# Optimal Execution in a General One-Sided Limit-Order Book

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Joint work with
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- Objective: Minimize total cost of purchase.

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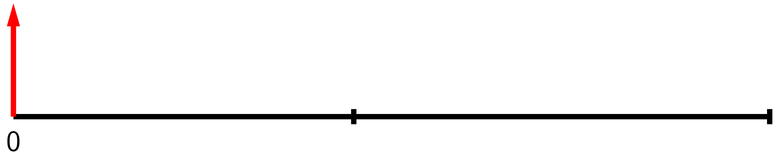
- Make an initial lump purchase at time zero.

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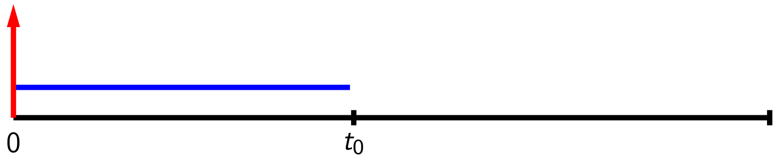


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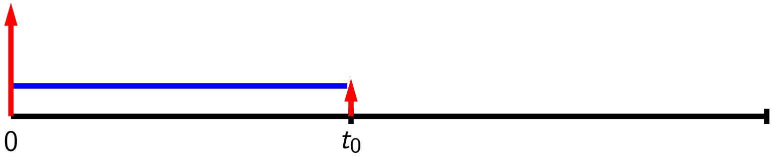


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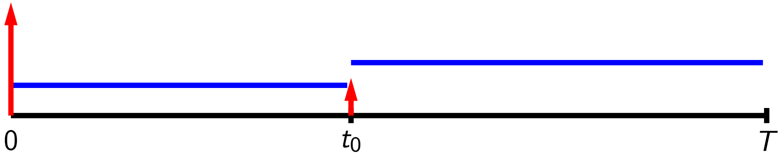


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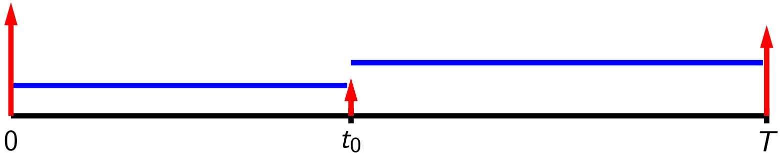
- At time $T$, make a final lump purchase.

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- Alfonsi, Fruth and Schied (2010). Same as Obizhaeva & Wang, except more general shape of limit-order book.

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- Trading takes place in continuous time.
- Immediate price impact depends on the order book shape. No restrictions are placed on the order book shape.
- Order book has resilience, with no permanent price impact.
- Purchasing at a constant rate is the continuous-time analogue of the results of the earlier papers.
- For order book shapes that fall outside the class studied previously, the optimal strategy can exhibit an intermediate lump purchase.

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- $E_t, 0 \le t \le T$ — Residual effect process. This is the quantity of orders missing from the order book because of the combined effect of agent's purchases and book's resilience:

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- $D_t, 0 \le t \le T$ — Price displacement due to the combined effect of agent's purchases and book's resilience.

# Price displacement

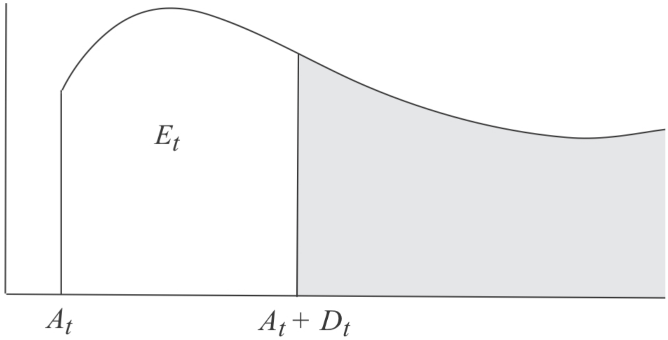


Figure: Shadow limit order book at time $t$.

- The shaded area shows the orders in the book. This is the actual limit order book.
- The white area $E_t$ shows orders missing from the shadow book.
- The current ask price is $A_t + D_t$.

# Formula for price displacement

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- $D_t \triangleq \psi(E_t), 0 \leq t \leq T$.

# Cost of execution

Suppose for the moment that $A_t \equiv 0$ and no purchases have been made prior to the present time.

