Is asymptotic extremal graph theory of dense graphs trivial?

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joint work with Sergey Norin

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Intro

Introduction

- Asymptotic extremal graph theory has been studied for more than a century (Mantel 1908, Turán, Erdös, Lovász, Szemeredi, Bollobás, Simonovits,. . .).

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- Recently, there has been several developments explaining this: Freedman, Lovász, Schrijver, Szegedy, Razborov, Chayes, Borgs, Sös, Vesztergombi, ...

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- Discovery of rich algebraic structure underlying many of these techniques.
- Neater proofs with no low-order terms.
- Methods for applying these techniques in semi-automatic ways.

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algebraic inequalities

Many fundamental theorems in extremal graph theory can be expressed as algebraic inequalities between subgraph densities.

Theorem (Freedman, Lovász, Schrijver 2007)

Every such inequality follows from the positive semi-definiteness of a certain infinite matrix.

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Razborov's flag algebras

A formal calculus capturing many standard arguments (induction, Cauchy-Schwarz,...) in the area.

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Razborov's flag algebras

A formal calculus capturing many standard arguments (induction, Cauchy-Schwarz,...) in the area.

Observation (Lovász-Szegedy and Razborov)

Every algebraic inequality between subgraph densities can be approximated by a finite number of applications of the Cauchy-Schwarz inequality.

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Automatic methods for proving theorems (based on SDP):

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- Razborov: Significant improvement over the known bounds on Turán's Hypergraph Problem.

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Applications

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- Hladky, Kral, Norin: Improved bounds for Caccetta-Haggkvist conjecture.

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Applications

Automatic methods for proving theorems (based on SDP):

- Razborov: Significant improvement over the known bounds on Turán's Hypergraph Problem.
- Hladky, Kral, Norin: Improved bounds for Caccetta-Haggkvist conjecture.
- HH, Hladky, Kral, Norin, Razborov: A question of Sidorenko and Jagger, Šťovíček and Thomason.
- HH, Hladky, Kral, Norin, Razborov: A conjecture of Erdös.

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Application

SDP methods + thinking:

Application

SDP methods + thinking:

- Razborov: Minimal density of triangles.

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Application

SDP methods + thinking:

- Razborov: Minimal density of triangles.
- Razborov: Turán's hypergraph problem under mild extra conditions.

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Application

SDP methods + thinking:

- Razborov: Minimal density of triangles.
- Razborov: Turán's hypergraph problem under mild extra conditions.
- HH, Hladky, Kral, Norin: Improved bounds on some problems.

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How far can we go?

Is asymptotic extremal graph theory trivial? Is lack of enough computational power the only barrier?

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Intro

How far can we go?

Is asymptotic extremal graph theory trivial? Is lack of enough computational power the only barrier?

Question (Razborov)

Can every true algebraic inequality between subgraph densities be proved using a finite amount of manipulation with subgraph densities of finitely many graphs?

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Homomorphisms

Homomorphism Densities

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Homomorphisms

Extremal graph theory

Studies the relations between the number of occurrences of different subgraphs in a graph *G*.

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Homomorphisms

Extremal graph theory

Studies the relations between the number of occurrences of different subgraphs in a graph G.

Equivalently one can study the relations between the "homomorphism densities".

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Homomorphisms

Homomorphism Density

Definition

- Throw the vertices of H on the vertices of G at random.

