H be a digraph. If for every pair of vertices a, b of H there exists a polymorphism of H which is either ternary and is majority, or Maltsev, over a, b, or is binary and is semilattice over a, b then LHOM(H) is polynomial time solvable.

Otherwise, LHOM(H) is NP-complete.

The k-*th power* of H is the digraph H^k with vertex set $V(H)^k$ in which $(u_1, u_2, \ldots, u_k)(v_1, v_2, \ldots, v_k)$ is an arc just if each $u_i v_i$ is an arc of H.

A *polymorphism of order* k is a homomorphism of H^k to H.

If $u_1v_1, u_2v_2, ..., u_kv_k \in E(H)$ then $f(u_1, u_2, ..., u_k)f(v_1, v_2, ..., v_k) \in E(H)$.

A polymorphism f is *conservative* if $f(u_1, u_2, \ldots, u_k)$ always is one of $u_1, u_2, \ldots, u_k$.

From now on we shall use the word polymorphism to mean a conservative polymorphism.

A polymorphism f of order two is *commutative* if $f(u,v) = f(v,u)$ for any u,v, and it is associative if $f(u, f(v,w)) = f(f(u,v),w)$.

A polymorphism f of order two is *semilattice* if it is commutative and associative.

A ternary polymorphism $g: H^3 \to H$ is *majority over* a,b, if

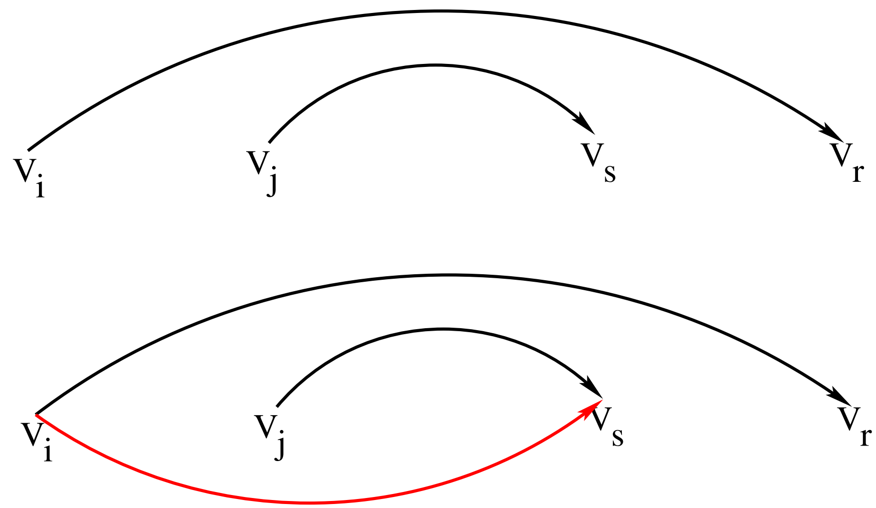
$$g(a,a,b) = g(a,b,a) = g(b,a,a) = a,$$

$$g(b,b,a) = g(b,a,b) = g(a,b,b) = b.$$

A ternary polymorphism $h: H^3 \to H$ is *maltsev over* a,b if

$$h(a,a,b) = h(b,a,a) = b, \quad h(a,b,b) = h(b,b,a) = a.$$

An ordering $<$ of vertices H, $v_1, v_2, ..., v_p$ is *min-ordering* if it satisfies the following property: if $v_i v_r \in E(H)$ and $v_j v_s \in E(H)$, then $v_{\min(i,j)} v_{\min(r,s)} \in E(H)$.



Now $f(v_i, v_j) = f(v_j, v_i) = v_i$, $i \leq j$ is a commutative and associative polymorphism.

It is known [Feder and Vardi (1998)] that if H admits any binary commutative polymorphism (e.g. a min-ordering), then the problem LHOM(H) is polynomial time solvable.

If H admits a majority polymorphism (over all pairs) then LHOM(H) is polynomial.

If H admits a matlsev polymorphism (over all pairs) then LHOM(H) is polynomial.

