Filling the Gaps

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Abstract

We overview local and stochastic volatility models and concentrate on calibration of parametric local volatility models

We describe a semi-analytical method to compute option prices in local volatility models using Laplace transform and Sturm-Liouville methods for ODEs

We apply this method to solve the calibration problem for a specific version of parametric local volatility: tiled local volatility with the number of volatility parameters equal to the number of option quotes

Acknowledgements

Black-Scholes (BS) paradigm

BS dynamics (1973) is the standard GBM:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t$$

BS equation for the call price:

$$C_t + rSC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} - rC = 0, \ \ C(T,S) = (S-K)_+$$

BS formula:

$$C^{(BS)}(0, S_0, T, K; \sigma, r) = SN\left(d_+\right) - e^{-rT}KN\left(d_-\right), \ \ \Delta = N\left(d_+\right)$$

$$d_\pm = \frac{\ln(S_0/K) + rT}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}$$

Black-Scholes (BS) paradigm

Alternative representation of the BS formula (Lipton (2000)):

$$C^{(BS)}(0, S_0, T, K; \sigma, r) = S - \frac{e^{-rT}K}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\left(-iu+\frac{1}{2}\right)(\ln(S_0/K)+rT)-\frac{\sigma^2 T}{2}\left(u^2+\frac{1}{4}\right)}}{u^2 + \frac{1}{4}} du$$

This formula is derived by representing payoff of a call option in the form

$$(S - K)_+ = S - \min\{S, K\}$$

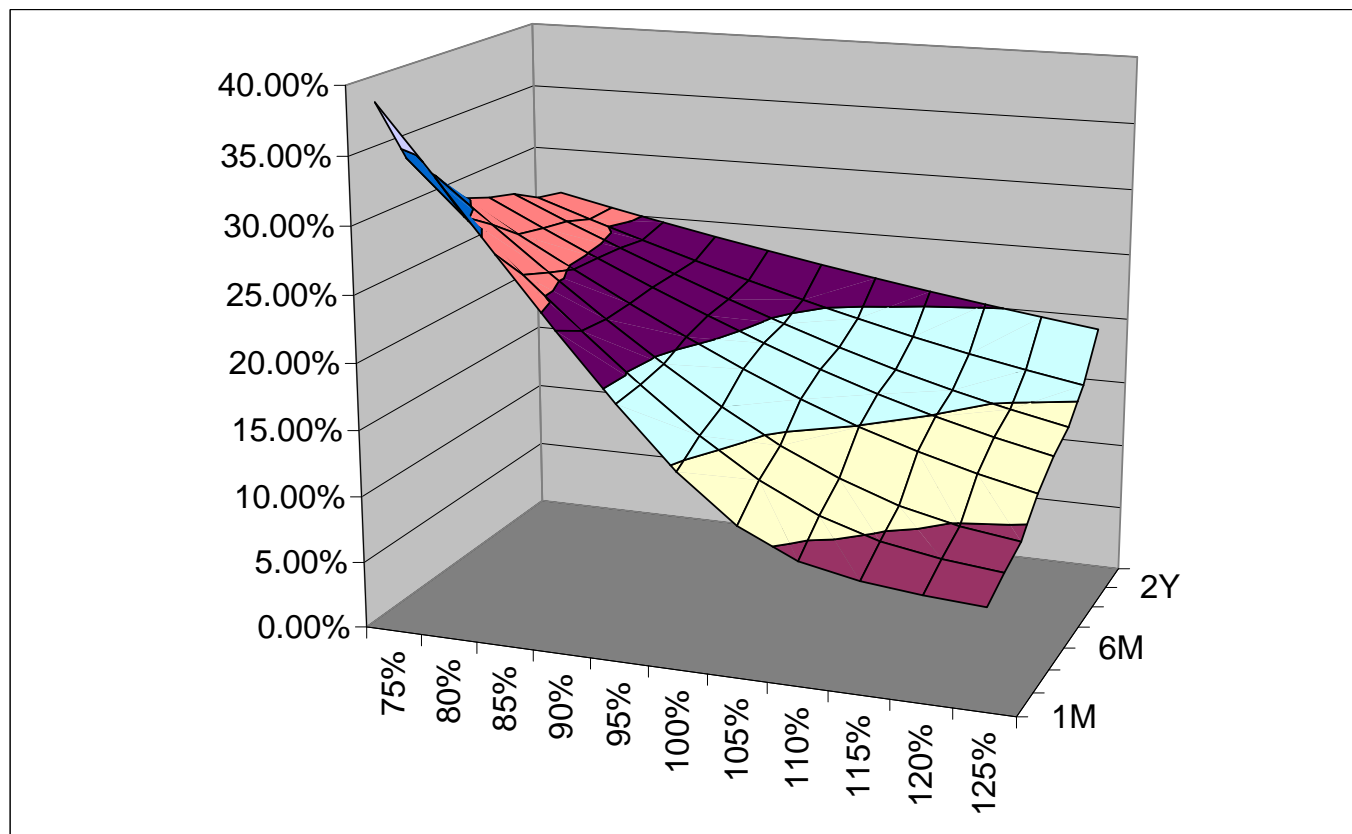and dealing with the bounded component of the payout by changing the measure.

This expression has very useful generalization known as the Lewis-Lipton formula which has important implications.
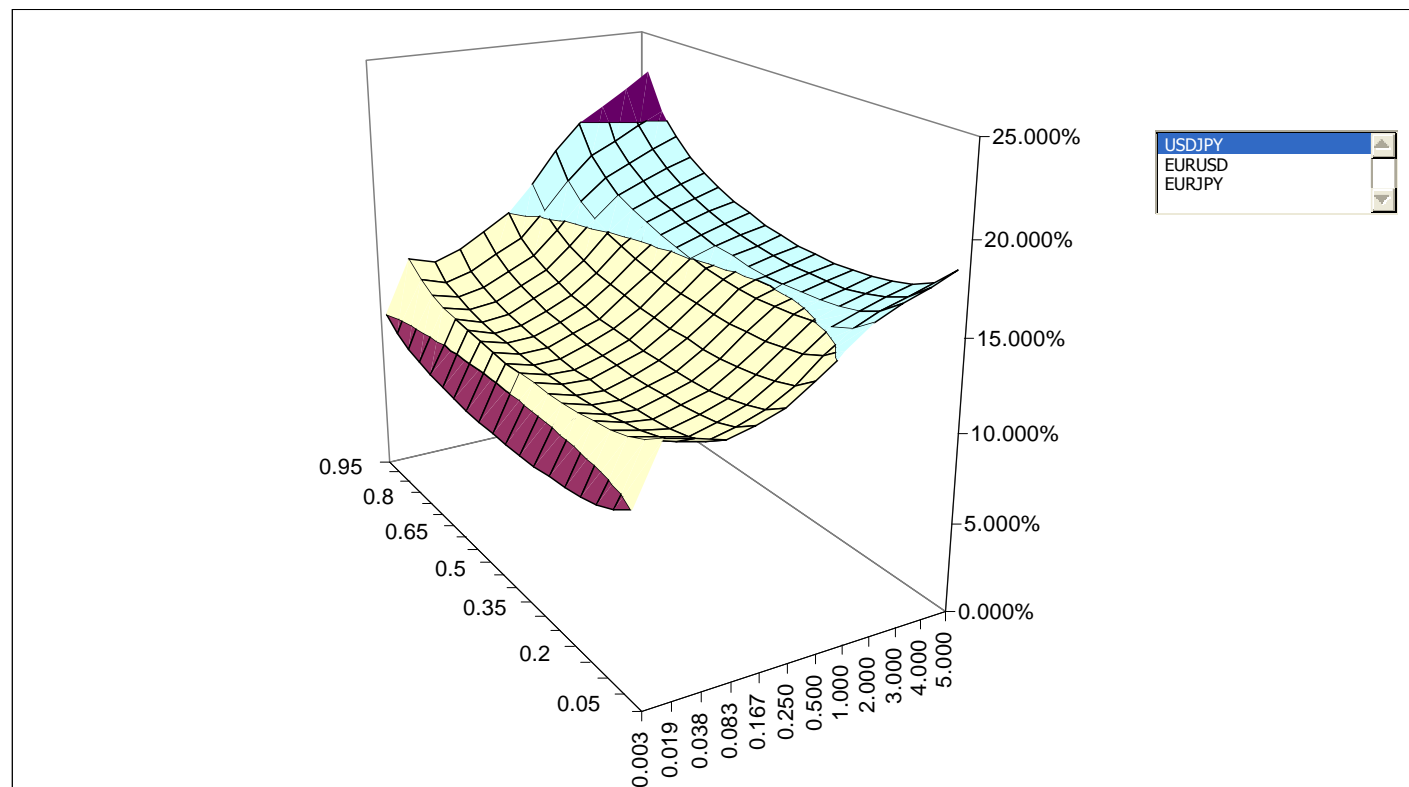
Implied volatility

"A wrong number which is substituted in a wrong formula to get the right price"

$$C^{(Mrkt)}(T, K) = C^{(BS)}(0, S_0, T, K; \sigma_{imp}(T, K), r)$$

Idealized equity market

Idealized forex market

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Real equity market

K\T	0.101	0.197	0.274	0.523	0.772	1.769	2.267	2.784	3.781	4.778	5.774
51.31								33.66	32.91		
58.64								31.78	31.29	30.08	
65.97								30.19	29.76	29.75	
73.30								28.63	28.48	28.48	
76.97			32.62	30.79	30.01	28.43					
80.63			30.58	29.36	28.76	27.53	27.13	27.11	27.11	27.22	28.09
84.30			28.87	27.98	27.50	26.66					
86.13											
87.96	29.06	27.64	27.17	26.63	26.37	25.75	25.55	25.80	25.85	26.11	26.93
89.97	27.97	26.72									
91.63	26.90	25.78	25.57	25.31	25.19	24.97					
93.46	25.90	24.89									
95.29	24.88	24.05	24.07	24.04	24.11	24.18	24.10	24.48	24.69	25.01	25.84
97.12	23.90	23.29									
98.96	23.00	22.53	22.69	22.84	22.99	23.47					
100.79	22.13	21.84									
102.62	21.40	21.23	21.42	21.73	21.98	22.83	22.75	23.22	23.84	23.92	24.86
104.45	20.76	20.69									
106.29	20.24	20.25	20.39	20.74	21.04	22.13					
108.12	19.82	19.84									
109.95	19.59	19.44	19.62	19.88	20.22	21.51	21.61	22.19	22.69	23.05	23.99
111.78	19.29	19.20									
113.62			19.02	19.14	19.50	20.91					
117.28			18.85	18.54	18.88	20.39	20.58	21.22	21.86	22.23	23.21
120.95			18.67	18.11	18.39	19.90					
124.61			18.71	17.85	17.93	19.45		20.54	21.03	21.64	22.51
131.94								19.88	20.54	21.05	21.90
139.27								19.30	20.02	20.54	21.35
146.60								18.49	19.64	20.12	