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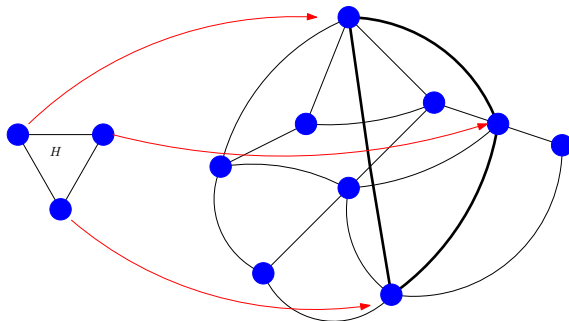
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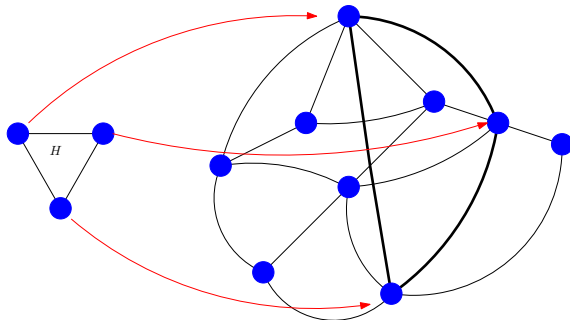
Homomorphisms

Homomorphism Density

> ## Definition
>
> - Throw the vertices of H on the vertices of G at random.
>
> $t_H(G) := \Pr[\text{edges go to edges}].$

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Homomorphisms

Definition

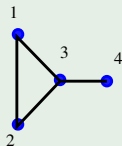A map $f : H \to G$ is called a homomorphism if it maps edges to edges.

Definition

A map $f : H \to G$ is called a homomorphism if it maps edges to edges.

$$t_H(G) = \Pr[f : H \to G \text{ is a homomorphism}].$$

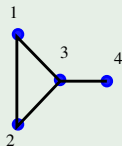Introduction   **Homomorphism densities**   Graph Algebras   Results   PSD method
○○○○○○○   ○○○○●○○   ○○○○○○○○○○○○○○   ○○○○○○○○○○○○○○ ○○○○○○○○○○○○○

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Homomorphism Density

Example

$$t_{K_2}(G) = \frac{2 \times 4}{4^2} = \frac{1}{2}.$$

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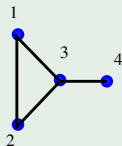
Homomorphisms

Homomorphism Density

Example

$$t_{K_2}(G) = \frac{2 \times 4}{4^2} = \frac{1}{2}.$$

Example

$$t_{K_3}(G) = \frac{3!}{4^3}.$$

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Homomorphisms

- Asymptotically $t_H(\cdot)$ and subgraph densities are equivalent.

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Homomorphisms

- Asymptotically $t_H(\cdot)$ and subgraph densities are equivalent.
- The functions t_H have nice algebraic structures:

$$t_{H_1 \sqcup H_2}(G) = t_{H_1}(G) t_{H_2}(G).$$

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Homomorphisms

- Many fundamental theorems in extremal graph theory can be expressed as algebraic inequalities between homomorphism densities.

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Homomorphisms

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Example (Goodman's bound 1959)

$$t_{K_3}(G) \geq 2t_{K_2}(G)^2 - t_{K_2}(G).$$

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Homomorphisms

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Example (Goodman's bound 1959)

$$t_{K_3}(G) \geq 2t_{K_2}(G)^2 - t_{K_2}(G).$$

- Every such inequality can be turned to a linear inequality:

$$a_1 t_{H_1}(G) + \ldots + a_m t_{H_m}(G) \geq 0.$$

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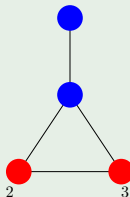
Example (Goodman's bound 1959)

$$t_{K_3}(G) - 2t_{K_2 \sqcup K_2}(G) + t_{K_2}(G) \geq 0.$$

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Graph Algebras

Algebra of Partially labeled graphs

Definition

A partially labeled graph is a graph in which some vertices are labeled by *distinct* natural numbers.