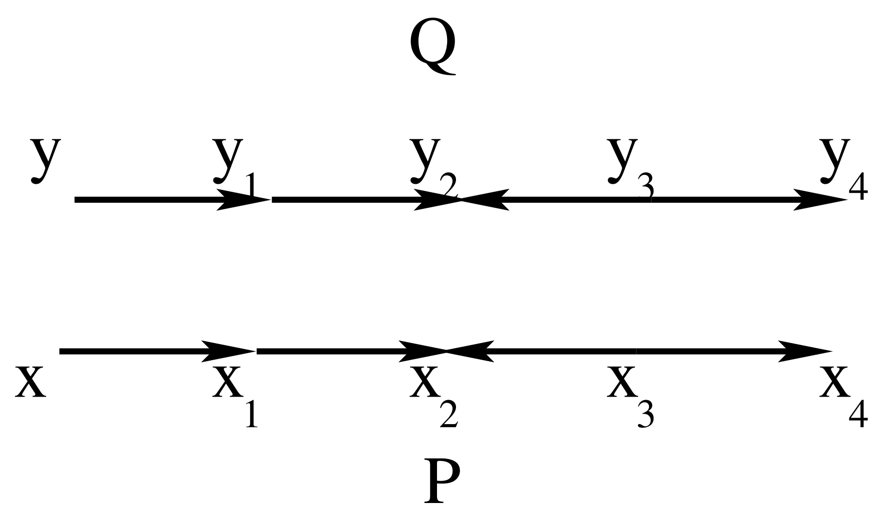
New Theorem(HR(2010-2011))

Let H be a digraph.

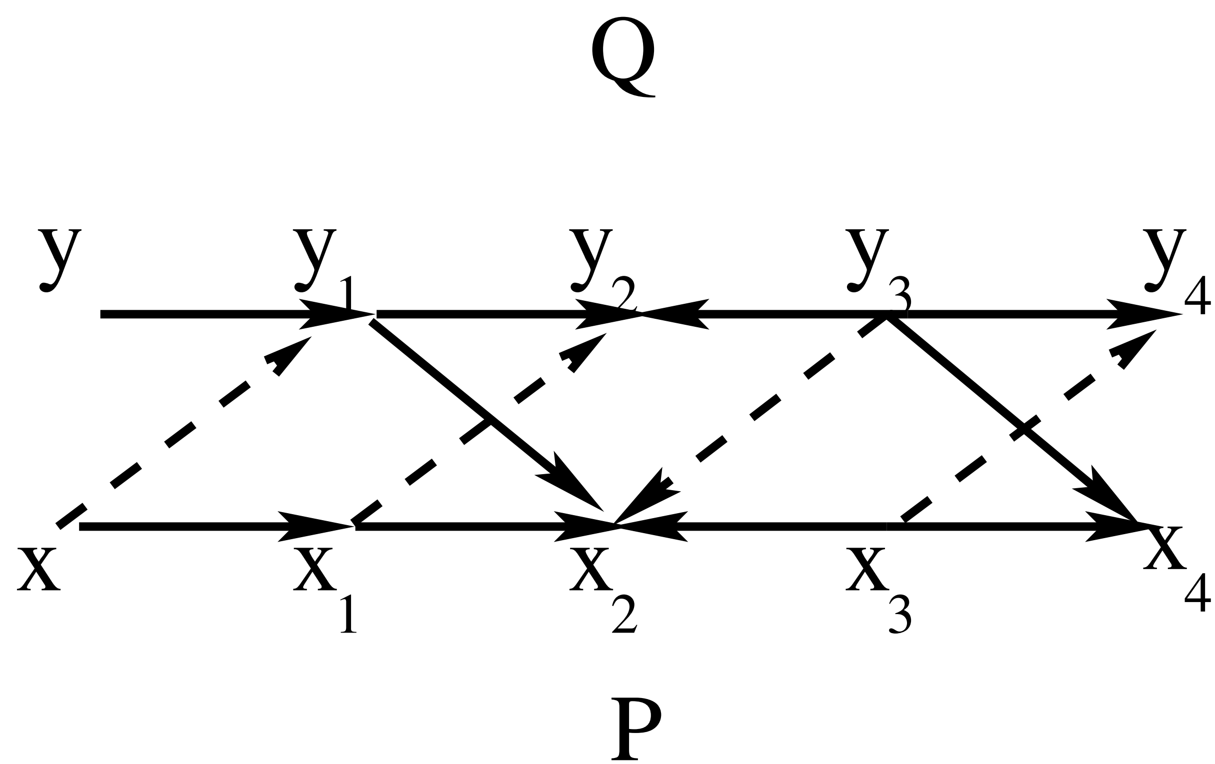
If H contains a DAT then LHOM(H) is NP-complete.