Real forex market
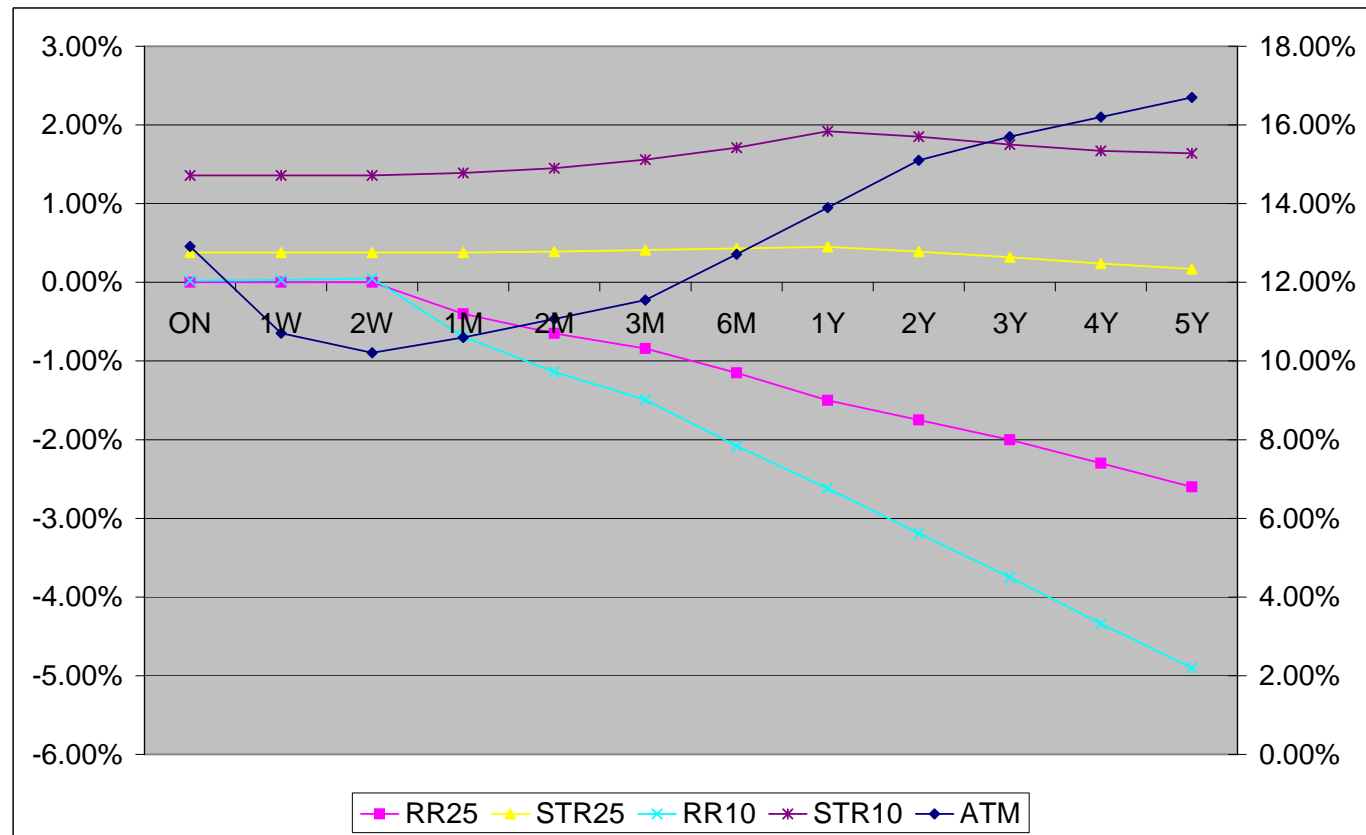
Imagination versus reality

Implied volatility set or implied volatility surface.

Vol versus strike or vol versus delta? And if so, which delta?

How to preserve no arbitrage condition? And what should it be?

At the very least we have to have

$$C_T(T, K) \geq 0, \ \ C_{KK}(T, K) \geq 0$$

Something needs to be done, but what?

It is clear that we need to alter the basic premises of the BS theory.

Several possibilities present themselves:
(A) Parametric local volatility;
(B) Non-parametric local volatility;
(C) Stochastic volatility;
(D) Jumps;
(E) Regime switching;
(F) Various combinations of the above.

Parametric local volatility

Replacing GBM with a dierent process was the earliest approach. In fact, it far predates BS approach. For instance, Bachelier (1900) postulated that stock price is governed by AMB:

$$\frac{dS_t}{S_t} = rdt + \frac{\sigma}{S_t}dW_t$$

Later, several other possibilities have been considered, notably, CEV (Cox (1975), Cox & Ross (1976), Emanuel & MacBeth (1982)):

$$\frac{dS_t}{S_t} = rdt + \sigma S_t^{\beta-1}dW_t$$

displaced diffusion (Rubinstein (1983)):

$$\frac{dS_t}{S_t} = rdt + \sigma\frac{(S_t + \beta)}{S_t}dW_t$$

hyperbolic diffusion (Lipton (2000)),

$$\frac{dS_t}{S_t} = rdt + \sigma\left(\alpha S_t + \beta + \frac{\gamma}{S_t}\right)dW_t$$

Non-parametric local volatility

In general, the so-called alternative stochastic processes do not match market prices exactly (although in many cases they come quite close).

Accordingly, an idea to consider processes with unknown local volatility to be calibrated to the market somehow had been proposed by several researchers (Derman & Kani (1994), Dupire (1994), Rubinstein (1994)).

The corresponding dynamics is

$$\frac{dS_t}{S_t} = rdt + \sigma_{loc}(t, S_t)dW_t$$

The first and the third approaches were formulated via implied trees, while the second one in terms of PDEs.

Stochastic volatility

Alternatively, several researchers suggested that volatility itself is stochastic. Several choices have been discussed in the literature:

(A) Hull & White (1988) model:

$$\frac{d\sigma_t}{\sigma_t} = \alpha dt + \gamma dW_t,$$

(B) Scott (1987) and Wiggins (1987) model:

$$\frac{d\sigma_t}{\sigma_t} = (\alpha - \beta\sigma_t)dt + \gamma dW_t,$$

(C) Stein & Stein (1991) model:

$$d\sigma_t = (\alpha - \beta\sigma_t)dt + \gamma dW_t$$

Stochastic volatility

(D) Heston (1993) model:

$$d\sigma_t = \left(\frac{\alpha}{\sigma_t} - \beta\sigma_t\right) dt + \gamma dW_t$$

(E) Lewis (2000) model:

$$\frac{d\sigma_t}{\sigma_t^2} = \left(\frac{\alpha}{\sigma_t} - \beta\sigma_t\right) dt + \gamma dW_t$$

Occasionally, it is more convenient to deal with variance $v_t = \sigma_t^2$. The most popular assumption is that variance is driven by a square-root (Feller (1952)) process (Heston (1993)):

$$dv_t = \kappa(\theta - v_t)dt + \varepsilon\sqrt{v_t}dW_t$$

My personal favorite model is Stein-Stein. The reasons are partly practical (easy to simulate) and partly sentimental (Kelvin wave analogy). Usual objections (negative vol) are irrelevant (vol is not a sign definite quantity).

Stochastic volatility

More recently Bergomi (2005) proposed to use HJM-style equations for stochastic volatility. Closer inspection suggests (to me?) that his model is more or less equivalent to Scott and Wiggins model.

Jump-diffusion based models

Merton (1976) proposed to add jumps to the standard BS dynamics:

$$\frac{dS_t}{S_t} = (r - \lambda m)dt + \sigma dW_t + \left(e^J - 1\right) dN_t$$

where N_t is yhe Poisson process with intensity λ, $m = \mathbb{E}\{e^J - 1\}$.

Merton considered Gaussian distribution of jumps. Other distributions, such as exponential (Kou (2002) and others) and hyper-exponential (Lipton (2002)) have been popular as well.

In reality though, it is exceedingly difficult to distinguish between different distributions, so that discrete one is perfectly adequate for many applications.

Lévy process based models

Lévy process based models had been popularized by many researchers, for instance, Boyarchenko & Levendorsky (2000, 2002), Carr & Wu (2003, 2003), Cont and Tankov (2004), Eberlein (1995), and many others.

These models assume that St is an exponential Lvy process of the jump-diffusion type

$$S_t = S_0 e^{X_t}$$

where

$$dX_t = \gamma dt + \sigma dW_t + \int_{\mathbf{R}} (e^x - 1) (\mu(dt, dx) - \nu(dx)dt), X_0 = 0$$

Here the random measure $\mu(dt, dx)$ counting jumps in dz over the time-interval dt, must, from the properties of time-homogenous Lévy processes, have the form $dt \times \mu(dz)$, with expectation $dt \times \nu(dz)$.

Regime switching

Regime switching models have been less popular. However, some of them are quite good. For instance, ITO33 model which they modestly call nobody's model (ITO 33 (2004)) looks rather appealing.

Composite models

Each of the models considered above has its own attractions (as well as drawbacks). Hence several researchers tried to build combined models.

The need for such models is particularly strong in forex market because in this markets several exotics are liquid and can be used for calibration purposes.

In particular, a class of the so-called LSV models was developed by Jex, Henderson, Wang (1999), Blacher (2001), Lipton (2002). The first model is tree-based, the other two are PDE based.

The corresponding dynamics has the form

$$\frac{dS_t}{S_t} = rdt + \sqrt{v_t}\sigma(t, S_t)dW_t,$$
$$dv_t = \kappa(\theta - v_t)dt + \epsilon\sqrt{v_t}dZ_t,$$
$$dW_t dZ_t = \rho dt.$$

Composite models

Blacher (2001) assumes that log-normal volatility is quadratic:

$$\sigma(t, S_t) = \left(a + b S_t + c S_t^2 \right).$$

Lipton (2002) considers hyperbolic log-normal volatility:

$$\sigma(t, S_t) = \left(\frac{a}{S_t} + b + c S_t \right).$$

as well as purely non-parametric one.

Jäckel, Kahl (2010) consider other interesting possibilities.

Lipton's model is offered commercially by Murex (without proper ac-knowledgement).

SABR model

A very different composite model is proposed by Hagan et al. (2002) (the so-called SABR model). The corresponding dynamics is

$$\frac{dF_t}{F_t} = \sigma_t F_t^{\beta-1} dW_t,$$
$$d\sigma_t = \upsilon \sigma_t dZ_t,$$
$$dW_t dZ_t = \rho dt.$$

Although this model has several attractive features, including is scaling properties, it is clearly not dynamic in nature.