- The cost of purchasing all the shares available at prices in $[0, x)$ is

$$\varphi(x) \triangleq \int_{[0,x)} \xi \, dF(\xi).$$

- The cost of purchasing $y$ shares is

$$\Phi(y) \triangleq \varphi\big(\psi(y)\big) + \Big[y - F\big(\psi(y)\big)\Big]\psi(y).$$

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Suppose only that $A_t \equiv 0$. Recall that $\Delta X_t = \Delta E_t$.

- Then the cost of the purchasing strategy $X_t, 0 \le t \le T$, is
$$C(X) = \int_0^T D_t \, dX_t^c + \sum_{0 \le t \le T} \big[\Phi(E_t) - \Phi(E_{t-})\big].$$

# Cost of execution (continued)

On the previous page, when we assume that $A_t \equiv 0$, we have the cost of execution

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If we do not assume that $A_t \equiv 0$, then the cost of execution is

$$\begin{aligned} C(X) &= \int_0^T (A_t + D_t) \, dX_t^c + \sum_{0 \le t \le T} \left[ A_t \Delta X_t + \Phi(E_t) - \Phi(E_{t-}) \right] \\ &= \int_0^T D_t \, dX_t^c + \sum_{0 \le t \le T} \left[ \Phi(E_t) - \Phi(E_{t-}) \right] + \int_{[0,T]} A_t \, dX_t. \end{aligned}$$

# Cost simplification

Using integration by parts, we write the term containing $A_t$ in the cost as

$$\int_{[0.T]} A_t \, dX_t \quad = \quad A_T X_T - A_0 X_{0-} - \int_0^T X_t \, dA_t$$

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- The search for an optimal trading strategy can be restricted to deterministic strategies.

# Cost simplification (continued)

**Theorem**

*The cost of using trading strategy $X_t, 0 \leq t \leq T$,*

$$C(X) = \int_0^T D_t \, dX_t^c + \sum_{0 \leq t \leq T} \left[ \Phi(E_t) - \Phi(E_{t-}) \right]$$

*is equal to*

$$C(X) = \Phi(E_T) + \int_0^T D_t h(E_t) \, dt.$$

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and

$$\Phi'(y) = \varphi'\big(\psi(y)\big)\psi'(y) = \psi(y)f\big(\psi(y)\big)\psi'(y) = \psi(y).$$

## Step Two of the Proof: Chain Rule

$$\Phi(E_T) \;=\; \int_0^T D^-\Phi(E_t)\,dE_t^c + \sum_{0\le t\le T}\big[\Phi(E_t) - \Phi(E_{t-})\big]$$

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&= \Phi(E_T) + \int_0^T D_t h(E_t)\, dt,
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where the last step uses $E_t = X_t - \int_0^t h(E_s) ds$.

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$$C(X) = \Phi(E_T) + \int_0^T D_t h(E_t)\, dt,$$

where $X_{0-} = 0$, $X_T = \overline{X}$ and

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Set

$$g(y) \triangleq y\psi\big(h^{-1}(y)\big)$$

so that

$$g\big(h(E_t)\big) = h(E_t))\psi(E_t) = h(E_t)D_t.$$

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$$\begin{aligned} E_t &= X_t - \int_0^t h(E_s) ds, \\ \psi(y) &\triangleq \sup\{x \geq 0 | F(x) < y\}, \\ D_t &\triangleq \psi(E_t). \end{aligned}$$

Set

$$g(y) \triangleq y\psi\left(h^{-1}(y)\right)$$

so that

$$g\left(h(E_t)\right) = h(E_t))\psi(E_t) = h(E_t)D_t.$$

Then

$$C(X) = \Phi(E_T) + \int_0^T g\left(h(E_t)\right) dt.$$

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## Theorem

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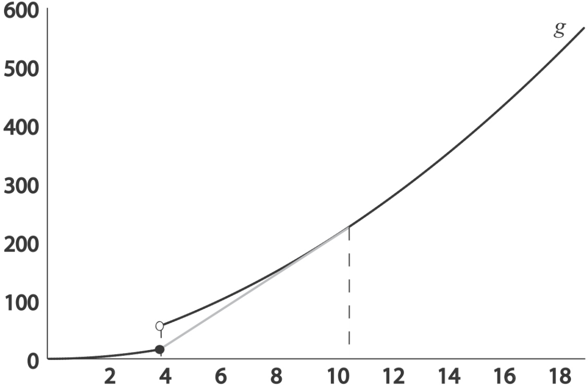
\end{aligned}$$

and equality holds if $h(E_t)$ is constant on $(0, T)$. Minimize the last expression over $E_T$ to determine the constant.

# Three-jump strategies

If $g$ is not convex, replace $g$ by its convex hull.