If H is DAT-free then LHOM(H) is polynomial time solvable.

We define two walks $P = x_0, x_1, \ldots, x_n$ and $Q = y_0, y_1, \ldots, y_n$ in H to be *congruent*, if they follow the same pattern of forward and backward arcs,



If P and Q are congruent walks, we say that P *avoids* Q, if there is no arc $x_i y_{i+1}$ in the same direction (forward or backward) as $x_i x_{i+1}$.



An *invertible pair* in H is a pair of vertices u, v such that

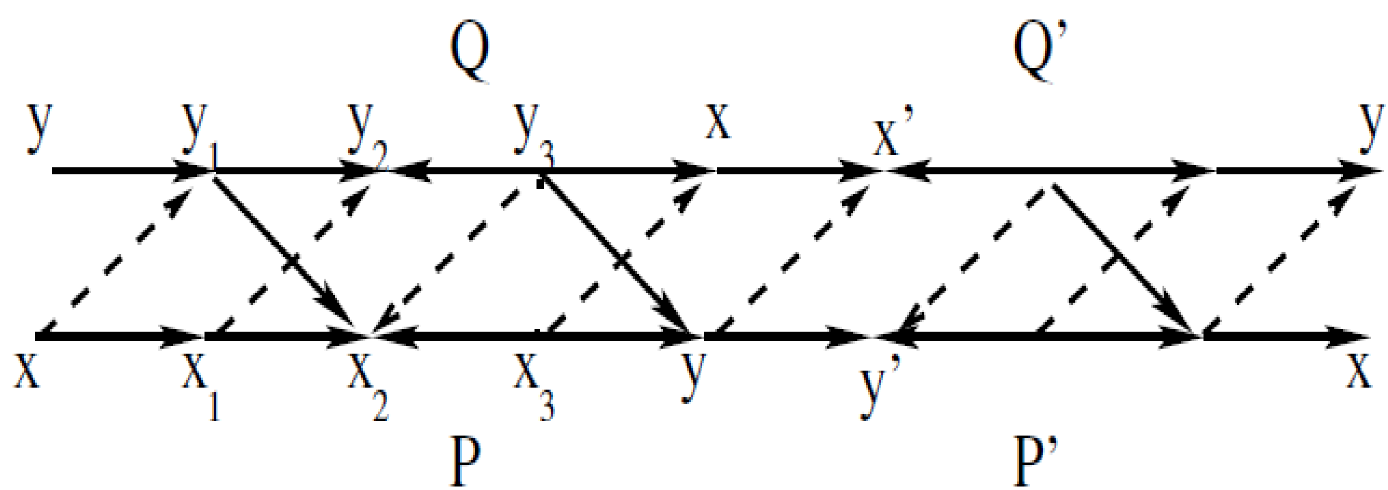
1. there exist congruent walks P from u to v and Q from v to u, such that P avoids Q,

2. and there exist congruent walks P' from v to u and Q' from u to v, such that P' avoids Q'.

Note that it is possible that P' is the reversal of P and Q' is the reversal of Q, as long as both P avoids Q and Q avoids P.

$$xx_1, yy_1 \in E(H)$$

$$f(x, y)\, f(x_1, y_1) \in E(H)$$



We reformulate these definitions in auxiliary digraph H^+ where $V(H^+) = \{(u, v) : u \neq v\}$, and arcs