Universal model

In 2002 Lipton proposed a Universal Vol Model. This model incorporates most of the attractive features of the models considered so far. The corresponding dynamics has the form

$$\frac{dS_t}{S_t} = (r - \lambda m)dt + \sqrt{v_t}\sigma_{loc}(t, S_t)dW_t + \left(e^J - 1\right)dN_t,$$

$$dv_t = \kappa(\theta - v_t)dt + \epsilon\sqrt{v_t}dZ_t \ (+\varphi dN_t)$$

$$dW_t dZ_t = \rho dt.$$

While this model is very attractive, it is very ambitious in its design and requires a lot of effort in order to be implemented properly.

Philosophical aside

Philosophical question: What kind of models are we looking for: plastic bags which assume the form of whatever goods are put into them, or cardboard boxes which can keep the form regardless and break if we put something too hard into them. Interesting ideas are developed by Aiyache (2004), and ITO 33 (2004).

Parametric local vol (such as CEV, quadratic, etc.) is a CB-style model. Might be difficult to match the market but can be good in other respects and provides a lot of insight.

Non-parametric local vol model is a PB-style model. It takes more or less arbitrary "market" prices (where it takes them from is entirely different question) and converts them into implied volatility.

Calibration of term structure model

We have to decide how to relate local vol and implied vol.

In the presence of term structure (but no skew) of the implied vol, there is a classical relation

$$\sigma_{loc}^2(T) = \frac{d\left(\sigma_{imp}^2(T)T\right)}{dT}$$

This formula is not as simple as it looks.

Generic calibration problem

For regular diffusion processes, the generic calibration problem reduces to defining the local volatility in such a way that market quotes for the corresponding option prices coincide with model prices.

The calibration problem for local volatility of the one-dimensional diffusion is represented by the Dupire equation for call prices $C(T, K)$ as functions of maturity time T and strike K (interest rates are zero):

$$
\begin{aligned}
C_T - \tfrac{1}{2}\sigma_{loc}^2(T, K)\, K^2 C_{KK} &= 0, \\
C(0, K) &= (S - K)_+
\end{aligned}
\tag{1}
$$

If market call prices $C(T, K)$ are known *for all* T, K, then $\sigma_{loc}^2(T, K)$ can found by inverting equation (1):

$$
\sigma_{loc}^2(T, K) = \frac{2 C_T(T, K)}{K^2 C_{KK}(T, K)}.
\tag{2}
$$

Generic calibration problem

If instead $\sigma_{loc}^2(T, K)$ is given by a particular functional form, prices of call options have to be obtained by solving equation (1).

The resulting calibration problem is solved in the least-squares sense.

While the Dupire equation (1) is one of the most ruthlessly efficient equations in the entire financial engineering field, most of the time the highly stylized equation (2) is used instead.

In view of the fact that under normal conditions $C(T, K)$ are known *only for a few* T_i, K_j, this usage is hard to justify.

Forward PDE

For computational purposes, it is more convenient to deal with covered calls $\bar{C}(T, K) = S - C(T, K)$, which solve the following problem

$$\bar{C}_T - \tfrac{1}{2}\sigma_{loc}^2(T, K) K^2 \bar{C}_{KK} = 0,$$
$$\bar{C}(0, K) = S - (S - K)_+. \tag{3}$$

We introduce a new independent variable X, $X = \ln(K/S)$, and a new dependent variable $B(T, X)$, $\bar{C}(T, X) = Se^{X/2}B(T, X)$:

$$B_T(T, X) - \tfrac{1}{2}v(T, X)\left(B_{XX}(T, X) - \tfrac{1}{4}B(T, X)\right) = 0,$$
$$B(0, X) = e^{X/2}\mathbf{1}_{\{X \leq 0\}} + e^{-X/2}\mathbf{1}_{\{X > 0\}}, \tag{4}$$

where $v(T, X) = \sigma_{loc}^2\left(T, Se^X\right)$.

Its solution can be represented as follows:

$$B(T, X) = \int_{-\infty}^{\infty} G(T, X, X')B(0, X')dX',$$

where $G(T, X, X')$ is the Green's function that solves equation (4) with initial condition given by delta function: $\delta(X - X')$.

Here X is a forward variable and X' is a backward variable.

Term structure of model parameters

Assume that v is a piece-wise constant function of time,

$$v(T, X) = v_i(X), \qquad T_{i-1} < T \leq T_i, \qquad 1 \leq i \leq I,$$

so that equation (4) can be solved by induction.

On each time interval $T_{i-1} < T \leq T_i$, $1 \leq i \leq I$, the corresponding problem is represented in the form

$$B_{i,\tau}(\tau, X) - \tfrac{1}{2}v_i(X)\left(B_{i,XX}(\tau, X) - \tfrac{1}{4}B_i(\tau, X)\right) = 0,$$
$$B_i(0, X) = B_{i-1}(X), \tag{5}$$

$$B_i(\tau, X) = B(T, X), \qquad \tau = T - T_{i-1}, \qquad B_{i-1}(X) = B(T_{i-1}, X).$$

Induction starts with

$$B_0(X) = e^{X/2}\mathbf{1}_{\{X \leq 0\}} + e^{-X/2}\mathbf{1}_{\{X > 0\}}.$$

The solution of problem (5) can be written as

$$B_i(\tau, X) = \int_{-\infty}^{\infty} G_i(\tau, X, X')B_{i-1}\left(X'\right)dX', \tag{6}$$

where G_i is the corresponding Green's function for the corresponding time interval.

Andreasen & Huge (2010)

As a crude approximation, the time-derivative $\partial/\partial\tau$ can be implicitly discretized and forward problem (5) can be cast in the form

$$B_i^{AH}(X) - \frac{1}{2}(T_i - T_{i-1})\, v_i(X)\left(B_{i,XX}^{AH}(X) - \frac{1}{4}B_i^{AH}(X)\right) = B_{i-1}^{AH}(X),$$

where $B_i^{AH}(X) \approx B(T_i, X)$.

This is the approach chosen by AH in the specific case of piecewise constant $v_i(X)$.

While intuitive and relatively simple to implement, this approach is not accurate, by its very nature, and its accuracy cannot be improved.

Moreover, for every τ, $0 < \tau \leq T_i - T_{i-1}$, a separate equation single step equation from time T_{i-1} to $T_{i-1} + \tau$ has to be solved. These equations are solved in isolation and are not internally consistent.

Below an alternative approach is proposed. This approach is based on representation (6); by construction it is exact in nature.

Laplace Transform

It turns out that the problem (5) can be solved exactly, rather than approximately, via the direct and inverse Laplace transform (for applications of the Laplace transform in derivatives pricing see Lipton, 2001).

After performing the direct Carson-Laplace transform

$$\widehat{B}_i\left(\lambda, X\right) = \lambda \mathcal{L}\left\{B_i\left(\tau, X\right)\right\},$$

the following Sturm-Liouville problem is obtained:

$$\widehat{B}_i\left(\lambda, X\right) - \frac{1}{2}\frac{1}{\lambda}v_i\left(X\right)\left(\widehat{B}_{i,XX}\left(\lambda, X\right) - \frac{1}{4}\widehat{B}_i\left(\lambda, X\right)\right) = B\left(T_{i-1}, X\right),$$
$$\widehat{B}_i\left(\lambda, X\right) \underset{X\to\pm\infty}{\to} 0. \tag{7}$$

It is clear that

$$B_i^{AH}\left(X\right) = \widehat{B}_i\left(\frac{1}{T_i - T_{i-1}}, X\right). \tag{8}$$

Sturm-Liouville equation

It is convenient to represent equation (7) in the standard Sturm-Liouville form

$$-\widehat{B}_{i,XX}\left(\lambda, X\right) + q_i^2\left(\lambda, X\right)\widehat{B}_i\left(\lambda, X\right) = \left(q_i^2\left(\lambda, X\right) - \tfrac{1}{4}\right)B\left(T_{i-1}, X\right),$$

$$\widehat{B}_i\left(\lambda, X\right)\underset{X\to\pm\infty}{\to}0,$$

where

$$q_i^2\left(\lambda, X\right) = \frac{2\lambda}{v_i\left(X\right)} + \frac{1}{4}.$$

The corresponding Green's function $\widehat{G}_i\left(\lambda, X, X'\right)$ solves the following adjoint Sturm-Liouville problems

$$-\widehat{G}_{i,XX}\left(\lambda, X, X'\right) + q_i^2\left(\lambda, X\right)\widehat{G}_i\left(\lambda, X, X'\right) = \delta\left(X - X'\right),$$

$$\widehat{G}_i\left(\lambda, X, X'\right)\underset{X\to\pm\infty}{\to}0,$$

$$-\widehat{G}_{i,X'X'}\left(\lambda, X, X'\right) + q_i^2\left(\lambda, X'\right)\widehat{G}_i\left(\lambda, X, X'\right) = \delta\left(X - X'\right),$$

$$\widehat{G}_i\left(\lambda, X, X'\right)\underset{X'\to\pm\infty}{\to}0. \tag{9}$$

ODE solution

To be concrete, backward problem (9) is considered and its fundamental solutions are denoted by $\widehat{g}_i^\pm (\lambda, X')$:

$$-\widehat{g}_{i,X'X'}^\pm \left(\lambda, X'\right) + q_i^2 \left(\lambda, X'\right) \widehat{g}_i^\pm \left(\lambda, X'\right) = 0, \qquad \widehat{g}_i^\pm \left(\lambda, X'\right) \underset{X' \to \pm\infty}{\to} 0.$$

These solutions are unique (up to a constant). It is well-known (see, e.g., Lipton (2001)) that

$$\widehat{G}_i \left(\lambda, X, X'\right) = \frac{1}{W(\lambda)} \begin{cases} \widehat{g}_i^+ \left(\lambda, X\right) \widehat{g}_i^- \left(\lambda, X'\right), & X' \le X, \\ \widehat{g}_i^- \left(\lambda, X\right) \widehat{g}_i^+ \left(\lambda, X'\right), & X' > X, \end{cases}$$

where $W(\lambda)$ is the so-called Wronskian

$$W(\lambda) = \widehat{g}_i^- \left(\lambda, X\right) \widehat{g}_{i,X'}^+ \left(\lambda, X'\right) - \widehat{g}_i^+ \left(\lambda, X\right) \widehat{g}_{i,X'}^- \left(\lambda, X'\right).$$

Summary

Once the Green's function is found, the solution of equation (7) can be represented in the form

$$\hat{B}_i\left(\lambda, X\right) = \int_{-\infty}^{\infty} \hat{G}_i\left(\lambda, X, X'\right)\left(q_i^2\left(\lambda, X'\right) - \frac{1}{4}\right) B_{i-1}\left(X'\right) dX'. \qquad (10)$$

This is a generic formula in models where the Green's function $\hat{G}\left(\lambda, X, X'\right)$ is known in the closed form.