Example

Definition

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Recall

$$t_H(G) := \Pr\left[f : H \to G \text{ is a homomorphism}\right].$$

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Recall

$$t_H(G) := \Pr[f : H \to G \text{ is a homomorphism}].$$

Definition

Let H be partially labeled with labels L. For $\phi : L \to G$, define

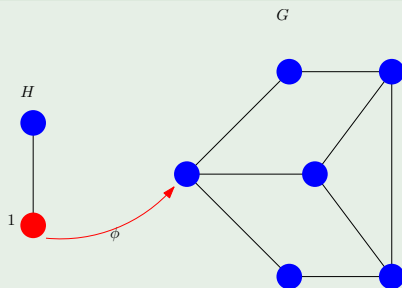$$t_{H,\phi}(G) := \Pr[f : H \to G \text{ is a hom.} \mid f|_L = \phi].$$

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Graph Algebras

Definition

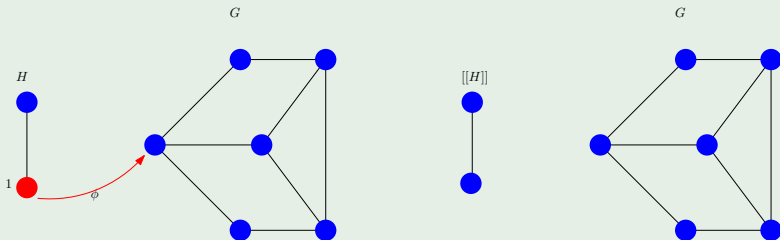Let H be partially labeled with labels L. For $\phi : L \to G$, define

$$t_{H,\phi}(G) := \Pr\left[f : H \to G \text{ is a hom.} \mid f|_L = \phi\right].$$

Example

$t_{H,\phi}(G) = \frac{3}{6} = \frac{1}{2}.$

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Graph Algebras

Example

Definition

Let $[\![H]\!]$ be H with no labels.

Example
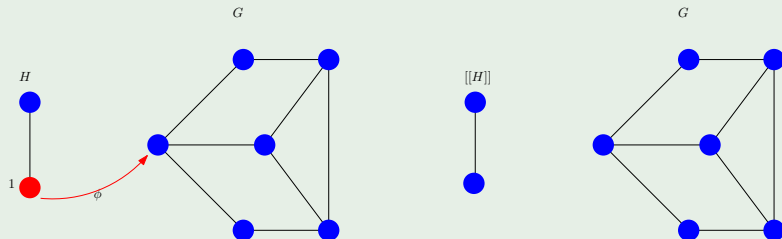
Definition

Let $[\![H]\!]$ be H with no labels.

$$\mathbb{E}_\phi \left[t_{H,\phi}(G) \right] = t_{[\![H]\!]}(G)$$

- Recall that:

$$t_{H_1 \sqcup H_2}(G) = t_{H_1}(G) t_{H_2}(G).$$

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Graph Algebras
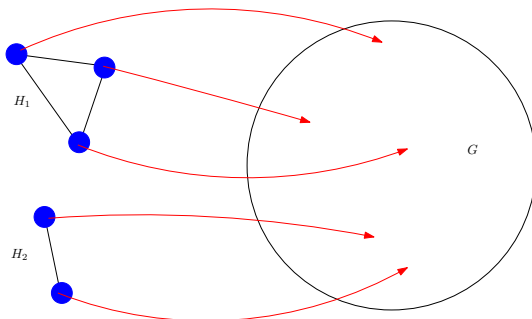
- Recall that:

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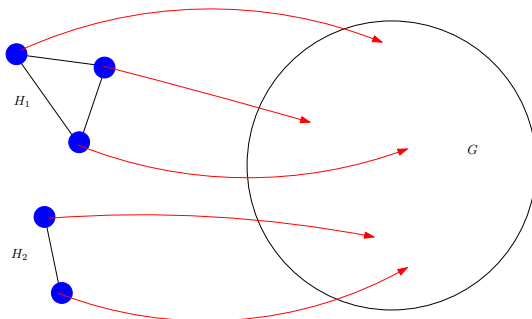
- This motivates us to define $H_1 \times H_2 := H_1 \sqcup H_2$.

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Graph Algebras

Definition

The product $H_1 \cdot H_2$ of partially labeled graphs H_1 and H_2:

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- Take their disjoint union, and then identify vertices with the same label.
- If multiple edges arise, only one copy is kept.