$$E(H^+) = \{(u, v)(u', v') :$$

where

$$uu', vv' \in E(H) \text{ and } uv' \notin E(H)$$

or

$$u'u, v'v \in E(H) \text{ and } u'v \notin E(H)\}.$$

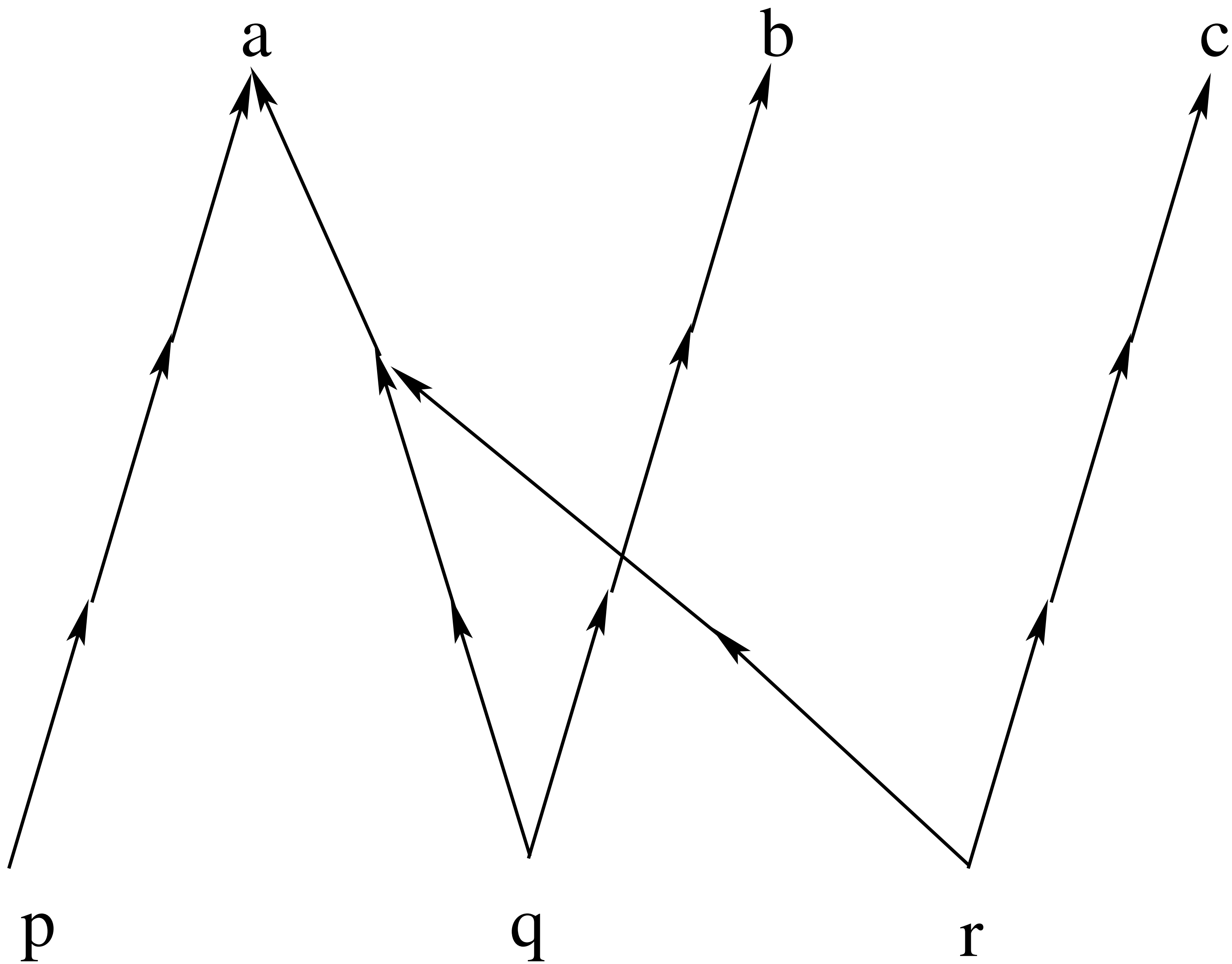
1) u, v is an invertible pair if and only if $(u, v), (v, u)$ are in the same strong component of H^+.

2) H^+ contains an arc from (u, v) to (u', v') if and only if it contains an arc from (v', u') to (v, u).

3) if u, v is an invertible pair in H and (p, q) is in the same strong component of H^+ as (u, v) then p, q is also an invertible pair in H.