The inverse Carson-Laplace transform yields $B\left(T_{i-1} + \tau, X\right)$ for $0 < \tau \leq T_i - T_{i-1}$, including $B_i\left(X\right)$.

In order to compute the integral in equation (10) it is assumed that X and X' are defined on the same grid $X_{\mathsf{min}} < X < X_{\mathsf{max}}$ and the trapezoidal rule is applied.

Once $B_i\left(X\right)$ is computed for a given $v_i\left(X\right)$, the latter function is changed until market prices are reproduced. The latter operation is non-linear in nature and might or might not be feasible. This depends on whether or not market prices are internally consistent.

Calibration problem for a tiled local volatility case

We apply the generic calibration method to the case of tiled local volatility considered by AH.

Given a *discrete* set of market call prices $C_{mrkt}\left(T_i, K_j\right)$, $0 \le i \le I$, $0 \le j \le J_i$, we consider a tiled local volatility $\sigma_{loc}\left(T, K\right)$

$$\sigma_{loc}\left(T, K\right) = \sigma_{ij}, \quad T_{i-1} < T \le T_i, \quad \bar{K}_{j-1} < K \le \bar{K}_j, \quad 1 \le i \le I, \quad 0 \le j \le J_i$$
$$\bar{K}_{-1} = 0, \quad \bar{K}_j = \tfrac{1}{2}\left(K_j + K_{j+1}\right), \quad 0 \le j \le J_i - 1, \quad \bar{K}_{J_i} = \infty,$$

Clearly, non-median break points can be chosen if needed. Equivalently, $\sigma_{loc}\left(T, X\right)$ has the form

$$\sigma_{loc}\left(T, X\right) = v_{ij}, \quad T_{i-1} < T \le T_i, \quad \bar{X}_{j-1} < X \le \bar{X}_j, \quad \bar{X}_j = \ln\left(\bar{K}_j / S\right).$$

By construction, for every T_i, $\sigma_{loc}\left(T_i, K\right)$ depends on as many parameters as there are market quotes.

On every step of the calibration procedure these parameters are adjusted in such way that the corresponding model prices $C_{mdl}\left(T_i, K_j\right)$ and market prices $C_{mrkt}\left(T_i, K_j\right)$ coincide within prescribed accuracy.

Calibration problem (9) for a tiled local volatility case

For calibration it is sufficient to consider $X = X_j$, where $X_j = \ln\left(S_j/K\right)$; however, to propagate the solution forward from T_{i-1} to T_i, it is necessary to consider all X.

A new set of ordered points is introduced

$$\{Y_k\} = \left\{\bar{X}_j\right\} \cup X, \quad -1 \leq k \leq J_1 + 1, \quad Y_{-1} = -\infty, \quad Y_{J_1+1} = \infty,$$

and it is assumed that $X = Y_{k*}$.

On each interval

$$\mathcal{J}_k = \left\{X' \middle| Y_{k-1} \leq X' \leq Y_k\right\},$$

except for the first and the last one, the general solution of equation (9) has the form

$$g_k\left(X'\right) = \alpha_{k,+} e^{q_k\left(X'-Y_{k*}\right)} + \alpha_{k,-} e^{-q_k\left(X'-Y_{k*}\right)},$$

while on the first and last intervals it has the form

$$g_0\left(X'\right) = \alpha_{0,+} e^{q_0\left(X'-Y_{k*}\right)}, \quad g_{J_1+1}\left(X'\right) = \alpha_{J_1+1,-} e^{-q_{J_1+1}\left(X'-Y_{k*}\right)}$$

so that the corresponding Green's function decays at infinity. Here q_k are constant values of the potential on the corresponding interval.

Calibration problem for a tiled local volatility case

For $k \neq k^*$ both $\widehat{G}$ and $\widehat{G}_X$ have to be continuous, while for $k = k^*$ only $\widehat{G}$ is continuous, while $\widehat{G}_X$ has a jump of size -1.

Thus, the following system of $2\,(J_1 + 1)$ linear equations can be obtained:

$$
\begin{pmatrix} -E^+_{kk} & -E^-_{kk} & E^+_{k+1k} & E^-_{k+1k} \\ -q_k E^+_{kk} & q_k E^-_{kk} & q_{k+1} E^+_{k+1k} & -q_{k+1} E^-_{k+1k} \end{pmatrix} \begin{pmatrix} \alpha_{k,+} \\ \alpha_{k,-} \\ \alpha_{k+1,+} \\ \alpha_{k+1,-} \end{pmatrix} = \begin{pmatrix} 0 \\ -\delta_{kk^*} \end{pmatrix}.
$$

Here

$$
E^\pm_{kl} = e^{\pm q_k (Y_l - Y_{k^*})},
$$

and δ_{kk^*} is the Kronecker symbol.

Calibration problem for a tiled local volatility case

In matrix form these equations can be written as

$$\mathcal{R}\vec{A} = \vec{B}_{k^*}, \tag{11}$$

For instance, for $J = 3, k^* = 1$, we have

$$\mathcal{R} = \begin{pmatrix} -E_{00}^+ & E_{10}^+ & E_{10}^- & & & & & \\ -q_0 E_{00}^+ & q_1 E_{10}^+ & -q_1 E_{10}^- & & & & & \\ & -1 & -1 & 1 & 1 & & & \\ & -q_1 & q_1 & q_2 & -q_2 & & & \\ & & & -E_{22}^+ & -E_{22}^- & E_{32}^+ & E_{32}^- & \\ & & & -q_2 E_{22}^+ & q_2 E_{22}^- & q_3 E_{32}^+ & -q_3 E_{32}^- & \\ & & & & & -E_{33}^+ & -E_{33}^- & E_{43}^- \\ & & & & & -q_3 E_{33}^+ & q_3 E_{33}^- & -q_4 E_{43}^- \end{pmatrix}$$

$$\vec{A} = \begin{pmatrix} \alpha_{0,+} & \alpha_{1,+} & \alpha_{1,-} & \alpha_{2,+} & \alpha_{2,-} & \alpha_{3,+} & \alpha_{3,-} & \alpha_{4,-} \end{pmatrix}^T,$$

$$\vec{B}_{k^*} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}^T,$$

Calibration problem for a tiled local volatility case

Although matrix equation (11) is five-diagonal (rather than tri-diagonal), it can still be solved very efficiently via forward elimination and backward substitution.

First, $\left(\alpha_{1,+}, \alpha_{1,-}\right)^T$ is eliminated in favor of $\alpha_{0,+}$ and $\left(\alpha_{J,+}, \alpha_{J,-}\right)^T$ is eliminated in favor of $\alpha_{J+1,-}$:

$$\begin{pmatrix} \alpha_{1,+} \\ \alpha_{1,-} \end{pmatrix} = \frac{1}{2q_1} \begin{pmatrix} (q_1 + q_0)\, E_{10}^- E_{00}^+ \\ (q_1 - q_0)\, E_{10}^+ E_{00}^+ \end{pmatrix} \alpha_{0,+} \equiv \vec{C}_1 \alpha_{0,+},$$

$$\begin{pmatrix} \alpha_{J,+} \\ \alpha_{J,-} \end{pmatrix} = \frac{1}{2q_J} \begin{pmatrix} \left(q_J - q_{J+1}\right) E_{JJ}^- E_{J+1J}^- \\ \left(q_J + q_{J+1}\right) E_{JJ}^+ E_{J+1J}^- \end{pmatrix} \alpha_{J+1,-} \equiv \vec{D}_J \alpha_{J+1,-},$$

Next, $\left(\alpha_{k,+}, \alpha_{k,-}\right)^T$ is eliminated in favor of $\left(\alpha_{k-1,+}, \alpha_{k-1,-}\right)^T$, $2 \leq k \leq k^*$ and $\left(\alpha_{k,+}, \alpha_{k,-}\right)^T$ is eliminated in favor of $\left(\alpha_{k+1,+}, \alpha_{k+1,-}\right)^T$, $k^* + 1 \leq k \leq J - 1$:

$$\begin{pmatrix} \alpha_{k,+} \\ \alpha_{k,-} \end{pmatrix} = \frac{1}{2q_k} \begin{pmatrix} (q_k + q_{k-1})\, E_{kk-1}^- E_{k-1k-1}^+ & (q_k - q_{k-1})\, E_{kk-1}^- E_{k-1k-1}^- \\ (q_k - q_{k-1})\, E_{kk-1}^+ E_{k-1k-1}^+ & (q_k + q_{k-1})\, E_{kk-1}^+ E_{k-1k-1}^- \end{pmatrix} \begin{pmatrix} \alpha_k \\ \alpha_k \end{pmatrix}$$

Calibration problem for a tiled local volatility case

$$\begin{pmatrix} \alpha_{k,+} \\ \alpha_{k,-} \end{pmatrix} = \frac{1}{2q_k} \begin{pmatrix} \left(q_k + q_{k+1}\right) E_{kk}^- E_{k+1k}^+ & \left(q_k - q_{k+1}\right) E_{kk}^- E_{k+1k}^- \\ \left(q_k - q_{k+1}\right) E_{kk}^+ E_{k+1k}^+ & \left(q_k + q_{k+1}\right) E_{kk}^+ E_{k+1k}^- \end{pmatrix} \begin{pmatrix} \alpha_{k+1,+} \\ \alpha_{k+1,-} \end{pmatrix}$$

and a recursive set of vectors is computed

$$\vec{C}_k = \mathcal{S}_k \vec{C}_{k-1}, \quad 2 \le k \le k^*, \qquad \vec{D}_k = \mathcal{T}_k \vec{D}_{k+1}, \quad k^* + 1 \le k \le J - 1.$$

Finally, a system of 2×2 equations for $\alpha_{0,+}, \alpha_{J+1,-}$ is obtained and solved

$$\begin{pmatrix} -1 & -1 \\ -q_{k^*} & q_{k^*} \end{pmatrix} \vec{C}_{k^*} \alpha_{0,+} + \begin{pmatrix} 1 & 1 \\ q_{k^*+1} & -q_{k^*+1} \end{pmatrix} \vec{D}_{k^*+1} \alpha_{J+1,-} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{12}$$

Once $\alpha_{0,+}, \alpha_{J+1,-}$ are determined, $\left(\alpha_{k,+}, \alpha_{k,-}\right)^T$ are calculated by using vectors $\vec{C}_k$ or $\vec{D}_k$.

This procedure is just icing on the cake since the size of the corresponding system (determined by the number of market quotes) is quite small and is not related to the size of the interpolation grid.