Example
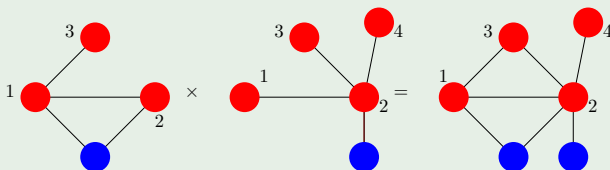
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Graph Algebras

- Let H_1 and H_2 be partially labeled with labels L_1 and L_2.

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Graph Algebras

- Let H_1 and H_2 be partially labeled with labels L_1 and L_2.
- Let $\phi : L_1 \cup L_2 \rightarrow G$.

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Graph Algebras

- Let H_1 and H_2 be partially labeled with labels L_1 and L_2.
- Let $\phi : L_1 \cup L_2 \to G$.
- We have $t_{H_1, \phi}(G) t_{H_2, \phi}(G) = t_{H_1 \times H_2, \phi}(G)$.

Example
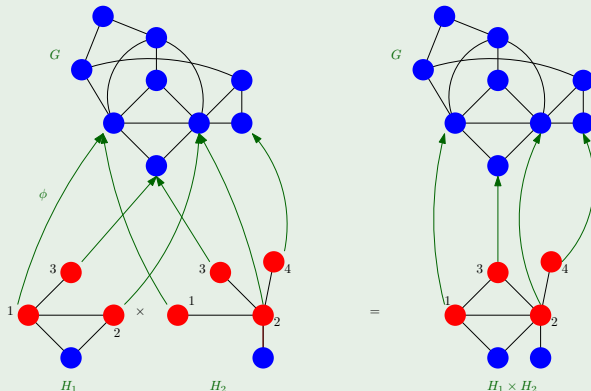
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Graph Algebras

Our main goal is to understand the set of all valid inequalities of the form: For all G $\qquad a_1 t_{H_1}(G) + \ldots + a_k t_{H_k}(G) \geq 0.$

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- Let $H_1, \ldots, H_k$ be partially labeled graphs with the set of labels L.
- Let $b_1, \ldots, b_k$ be real numbers and $\phi : L \to G$.

$$
\begin{aligned}
0 &\leq \left(\sum b_i t_{H_i, \phi}(G) \right)^2 = \sum b_i b_j t_{H_i, \phi}(G) t_{H_j, \phi}(G) \\
&= \sum b_i b_j t_{H_i \times H_j, \phi}(G)
\end{aligned}
$$

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Graph Algebras

Our main goal is to understand the set of all valid inequalities of the form: For all G $\qquad a_1 t_{H_1}(G) + \ldots + a_k t_{H_k}(G) \geq 0$.

- Let $H_1, \ldots, H_k$ be partially labeled graphs with the set of labels L.
- Let $b_1, \ldots, b_k$ be real numbers and $\phi : L \to G$.
-

$$\sum b_i b_j t_{H_i \times H_j, \phi}(G) \geq 0$$

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Graph Algebras

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-
$$\sum b_i b_j t_{[\![H_i \times H_j]\!]}(G) \geq 0$$

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Graph Algebras

Abstraction

- A quantum graph is a formal linear combination of graphs:

$$a_1 H_1 + \ldots + a_k H_k.$$

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- A quantum graph $a_1 H_1 + \ldots + a_k H_k$ is called positive, if for all G,

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Graph Algebras

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- Goodman:

$$K_3 - 2(K_2 \sqcup K_2) + K_2 \geq 0.$$

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- We want to understand the set of all positive quantum graphs.