To achieve a constant purchasing rate on the graph of the convex hull that is not on the graph of $g$, say at 6, purchase a while at rate 4 and a while at rate 10.324. The switch from 4 to 10.324 creates an intermediate jump.

## Example (Block order book)

Let $q$ and $\rho$ be a positive constants. Set

$$F(x) = qx, \quad h(x) = \rho x.$$

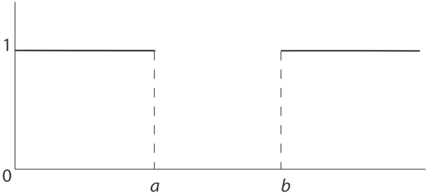
Then

$$\psi(y) = \frac{y}{q}, \quad \Phi(y) = \frac{y^2}{2q}, \quad g(y) = \frac{y^2}{\rho q}.$$

Optimal strategy:

- Initial lump purchase of size $\frac{\overline{X}}{2+\rho T}$,
- Intermediate purchases at rate $\frac{\rho \overline{X}}{2+\rho T}$,
- Terminal lump purchase of size $\frac{\overline{X}}{2+\rho T}$.

# Example (Modified block order book)



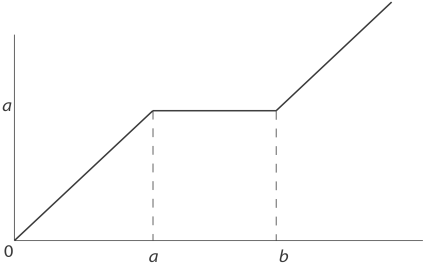




Figure: Density and cumulative distribution of the modified block order book

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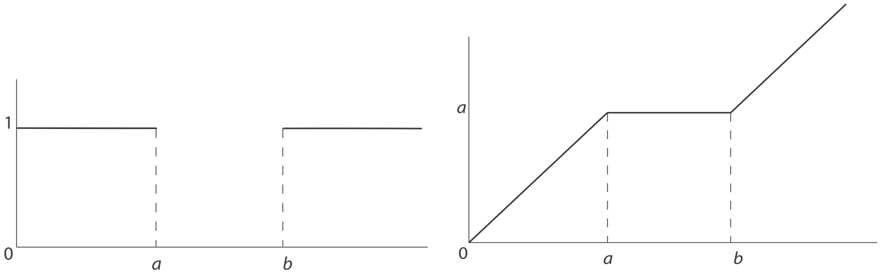


Figure: Density and cumulative distribution of the modified block order book

$$\psi(y) = \begin{cases} y, & 0 \le y \le a, \\ y + b - a, & a < y < \infty, \end{cases}$$

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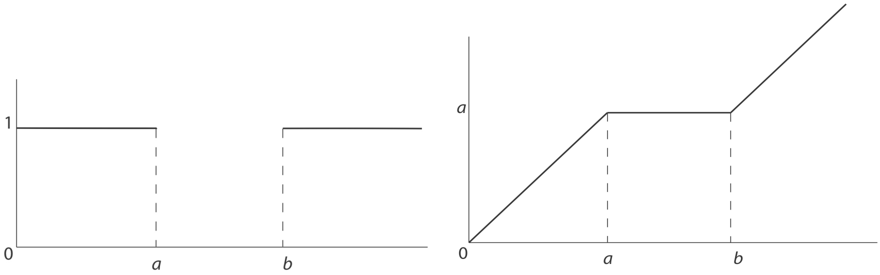


Figure: Density and cumulative distribution of the modified block order book

$$\psi(y) = \begin{cases} y, & 0 \leq y \leq a, \\ y + b - a, & a < y < \infty, \end{cases}$$

$$\Phi(y) = \begin{cases} \frac{1}{2} y^2, & 0 \leq y \leq a, \\ \frac{1}{2}\left((y + b - a)^2 + a^2 - b^2\right), & a \leq y < \infty. \end{cases}$$

# Example (Modified block order book, continued)

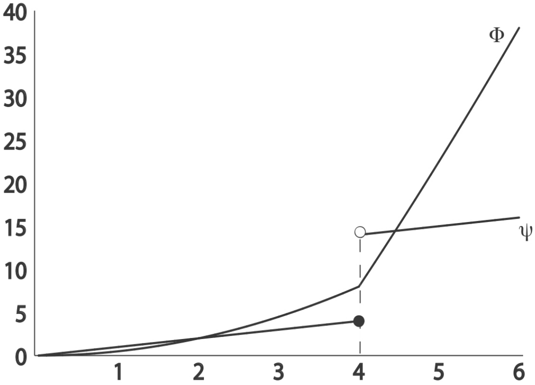


Figure: Functions $\Phi$ and $\psi$ for the modified block order book with parameters $a = 4$ and $b = 14$