Calibration problem for a tiled local volatility case

Once the coefficient vector $\vec{A}$ is known, $\hat{B}_i(\lambda, X)$ can be computed semi-analytically via equation (10):

$$
\begin{aligned}
\hat{B}_i(\lambda, X) = {} & \left(q_0^2 - \tfrac{1}{4}\right) \alpha_{0,+} e^{-q_0 Y_{k*}} \int_{-\infty}^{Y_0} e^{q_0 X'} B_{i-1}(X')\, dX' \\
& + \sum_{k=1}^{J_i} \left(q_k^2 - \tfrac{1}{4}\right) \left(\alpha_{k,+} e^{-q_k Y_{k*}} \int_{Y_{k-1}}^{Y_k} e^{q_k X'} B_{i-1}(X')\, dX' \right. \\
& \qquad\qquad \left. + \alpha_{k,-} e^{q_k Y_{k*}} \int_{Y_{k-1}}^{Y_k} e^{-q_k(X' - Y_{k*})} B_{i-1}(X')\, dX' \right) \\
& + \left(q_{J_i+1}^2 - \tfrac{1}{4}\right) \alpha_{J_i+1,-} e^{q_{J_i+1} Y_{k*}} \int_{Y_{J_1}}^{\infty} e^{-q_{J_i+1} X'} B_{i-1}(X')\, dX'.
\end{aligned}
\tag{13}
$$

i) The corresponding integrals are computed via the trapezoidal rule.
ii) Variables X and X' are defined on the same dense spatial grid $X_{\min} < X, X' < X_{\max}$; this grid is similar to the one used in a conventional finite-difference solver.
iii) For $X' > X_{\max}$ or $X' < X_{\min}$ it is assumed that $B_{i-1}(X') = e^{-|X|/2}$.
iV) Since these integrals are independent on Y_{k*}, they can be pre-computed for all X; thus the corresponding calculation has complexity linear in J_i.
v) For $i = 1$, the corresponding integrals can be computed analytically.

Calibration problem for a tiled local volatility case

The inverse Carson-Laplace transform generates $B(T, X)$:

$$B(T_{i-1} + \tau, X) = \mathcal{L}_\tau^{-1} \left\{ \frac{\hat{B}(\lambda, X)}{\lambda} \right\}. \tag{14}$$

This transform can be performed efficiently via the Stehfest algorithm:

$$B(T_{i-1} + \tau, X) = \sum_{k=1}^{N} \frac{St_k^N}{k} \hat{B}(k\Lambda, X), \quad \Lambda = \frac{\ln 2}{\tau}. \tag{15}$$

Choosing $N = 12$ is sufficient. Coefficients St_k^{12} are given below

1	2	3	4	5	6
$-0.01(6)$	$16.01(6)$	-1247	$27554.(3)$	$-263280.8(3)$	1324138.7
7	8	9	10	11	12
$-3891705.5(3)$	$7053286.(3)$	-8005336.5	5552830.5	-2155507.2	359251.2

It is obvious that these coefficients are very stiff.
The above procedure allows one to calculate $B(T_i, X_j)$ *for given* v_{ij}.
To calibrate the model to the market, v_{ij} are changed until model and market prices agree. It is worth noting that, as always, vectorizing $\hat{B}(\lambda, X)$ makes computation more efficient.

Calibration problem for a tiled local volatility case

The calibration algorithm is summarized as follows:

(A) At initialization, $B_0(X)$ given by equation (4) is computed on the spatial grid;

(B) At time T_{i+1}, given $B_i(X)$, equations (13) and (14) are used to compute $B(\lambda, X_j)$ and $B\left(T_i, X_j\right)$ at specified market strikes only; $\{v_{ij}\}$ are adjusted until model prices match market prices;

(C) After calibration at time T_{i+1} is complete, $B\left(T_i, X\right)$ is computed on the entire spatial grid using new model parameters at time T_{i+1};

(D) The algorithm is repeated for the next time slice.

If so desired, $B\left(T, X\right)$ can be calculated on the entire temporal-spatial grid with very limited additional effort.

Illustration

For illustrative purposes a tiled local volatility model is calibrated to SX5E equity volatility data as of 01/03/2010. This data is taken from AH.

Depending on maturity, one needs up to 13 tiles to be able to calibrate the model to the market.

In Figure 1 model and market implied volatilities for the index are shown graphically, while in Table 1 the same volatilities are presented numerically.

In Figure 2 the calibrated tiled local volatility is shown.

In Figure 3, the Laplace transforms of the Green's functions $\hat{G}_1(\lambda, X_j, X')$ are shown as functions of X' for fixed λ.

In Figure 4, the Laplace transforms of option prices $\hat{B}(\lambda, X_j)$ are shown as functions of λ.

In Figure 5, functions $B(T_i, X)$ are shown as functions of X.

In all these Figures model parameters calibrated to the SX5E volatility surface are used.

K/T	0.101		0.197		0.274		0.523		0.772		1.769		2.267		2.784		3.781		4.778		5.774	
	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt
51.31			34.40		40.66		38.42		36.88		33.62		32.44		33.66	33.66	32.91	32.91	30.92		32.27	
58.64	35.80		34.29		39.25		36.60		35.21		32.30		31.25		31.78	31.78	31.29	31.29	30.18	30.08	31.30	
65.97	34.65		33.37		37.20		34.52		33.34		30.86		29.98		30.19	30.19	29.76	29.76	29.56	29.75	30.28	
73.30	33.80		32.10		34.39		32.17		31.22		29.30		28.63		28.63	28.63	28.49	28.48	28.54	28.48	29.22	
76.97	33.05		31.25		32.59	32.62	30.79	30.79	30.00	30.01	28.42	28.43	27.89		27.83		27.78		27.88		28.66	
80.63	32.06		30.27		30.66	30.58	29.37	29.36	28.75	28.76	27.53	27.53	27.13	27.13	27.11	27.11	27.11	27.11	27.25	27.22	28.09	28.09
84.30	30.76		29.09		28.82	28.87	27.96	27.98	27.51	27.50	26.63	26.66	26.33		26.44		26.46		26.66		27.51	
86.13	29.95		28.38		27.94		27.27		26.92		26.19		25.92		26.11		26.14		26.36		27.20	
87.96	29.10	29.06	27.65	27.64	27.16	27.17	26.64	26.63	26.38	26.37	25.79	25.75	25.55	25.55	25.80	25.80	25.85	25.85	26.10	26.11	26.93	26.93
89.97	27.90	27.97	26.66	26.72	26.26		25.88		25.71		25.30		25.11		25.42		25.49		25.77		26.60	
91.63	26.92	26.90	25.82	25.78	25.57	25.57	25.30	25.31	25.20	25.19	24.93	24.97	24.79		25.12		25.23		25.53		26.36	
93.46	25.86	25.90	24.86	24.89	24.76		24.64		24.63		24.54		24.43		24.78		24.94		25.26		26.08	
95.29	24.98	24.88	24.08	24.05	24.04	24.07	24.06	24.04	24.11	24.11	24.19	24.18	24.10	24.10	24.48	24.48	24.71	24.69	25.00	25.01	25.84	25.84
97.12	23.86	23.90	23.24	23.29	23.31		23.41		23.53		23.81		23.73		24.14		24.46		24.71		25.57	
98.96	22.98	23.00	22.58	22.53	22.69	22.69	22.83	22.84	23.00	22.99	23.47	23.47	23.40		23.82		24.26		24.45		25.33	
100.79	22.14	22.13	21.78	21.84	21.97		22.20		22.42		23.11		23.03		23.48		24.03		24.15		25.07	
102.62	21.38	21.40	21.24	21.23	21.43	21.42	21.72	21.73	21.98	21.98	22.81	22.83	22.75	22.75	23.22	23.22	23.81	23.84	23.93	23.92	24.86	24.86
104.45	20.78	20.76	20.70	20.69	20.89		21.22		21.50		22.48		22.45		22.95		23.55		23.69		24.63	
106.29	20.24	20.24	20.23	20.25	20.40	20.39	20.74	20.74	21.04	21.04	22.15	22.13	22.16		22.68		23.25		23.47		24.41	
108.12	19.84	19.82	19.86	19.84	19.95		20.28		20.59		21.81		21.87		22.43		22.95		23.24		24.19	
109.95	19.57	19.59	19.45	19.44	19.58	19.62	19.87	19.88	20.20	20.22	21.50	21.51	21.61	21.61	22.19	22.19	22.71	22.69	23.04	23.05	23.99	23.99
111.78	19.28	19.29	19.18	19.20	19.26		19.49		19.84		21.20		21.35		21.94		22.48		22.84		23.79	
113.62	19.04		18.98		19.01	19.02	19.13	19.14	19.50	19.50	20.91	20.91	21.08		21.69		22.27		22.63		23.60	
117.28	18.73		18.68		18.84	18.85	18.55	18.54	18.90	18.88	20.39	20.39	20.58	20.58	21.22	21.22	21.85	21.86	22.24	22.23	23.21	23.21
120.95	18.65		18.48		18.66	18.67	18.11	18.11	18.38	18.39	19.89	19.90	20.14		20.86		21.41		21.91		22.84	
124.61	18.92		18.34		18.71	18.71	17.85	17.85	17.92	17.93	19.45	19.45	19.79		20.54	20.54	21.03	21.03	21.62	21.64	22.51	22.51
131.94	19.85		18.19		18.72		17.59		17.32		18.77		19.22		19.88	19.88	20.54	20.54	21.06	21.05	21.90	21.90
139.27			18.12		18.61		17.47		17.00		18.29		18.79		19.30	19.30	20.02	20.02	20.53	20.54	21.35	21.35
146.60			17.31		18.63		17.43		16.83		17.92		18.44		18.49	18.49	19.64	19.64	20.12	20.12	20.90	