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Graph Algebras

- A partially labeled quantum graph is a formal linear combination of partially labeled graphs:

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Graph Algebras

- A partially labeled quantum graph is a formal linear combination of partially labeled graphs:

$$a_1 H_1 + \ldots + a_k H_k.$$

- Partially labeled quantum graphs form an algebra:

$$(a_1 H_1 + \ldots + a_k H_k) \cdot (b_1 L_1 + \ldots + b_\ell L_\ell) = \sum a_i b_j H_i \cdot L_j.$$

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Graph Algebras

Unlabeling operator

$[\![\cdot]\!]$: partially labeled quantum graph $\mapsto$ quantum graph

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Graph Algebras

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Recall

$$\llbracket (\sum b_i H_i)^2 \rrbracket = \sum b_i b_j \llbracket H_i \times H_j \rrbracket \geq 0$$

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Equivalently

For every partially labeled quantum graph g we have $\llbracket g^2 \rrbracket \geq 0$.

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Graph Algebras

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$$[\![(\sum b_i H_i)^2]\!] = \sum b_i b_j [\![H_i \times H_j]\!] \geq 0$$

Equivalently

For every partially labeled quantum graph g we have $[\![g^2]\!] \geq 0$.

Corollary

Always

$$[\![g_1^2 + \ldots + g_k^2]\!] \geq 0.$$

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Graph Algebras

Question (Lovász's 17th Problem, Lovász-Szegedy)

Is it true that every $f \geq 0$ is of the form

$$f = [\![g_1^2 + g_2^2 + \ldots + g_k^2]\!]$$

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Does every valid inequality between homomorphism densities have a proof in Cauchy-Schwarz calculus?

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Does every valid inequality between homomorphism densities have a proof in Cauchy-Schwarz calculus?

Theorem (HH and Norin)

The answer to the above questions is negative.

Reflection positivity

Theorem (Freedman, Lovasz, Shrijver)

Every valid inequality between homomorphism densities follows from an infinite number of applications of Cauchy-Schwarz inequality.

Reflection positivity

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Every valid inequality between homomorphism densities follows from an infinite number of applications of Cauchy-Schwarz inequality.

Observation (Lovasz-Szegedy and Razborov)

If $f \geq 0$ and $\epsilon > 0$, there exists a positive integer k and quantum labeled graphs $g_1, g_2, \ldots, g_k$ such that

$$-\epsilon \leq f - [\![g_1^2 + g_2^2 + \ldots + g_k^2]\!] \leq \epsilon.$$

positive polynomials

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Results

- Polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is called positive if it takes only non-negative values.

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Results

- Polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is called positive if it takes only non-negative values.
- $p_1^2 + \ldots + p_k^2$ is always positive.

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Theorem (Hilbert 1888)

There exist 3-variable positive homogenous polynomials which are not sums of squares of polynomials.

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Results

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Example (Motzkin's polynomial)

$$x^4 y^2 + y^4 z^2 + z^4 x^2 - 6x^2 y^2 z^2 \geq 0.$$

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Extending to quantum graphs

Theorem (HH and Norin)

There are positive quantum graphs f which are not sums of squares. That is, always $f \neq [\![g_1^2 + \ldots + g_k^2]\!]$.
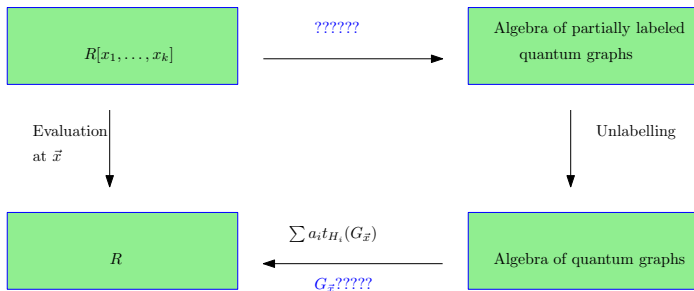
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There are positive quantum graphs f which are not sums of squares. That is, always $f \neq [\![g_1^2 + \ldots + g_k^2]\!]$.

- Convert $x^4y^2 + y^4z^2 + z^4x^2 - 6x^2y^2z^2$ to a quantum graph.

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- Convert $x^4 y^2 + y^4 z^2 + z^4 x^2 - 6x^2 y^2 z^2$ to a quantum graph.