## Example (Modified block order book, continued)

$$g(y) = \begin{cases} y^2, & 0 \le y \le a, \\ y^2 + (b-a)y, & a < y < \infty. \end{cases}$$

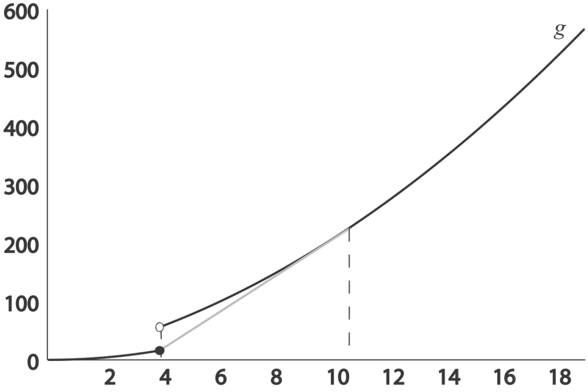


Figure: Function $g$ for the modified block order book with parameters $a = 4$ and $b = 14$. The convex hull $\widehat{g}$ is constructed by replacing a part $\{g(y)\ , y \in (a, \beta)\}$ by a straight line connecting $g(a)$ and $g(\beta)$. Here $\beta = 10.324$

## Example (Discrete order book)

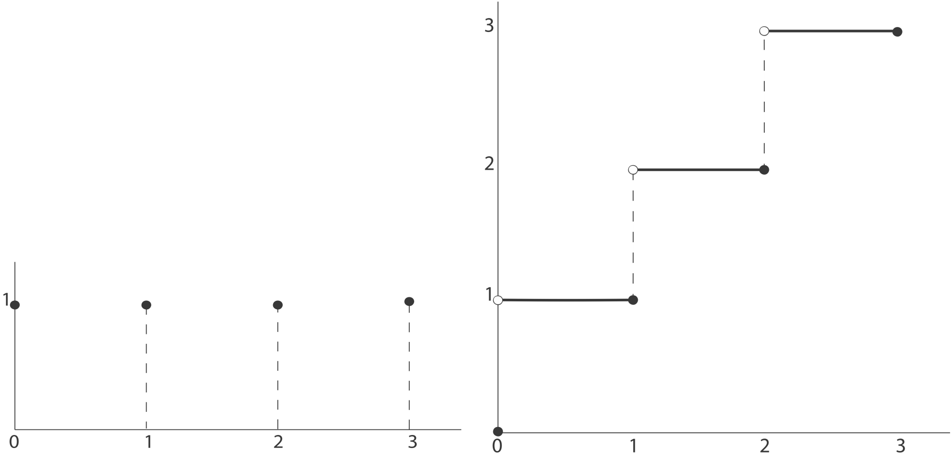


Figure: Measure and cumulative distribution function of the discrete order book

## Example (Discrete order book, continued)

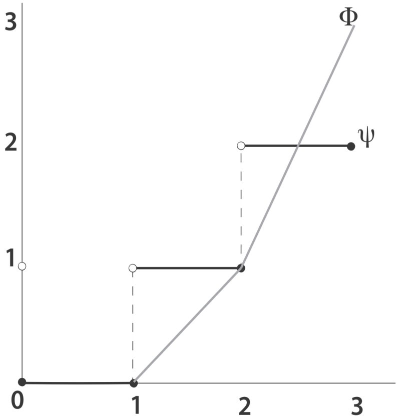


Figure: Functions $\Phi$ and $\psi$ for the discrete order book

# Example (Discrete order book, continued)

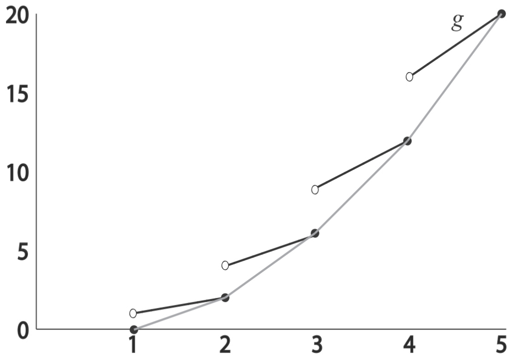


Figure: Function $g$ for the discrete order book. The convex hull $\widehat{g}$ interpolates linearly between the points $(k, (k-1)k)$ and $(k+1, k(k+1))$.