Table 1. The table shows market and model SX5E implied volatility quotes for March 1, 2010. The same market data has been used by AH.
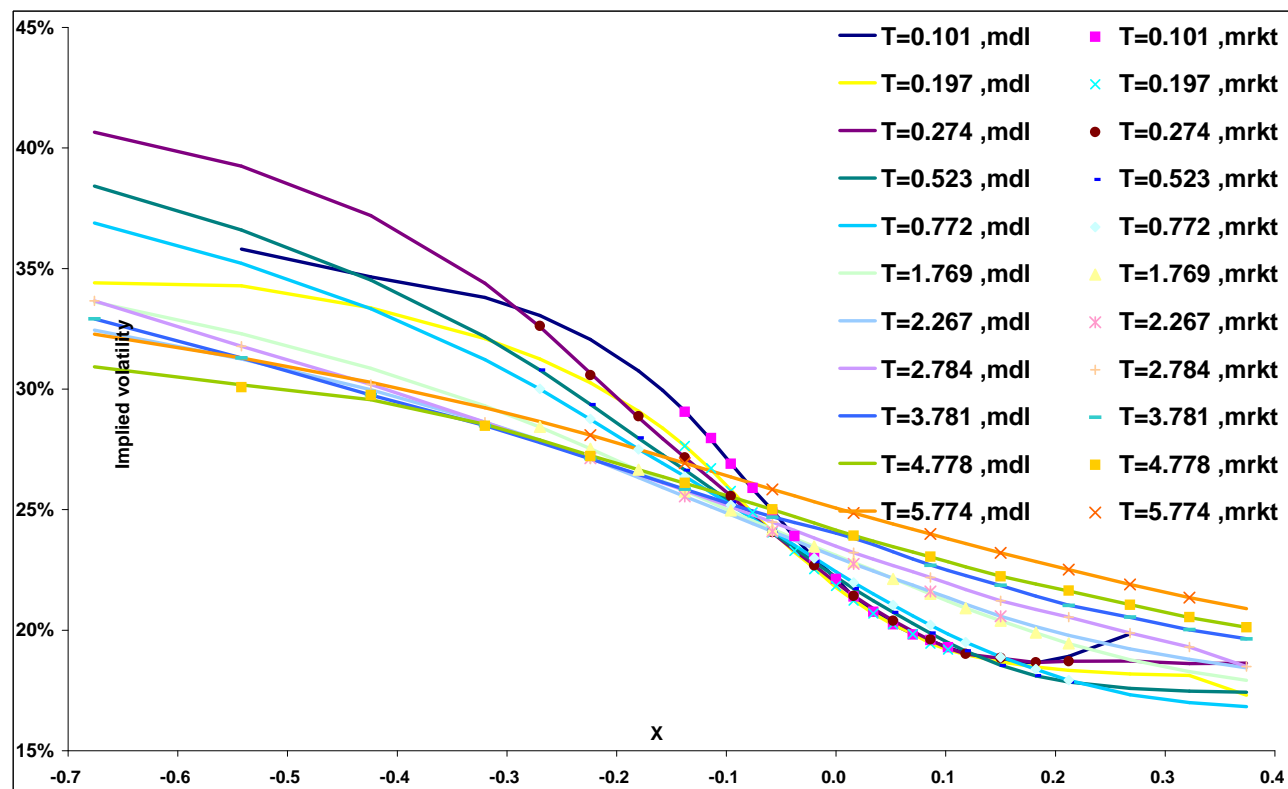
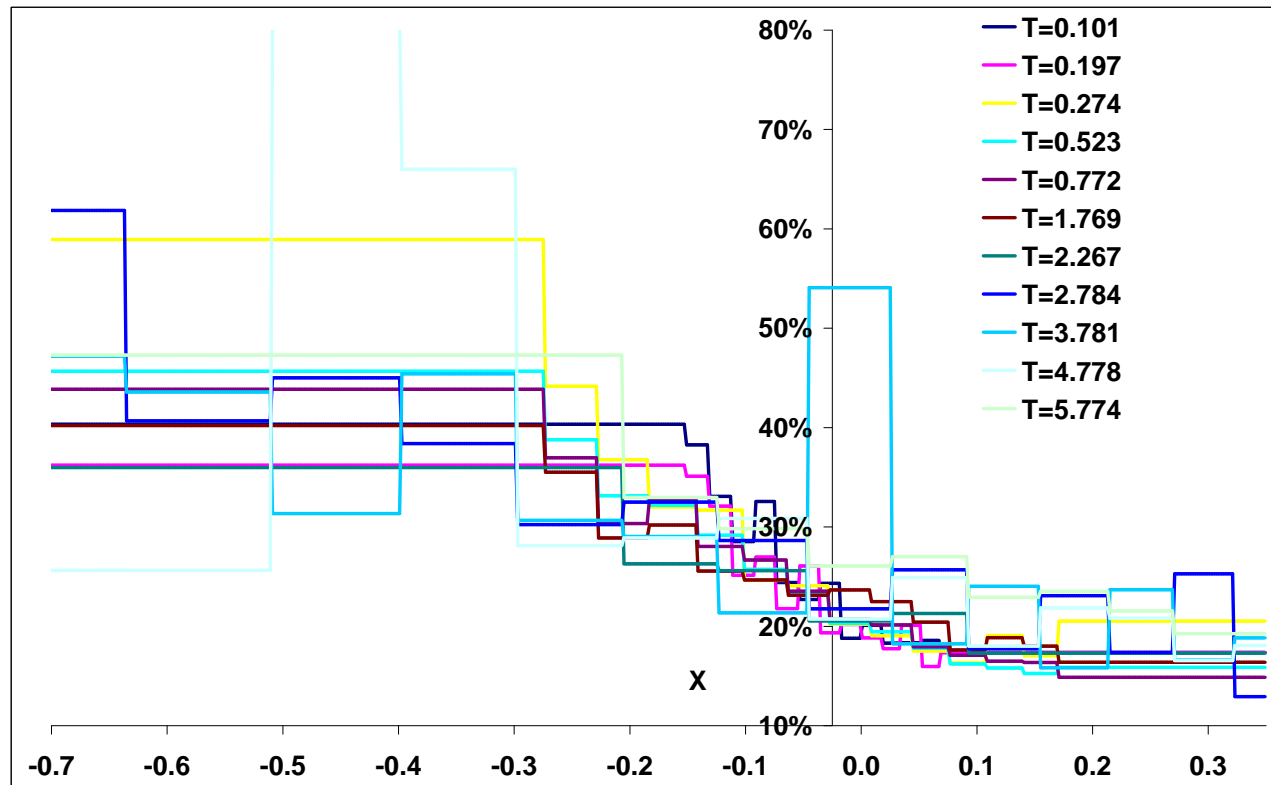Figure 1. Market and model SX5E implied volatility quotes for March 1, 2010.

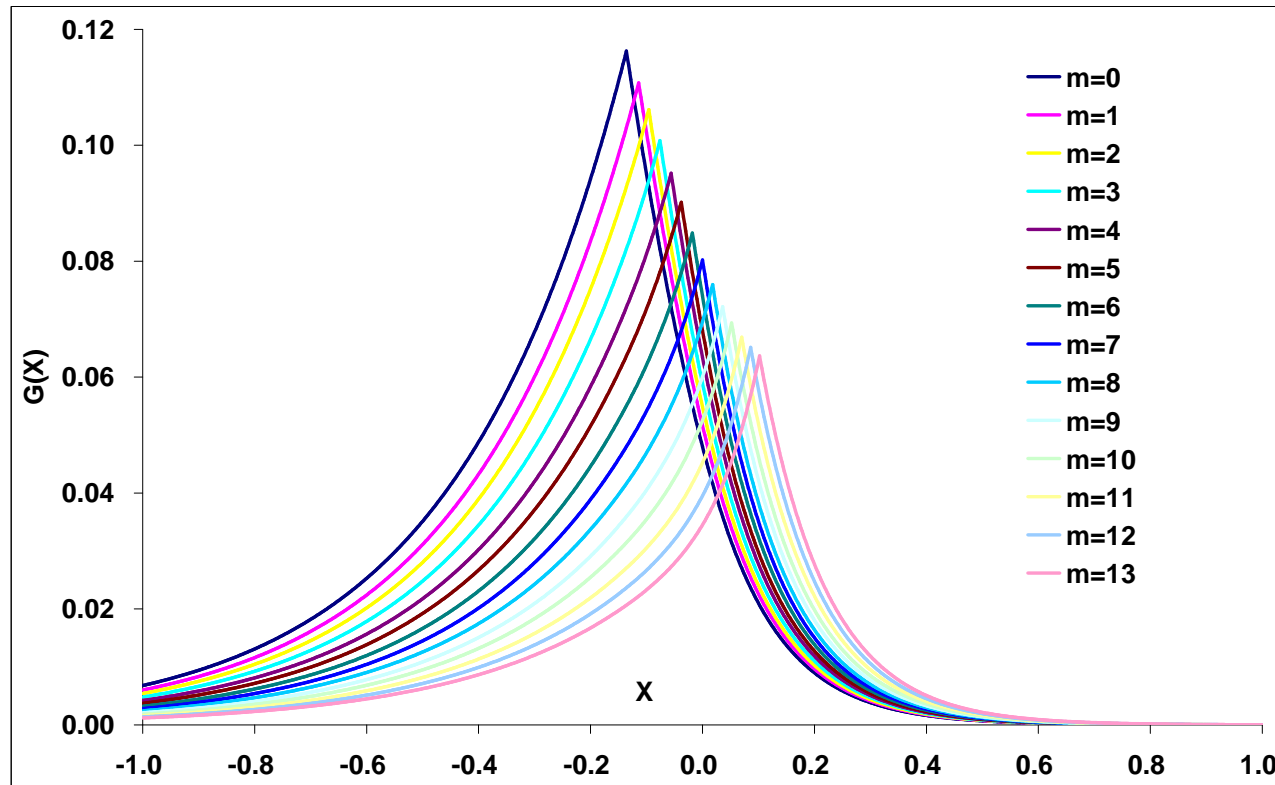Figure 2. Calibrated local volatility for March 1, 2010.