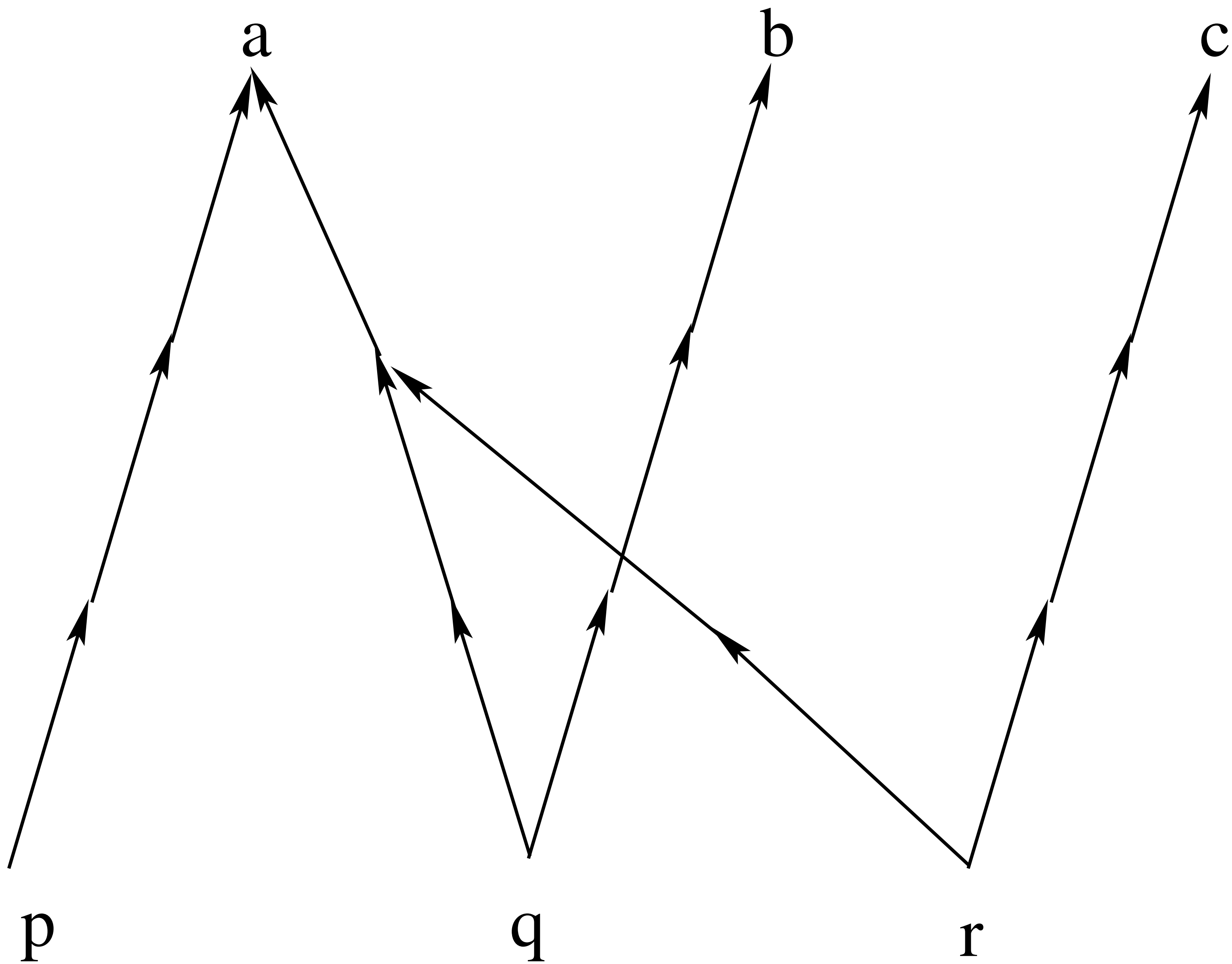
A *permutable triple* in H is a triple of vertices a, b, c together with six vertices $s(a), r(a)$, $s(b), r(b)$, $s(c), r(c)$, (base pairs) which satisfy the following condition.