Theorem (Artin 1927, Solution to Hilbert's 17th Problem)

Every positive polynomial is of the form

$$(p_1/q_1)^2 + \ldots + (p_k/q_k)^2.$$

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Results

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Corollary

The problem of checking the positivity of a polynomial is decidable.

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- Co-recursively enumerable : Try to find a point that makes *p* negative.

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Corollary

The problem of checking the positivity of a polynomial is decidable.

- Co-recursively enumerable : Try to find a point that makes *p* negative.
- recursively enumerable : Try to write $p = \sum(p_i/q_i)^2$.

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Results

- Our solution to Lovász's 17th problem was based on an analogy to polynomials.

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Results

- Our solution to Lovász's 17th problem was based on an analogy to polynomials.
- Since there are polynomials which are positive but not sums of squares, our theorem was expected.

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Results

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- Lovász: Does Artin's theorem (sums of rational functions) hold for graph homomorphisms?

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- Lovász: Does Artin's theorem (sums of rational functions) hold for graph homomorphisms?

- Maybe at least the decidability? (A 10th problem)

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Results

- Checking the positivity of $p \in \mathbb{R}[x_1, \ldots, x_k]$ on $[0,1]^k$ is decidable.

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- Matiyasevich 1970, Solution to Hilbert's 10th problem: Checking the positivity of $p \in \mathbb{R}[x_1, \ldots, x_k]$ on $\mathbb{Z}^k$ is undecidable.
- $\{t_H(G) : G\}$ is dense in $[0, 1]$, so maybe we should expect decidability for inequalities on homomorphism densities.

Theorem (HH and Norin)

The following problem is undecidable.

- QUESTION: *Does the inequality*
  $a_1 t_{H_1}(G) + \ldots + a_k t_{H_k}(G) \geq 0$ *hold for every graph G?*

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Corollary

Analogue of Artin's theorem does not hold for homomorphism densities.

Proof

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Equivalently

Theorem (HH and Norin)

The following problem is undecidable.

- INSTANCE: *A polynomial $p(x_1, \ldots, x_k)$ and graphs $H_1, \ldots, H_k$.*
- QUESTION: *Does the inequality $p(t_{H_1}(G), \ldots, t_{H_k}(G)) \geq 0$ hold for every graph G?*

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Instead I will prove the following theorem:

Theorem

*The following problem is undecidable.*

- INSTANCE: *A polynomial $p(x_1, \ldots, x_k, y_1, \ldots, y_k)$.*
- QUESTION: *Does the inequality $p(t_{K_2}(G_1), \ldots, t_{K_2}(G_k), t_{K_3}(G_1), \ldots, t_{K_3}(G_k)) \geq 0$ hold for every $G_1, \ldots, G_k$?*

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Results

- Hilbert 10th: Checking the positivity of $p \in \mathbb{R}[x_1, \ldots, x_k]$ on $\left\{ 1 - \frac{1}{n} : n \in \mathbb{Z} \right\}^k$ is undecidable.