Figure 3. the Laplace transforms of the Green's functions $\widehat{G}_1\left(\lambda, X_j, X'\right)$ as functions of X', where $\lambda = 1$ and $X = X_j$, $0 \leq j \leq 13$, are given in Table 1.
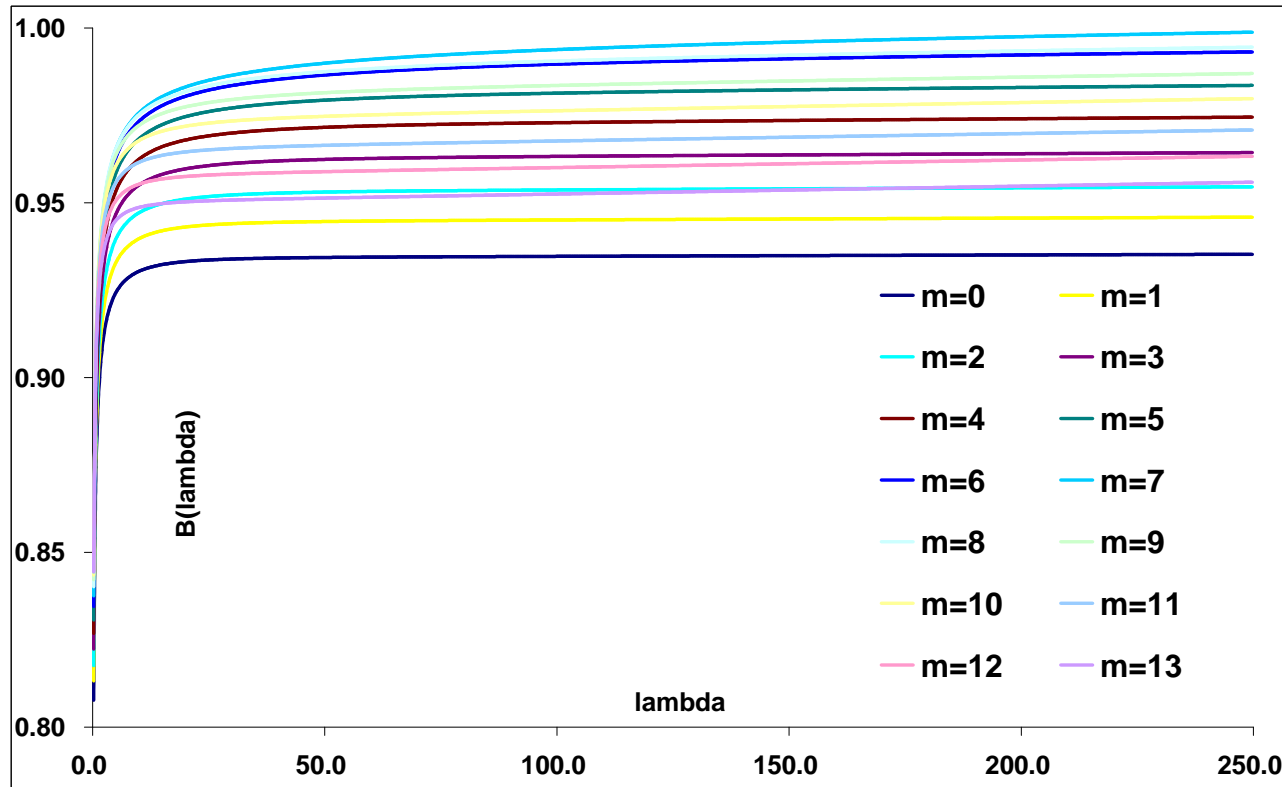
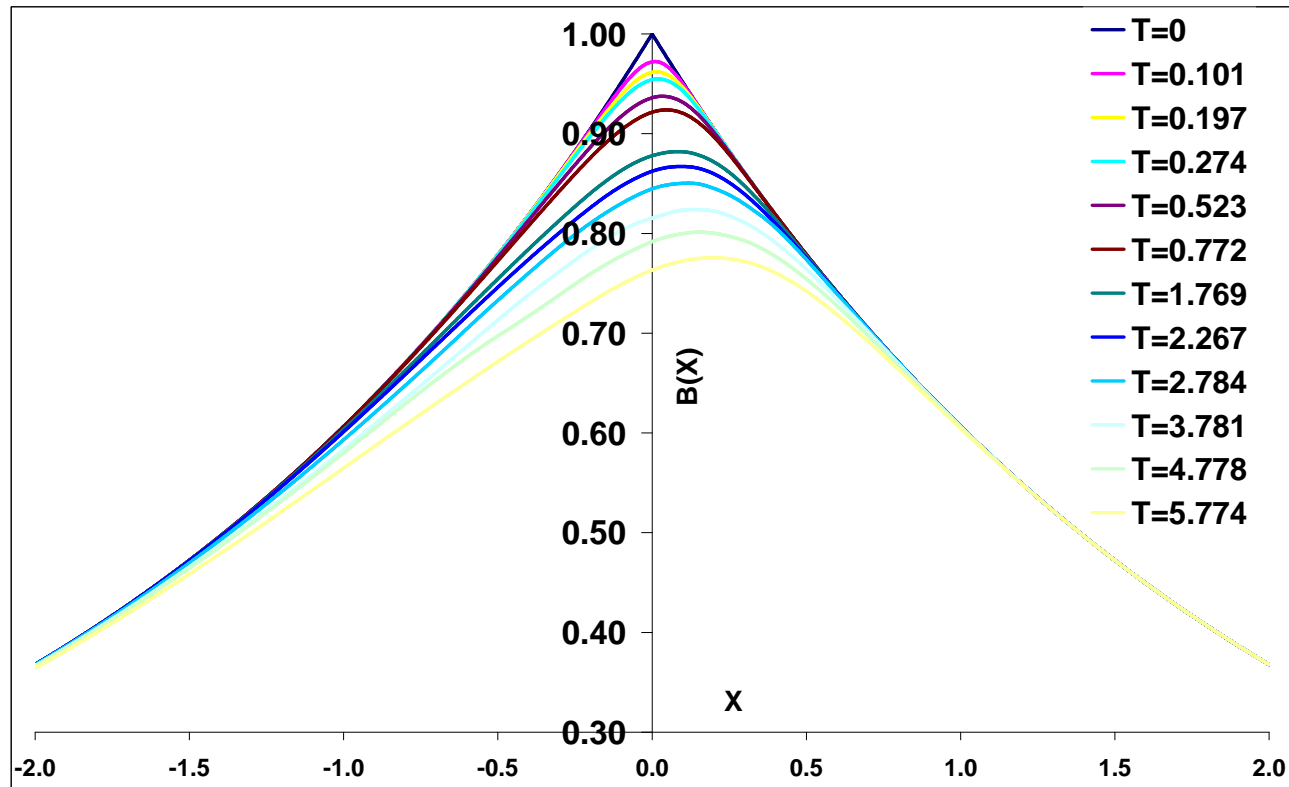Figure 4. Option prices $\hat{B}\left(\lambda, X_j\right)$ as functions of λ, where X_j are given in Table 1.

Figure 5. Option prices $B(T_i, X)$ as functions of X, where T_i are given in Table 1.

One-tile case. Exact Solution

Consider the one-tile case with $\sigma = \sigma_0$, which is the classical Black-Scholes case. In this case matrix equation (12) is trivial

$$\begin{pmatrix} -1 \\ -q_0 \end{pmatrix} \alpha_{0,+} + \begin{pmatrix} 1 \\ -q_0 \end{pmatrix} \alpha_{1,-} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad q_0 = \sqrt{\frac{2\lambda}{\sigma_0^2} + \frac{1}{4}}.$$

Accordingly, $\alpha_{0,+} = \alpha_{1,-} = 1/2q_0$.

The Green's function $\widehat{G}(\lambda)$ and the corresponding option price $\widehat{B}(\lambda)$ have the form

$$\widehat{G}\left(\lambda, X, X'\right) = \frac{e^{-q_0|X-X'|}}{q_0}, \quad \widehat{B}(\lambda, X) = e^{-\frac{|X|}{2}} - \frac{e^{-q_0|X|}}{2q_0}.$$

The inverse Carson-Laplace transform of $\widehat{B}(\lambda)$ yields $B(T)$:

$$B(T, X) = e^{-\frac{|X|}{2}} \Phi\left(\frac{|X| - \frac{\sigma_0^2 T}{2}}{\sqrt{\sigma_0^2 T}}\right) + e^{\frac{|X|}{2}} \Phi\left(-\frac{|X| + \frac{\sigma_0^2 T}{2}}{\sqrt{\sigma_0^2 T}}\right).$$

Here $\Phi(\xi)$, $\phi(\xi)$ are the cumulative density and density of the standard Gaussian variable.

One-tile case. Approximate Solution

While the above transforms can be computed in a closed form, in multi-tile case it is not possible.

Accordingly, an approximation valid for $\lambda \to \infty$ is useful:

$$q_0 \approx \frac{\zeta}{\sigma_0} + \frac{\sigma_0}{8\zeta}, \qquad e^{-q_0|X|} \approx e^{-\frac{\zeta}{\sigma_0}|X|}\left(1 - \frac{\sigma_0\,|X|}{8\zeta}\right),$$

$$\hat{B}_a\left(\lambda, X\right) \approx e^{-\frac{|X|}{2}} - \frac{\sigma_0 e^{-\frac{\zeta}{\sigma_0}|X|}}{2\zeta} + \frac{\sigma_0^2\,|X|\,e^{-\frac{\zeta}{\sigma_0}|X|}}{16\zeta^2},$$

where $\zeta = \sqrt{2\lambda}$. It is well-known that for $z < 0$

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{e^{\zeta z}}{\zeta^3}\right) &= \sqrt{T}\Psi_3\left(\frac{z}{\sqrt{T}}\right), \\
\mathcal{L}^{-1}\left(\frac{e^{\zeta z}}{\zeta^4}\right) &= T\Psi_4\left(\frac{z}{\sqrt{T}}\right),
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\Psi_3\left(\xi\right) &= \xi\Phi\left(\xi\right) + \phi\left(\xi\right), \\
\Psi_4\left(\xi\right) &= \tfrac{1}{2}\left(\left(\xi^2 + 1\right)\Phi\left(\xi\right) + \xi\phi\left(\xi\right)\right).
\end{aligned}$$

One-tile case. Approximate Solution

Equation (16) shows that the inverse Carson-Laplace transform of $\widehat{B}_a(\lambda)$ yields an approximate option price:

$$B_a(T, X) = e^{-\frac{|X|}{2}} - \sigma_0\sqrt{T}\Psi_3\left(-\frac{|X|}{\sigma_0\sqrt{T}}\right) + \frac{\sigma_0^2 T |X|}{8}\Psi_4\left(-\frac{|X|}{\sigma_0\sqrt{T}}\right). \quad (17)$$

Exact and approximate implied volatilities for several representative maturities are shown in Figure 6.

It is clear that exact implied volatility is equal to σ_0.