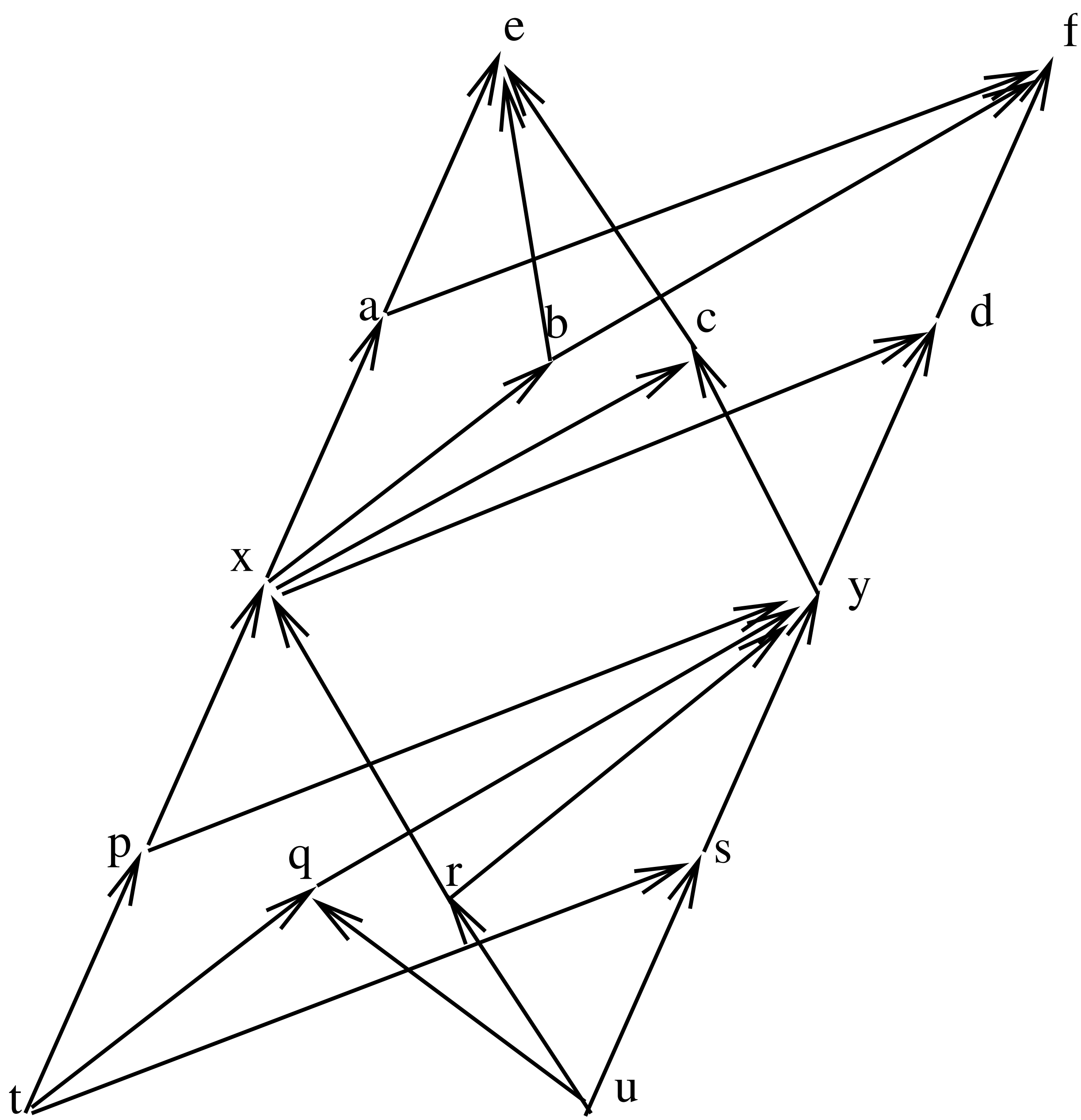
For any vertex x from a, b, c, there exists a walk $P(x, s(x))$ from x to $s(x)$ and two walks $P(y, r(x))$ (from y to $r(x)$), and $P(z, r(x))$ (from z to $r(x)$), congruent to $P(x, s(x))$, such that $P(x, s(x))$ avoids both $P(y, r(x))$ and $P(z, r(x))$ (here y and z are the other two vertices from a, b, c).


a
b
c
p
q
r

A *digraph asteroidal triple* (DAT) is a permutable triple u, v, w in which all the base pairs are the same and the base pair is invertible.

For any vertex x from u, v, w, there exists a walk $P(x, x)$ from x to s and two walks $P(y, r)$ (from y to r), and $P(z, r)$ (from z to r), congruent to $P(x, s)$, such that $P(x, s)$ avoids both $P(y, r)$ and $P(z, r)$


a
b
c
p
q
r


e
f
a
b
c
d
x
y
p
q
r
s
t
u

NP-completeness Proof : If H contains a DAT then Not-All-Equal 3-SAT (without negated variable) or 3-coloring is reduced to LHOM(H).