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Results

- Hilbert 10th: Checking the positivity of $p \in \mathbb{R}[x_1, \ldots, x_k]$ on $\left\{1 - \frac{1}{n} : n \in \mathbb{Z}\right\}^k$ is undecidable.
- Bollobás, Razborov: Goodman's bound is achieved only when $t_{K_2}(G) \in \left\{1 - \frac{1}{n} : n \in \mathbb{Z}\right\}$.
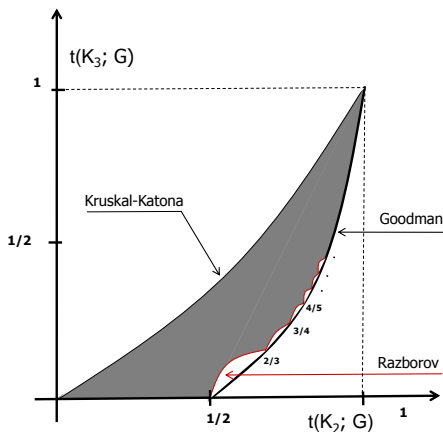
Let S be the grey area and $g(x) = 2x^2 - x$. (Goodman: $t_{K_3}(G) \geq 2t_{K_2}(G)^2 - t_{K_2}(G)$.)
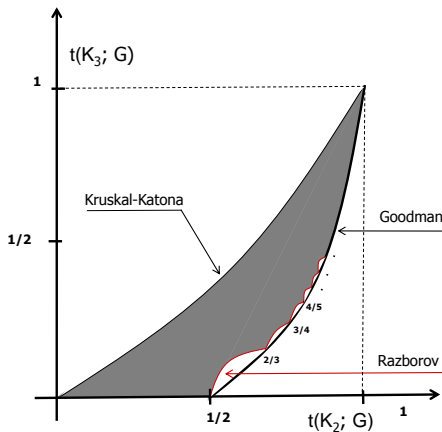
Let S be the grey area and $g(x) = 2x^2 - x$. (Goodman: $t_{K_3}(G) \geq 2t_{K_2}(G)^2 - t_{K_2}(G)$.)

Lemma

Let $p \in \mathbb{R}[x_1, \ldots, x_k]$. Define $q(x_1, \ldots, x_k, y_1, \ldots, y_k)$ as

$$q := p \prod_{i=1}^{k} (1 - x_i)^6 + C_p \times \left(\sum_{i=1}^{k} y_i - g(x_i) \right).$$

T.F.A.E.

Let S be the grey area and $g(x) = 2x^2 - x$. (Goodman: $t_{K_3}(G) \geq 2t_{K_2}(G)^2 - t_{K_2}(G)$.)

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T.F.A.E.

- $p < 0$ *for some* $x_1, \ldots, x_k \in \{1 - 1/n : n \in \mathbb{N}\}$.
  (undecidable)

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T.F.A.E.

- $p < 0$ *for some* $x_1, \ldots, x_k \in \{1 - 1/n : n \in \mathbb{N}\}$. *(undecidable)*
- $q < 0$ *for some* $(x_i, y_i) \in S$'s.
- $q < 0$ *for some* $x_i = t_{K_2}(G_i)$ *and* $y_i = t_{K_3}(G_i)$. *(reduction)*

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Proof by computers!

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PSD method

- Let $H_1, H_2, \ldots$ be all graphs.

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PSD method
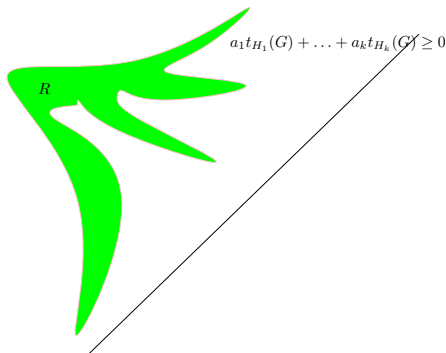
- Let $H_1, H_2, \ldots$ be all graphs.
- Let R be the closure of all points
  $(t_{H_1}(G), t_{H_2}(G), \ldots) \in [0, 1]^{\mathbb{N}}$.

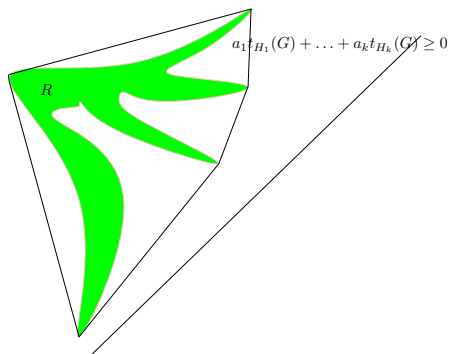| Introduction | Homomorphism densities | Graph Algebras | Results | PSD method |
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PSD method

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- What does the validity of $\sum a_i t_{H_i}(G) \geq 0$ mean?

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- What does the validity of $\sum a_i t_{H_i}(G) \geq 0$ mean?