Polynomial Proof : If H does NOT contain a DAT then there are two polymorphisms f and g, that f on some of the pairs of H is semmi-lattice and g on the other pairs of H is majority. There is no need for Maltsev!

The existence of DAT in H can be checked in polynomial time.

Digraphs with Majority Function

Theorem HR(2011)

H has a majority iff H does not contain any permutable triple.

Proof Idea:

If there is a permutable triple then there is no majority.

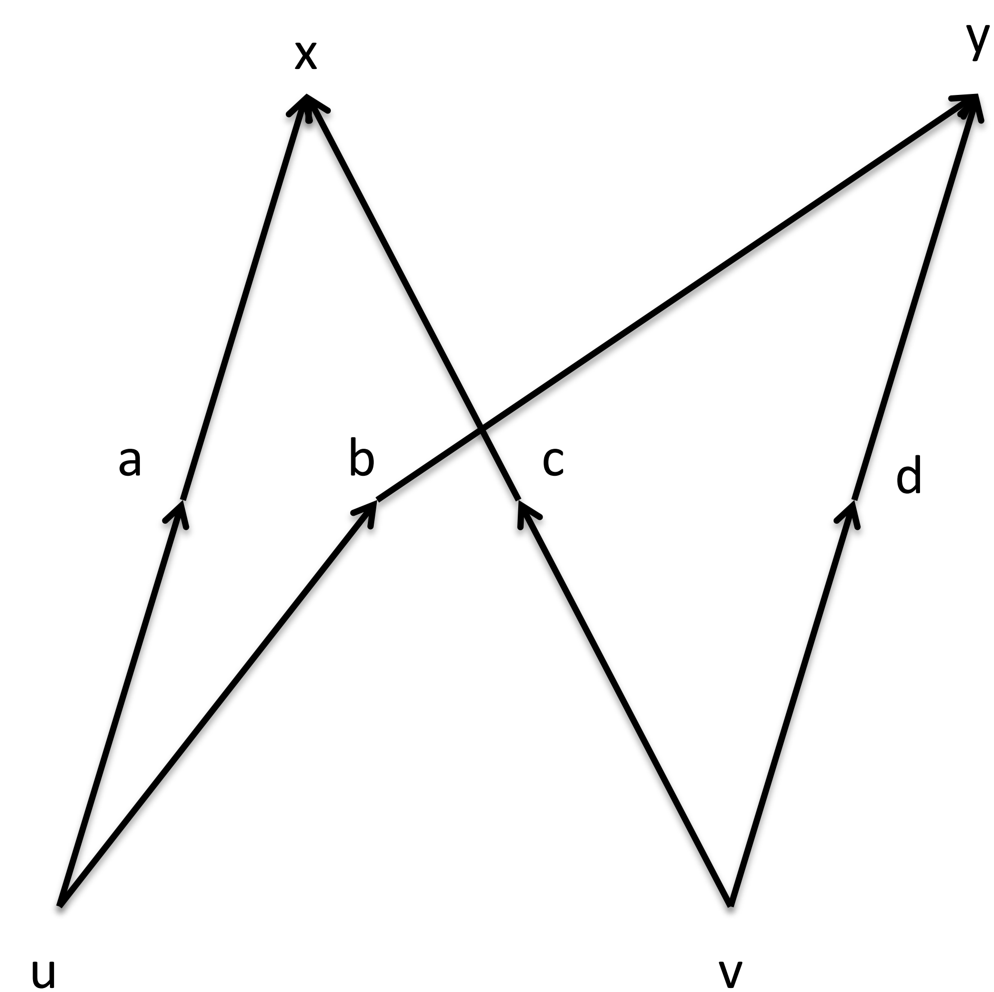
If there is no permutable triple then each triple x, y, z has a distinguisher.

We say that x is a *distinguisher* for x, y, z, if for any three mutually congruent walks from x, y, z to any s, b, b respectively, the first walk does not avoid one of the other two walks.

We define the values of f as follows: for a triple (u, v, w), we set $f(u, v, w)$ to be the first vertex from u, v, w, in this order, that is a distinguisher for u, v, w.

There is a majority

g(a,b,c)=a
g(a,b,d)=b
g(a,c,d)=c
g(b,c,d)=d



Thank You!