$a_1 t_{H_1}(G) + \ldots + a_k t_{H_k}(G) \geq 0$

R

Characterization of the convex hull of *R*

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PSD method

Characterization of the convex hull of R

Let $F_1, F_2, \ldots$ be all partially labeled graphs.

Characterization of the convex hull of R

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For all G:

- Condition I: $t_{K_1}(G) = 1$.

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PSD method

Characterization of the convex hull of R

Let $F_1, F_2, \ldots$ be all partially labeled graphs.
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Theorem (Freedman,Lovász,Schriver,Szegedy)

$(x_{H_1}, x_{H_2}, \ldots) \in [0, 1]^{\mathbb{N}}$ *belongs to the convex hull of R* $\Longleftrightarrow$

- *Condition I:* $x_{K_1} = 1$.
- *Condition II:* $x_{H \sqcup K_1} = x_H$ *for all graph H.*
- *Condition III: The infinite matrix whose ij-th entry is* $x_{[\![F_i \times F_j]\!]}$ *is positive semi-definite.*

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Proving $\sum a_i t_{H_i}(G) \geq 0$:

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Proving $\sum a_i t_{H_i}(G) \geq 0$:

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Minimize $\sum a_i x_{H_i}$

Subject to   (i) $x_{K_1} = 1$

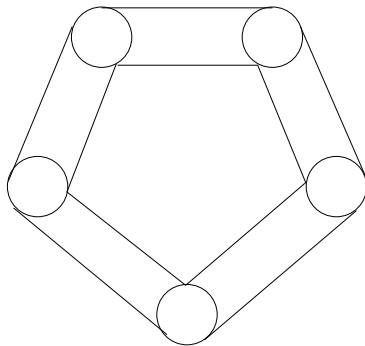   (ii) $x_{H \sqcup K_1} = x_H$ for all graphs H

                 (iii) The matrix $[x_{[\![F_i \times F_j]\!]}]_{ij}$ is p.s.d.

Application I: 5 cycles in triangle-free graphs

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Theorem (HH, Hladky, Kral, Norin, Razborov)

Erdös's Conjecture 1982: The maximum number of cycles of length five in a triangle-free graph is $(n/5)^5$.

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Application II: common graphs

Definition

A graph H on m graphs is called common, if

$$t_H(G) + t_H(\overline{G}) \geq 2^{1-m}$$

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- Thomason 1989: Any graph containing K_4 is not common.
- Jagger, Stovicek, Thomason 1994 and Sidorenko 1994: Are there common non-3-colorable graph? Is W_5 common?

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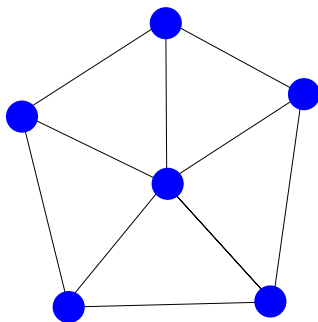0000000000●00

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Theorem (HH, Hladky, Kral, Norin, Razborov)

The wheel W_5 is common.

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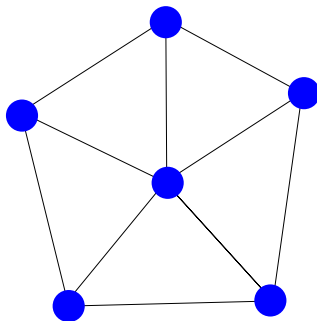
The wheel W_5 is common.

- The computer generated proof uses 150 Cauchy-Schwarz's.

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Open problems

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Recall

R is the closure of $\{(t_{H_1}(G), t_{H_2}(G), \ldots) \,:\, G\} \subset [0, 1]^{\mathbb{N}}$.

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- Lovász and Sös: Which points of R are unique solutions to some equality $p(x_{H_1}, \ldots, x_{H_k}) = 0$? (Finitely Forcible)

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- Lovász's collection of open problems.

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Thank you!