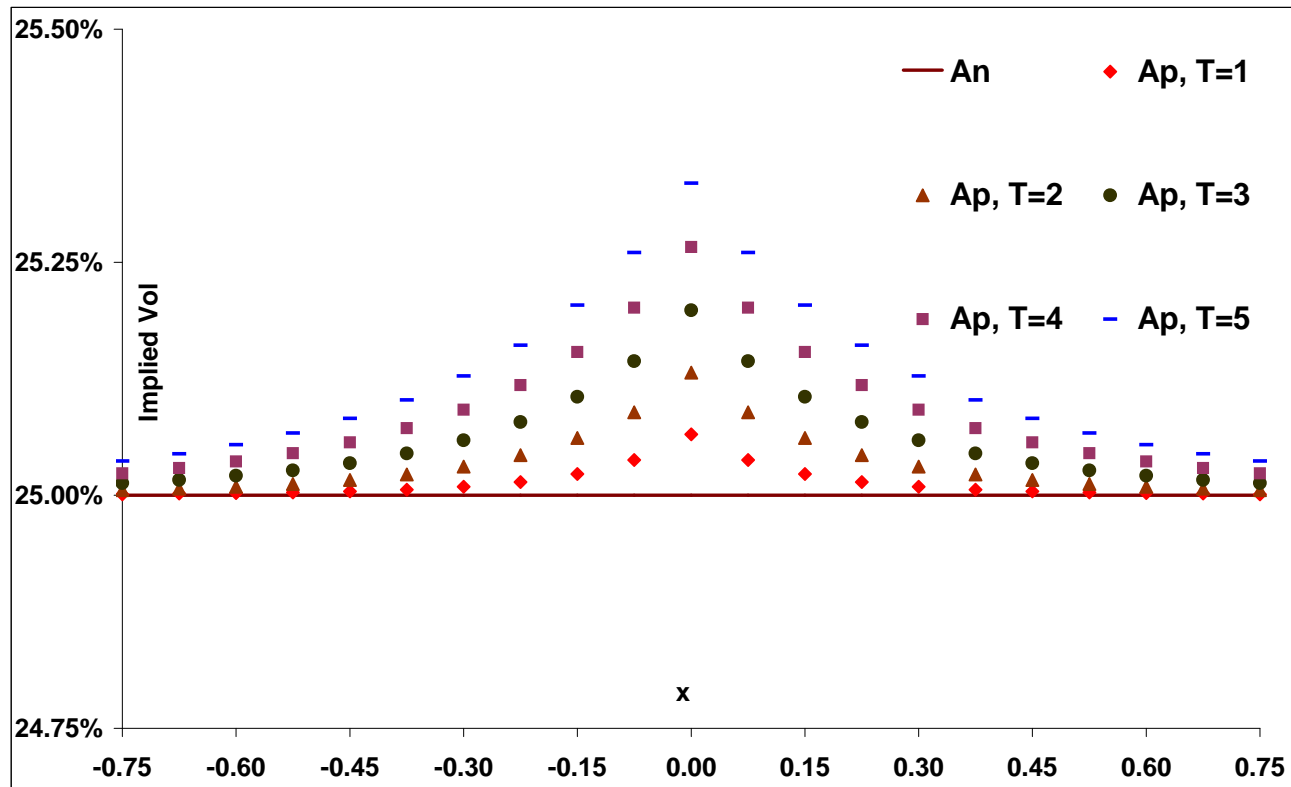This Figure shows that the above approximation is reasonably accurate provided that $\sigma_0^2 T$ is sufficiently small.

Figure 6. Implied volatility given by equation (17) with $\sigma_0 = 25\%$ vs. exact implied volatility σ_0.

Two-tile case

Consider two-tiled case:

$$\sigma(X) = \begin{cases} \sigma_0, & X \leq \bar{X}, \\ \sigma_1 & X > \bar{X}. \end{cases}$$

For concreteness, the case when $X < \bar{X}_0$ is considered. In the case in question equation (11) has the form

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ -q_0 & q_0 & -q_0 & 0 \\ 0 & -e^{q_0(\bar{X}_0-X)} & -e^{-q_0(\bar{X}_0-X)} & e^{-q_1(\bar{X}_0-X)} \\ 0 & -q_0 e^{q_0(\bar{X}_0-X)} & +q_0 e^{-q_0(\bar{X}_0-X)} & -q_1 e^{-q_1(\bar{X}_0-X)} \end{pmatrix} \begin{pmatrix} \alpha_{0,+} \\ \alpha_{1,+} \\ \alpha_{1,-} \\ \alpha_{2,-} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

so that

$$\begin{pmatrix} \alpha_{0,+} & \alpha_{1,+} & \alpha_{1,-} & \alpha_{2,-} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2q_0} + \frac{(q_0-q_1)}{2q_0(q_0+q_1)} e^{-2q_0(\bar{X}_0-X)} & \frac{(q_0-q_1)}{2q_0(q_0+q_1)} e^{-2q_0(\bar{X}_0-X)} & \frac{1}{2q_0} & \frac{1}{(q_0+q_1)} e^{(q_1-q_0)(} \end{pmatrix}$$

Two-tile case

Accordingly, when $X > \bar{X}_0$, $\widehat{G}(\lambda, X, X')$ can be written as

$$
\widehat{G}(\lambda, X, X') = \begin{cases} \frac{1}{2q_0}e^{-q_0|X'-X|} + \frac{(q_0-q_1)}{2q_0(q_0+q_1)}e^{q_0(X'+X-2\bar{X}_0)}, & X' \leq \bar{X}_0, \\ \frac{1}{(q_0+q_1)}e^{-q_1(X'-\bar{X}_0)-q_0(\bar{X}_0-X)}, & \bar{X}_0 < X'. \end{cases}
$$

By the same token, when $X > \bar{X}_0$, it can be written as

$$
\widehat{G}(\lambda, X, X') = \begin{cases} \frac{1}{(q_0+q_1)}e^{q_1(\bar{X}_0-X)+q_0(X'-\bar{X}_0)}, & X' \leq \bar{X}_0, \\ \frac{1}{2q_1}e^{-q_1|X'-X|} + \frac{(q_1-q_0)}{2q_1(q_1+q_0)}e^{-q_1(X'+X-2\bar{X}_0)}, & \bar{X}_0 < X'. \end{cases}
$$

As expected, $\widehat{G}(\lambda, X, X') = \widehat{G}(\lambda, X', X)$

It can be shown that

$$
\widehat{B}(\lambda, X) = e^{-\frac{|X|}{2}} - \frac{(q(X)+q(0)-(q_0+q_1))e^{-q(X)|X-\bar{X}_0|-q(0)|\bar{X}_0|}}{2q(0)(q_0+q_1)} - \frac{e^{-q(X)(|X|-|\bar{X}_0|)-q(0)|}}{(q(X)+q(0))}
$$

$$
(18)
$$

$$
q(X) = \sqrt{\frac{2\lambda}{\sigma^2(X)} + \frac{1}{4}} = \begin{cases} \sqrt{\frac{2\lambda}{\sigma_0^2} + \frac{1}{4}}, & X \leq \bar{X}_0, \\ \sqrt{\frac{2\lambda}{\sigma_1^2} + \frac{1}{4}}, & X > \bar{X}_0. \end{cases}
$$

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Two-tile case

When $\lambda \to \infty$, $\widehat{B}\left(\lambda, X\right)$ can be expanded as follows

$$\widehat{B}\left(\lambda, X\right) = e^{-\frac{|X|}{2}}$$

$$-\frac{\left(\left(\frac{1}{\sigma(0)}+\frac{1}{\sigma(X)}\right)-\left(\frac{1}{\sigma_0}+\frac{1}{\sigma_1}\right)\right)}{\frac{2}{\sigma(0)}\left(\frac{1}{\sigma_0}+\frac{1}{\sigma_1}\right)\zeta}e^{-\left(\frac{|X-\bar{X}_0|}{\sigma(X)}+\frac{|\bar{X}_0|}{\sigma(0)}\right)\zeta}\left(1-\frac{\sigma(X)|X-\bar{X}_0|+\sigma(0)|\bar{X}_0|}{8\zeta}\right)$$

$$-\frac{1}{\left(\frac{1}{\sigma(0)}+\frac{1}{\sigma(X)}\right)\zeta}e^{-\left(\frac{|X|-|\bar{X}_0|}{\sigma(X)}+\frac{|\bar{X}_0|}{\sigma(0)}\right)\zeta}\left(1-\frac{\sigma(X)(|X|-|\bar{X}_0|)+\sigma(0)|\bar{X}_0|}{8\zeta}\right).$$

$$\widehat{B}\left(\lambda, X\right) = e^{-\frac{|X|}{2}}$$

$$-\frac{\sigma(0)\left(\frac{\sigma_0\sigma_1(\sigma(0)+\sigma(X))}{\sigma(0)\sigma(X)}-(\sigma_0+\sigma_1)\right)}{2(\sigma_0+\sigma_1)\zeta}e^{-\left(\frac{|X-\bar{X}_0|}{\sigma(X)}+\frac{|\bar{X}_0|}{\sigma(0)}\right)\zeta}\left(1-\frac{\sigma(X)|X-\bar{X}_0|+\sigma(0)|\bar{X}_0|}{8\zeta}\right)$$

$$-\frac{\sigma(0)\sigma(X)}{(\sigma(0)+\sigma(X))\zeta}e^{-\left(\frac{|X|-|\bar{X}_0|}{\sigma(X)}+\frac{|\bar{X}_0|}{\sigma(0)}\right)\zeta}\left(1-\frac{\sigma(X)(|X|-|\bar{X}_0|)+\sigma(0)|\bar{X}_0|}{8\zeta}\right).$$

Two-tile case

The inverse Carson-Laplace transform using on equation (16) yields

$$B\left(T, X\right) = e^{-\frac{|X|}{2}} - \sigma\left(0\right)\sqrt{T}\left(\frac{\left(\sigma_0 + \sigma_1 - 2\sigma(0)\right)}{\left(\sigma_0 + \sigma_1\right)}\Psi_3\left(-\left(\frac{|X - \bar{X}_0|}{\sigma(X)\sqrt{T}} + \frac{|\bar{X}_0|}{\sigma(0)\sqrt{T}}\right)\right.\right.$$
$$\left.+\Psi_3\left(-\left(\frac{|X| - |\bar{X}_0|}{\sigma(X)\sqrt{T}} + \frac{|\bar{X}_0|}{\sigma(0)\sqrt{T}}\right)\right)\right)$$
$$+\frac{\sigma(0)\sigma(X)T}{8}\left(\frac{\left(\frac{\sigma_0\sigma_1(\sigma(0) + \sigma(X))}{\sigma(0)\sigma(X)} - (\sigma_0 + \sigma_1)\right)\left(\sigma(X)|X - \bar{X}_0| + \sigma(0)|\bar{X}_0|\right)}{(\sigma_0 + \sigma_1)\sigma(X)}\Psi_4\left(-\left(\frac{|X - \bar{X}_0|}{\sigma(X)\sqrt{T}} + \right.\right.\right.$$
$$\left.\left.\left.+\frac{2\left(\sigma(X)(|X| - |\bar{X}_0|) + \sigma(0)|\bar{X}_0|\right)}{(\sigma(0) + \sigma(X))}\Psi_4\left(-\left(\frac{|X| - |\bar{X}_0|}{\sigma(X)\sqrt{T}} + \frac{|\bar{X}_0|}{\sigma(0)\sqrt{T}}\right)\right)\right)\right).$$

$$(19)$$

Figure 7 shows the implied volatility computed using approximate formula (19) and the implied volatility computed by the exact algorithm. This formula provides adequate solution to the original problem. Note that the classical short-time approximation

$$\sigma_{imp}\left(X\right) = \frac{X}{\int_0^X \frac{d\xi}{\sigma_{loc}(\xi)}} = \frac{|X|}{\frac{|X| - |\bar{X}_0|}{\sigma(X)} + \frac{|\bar{X}_0|}{\sigma(0)}}, \qquad (20)$$

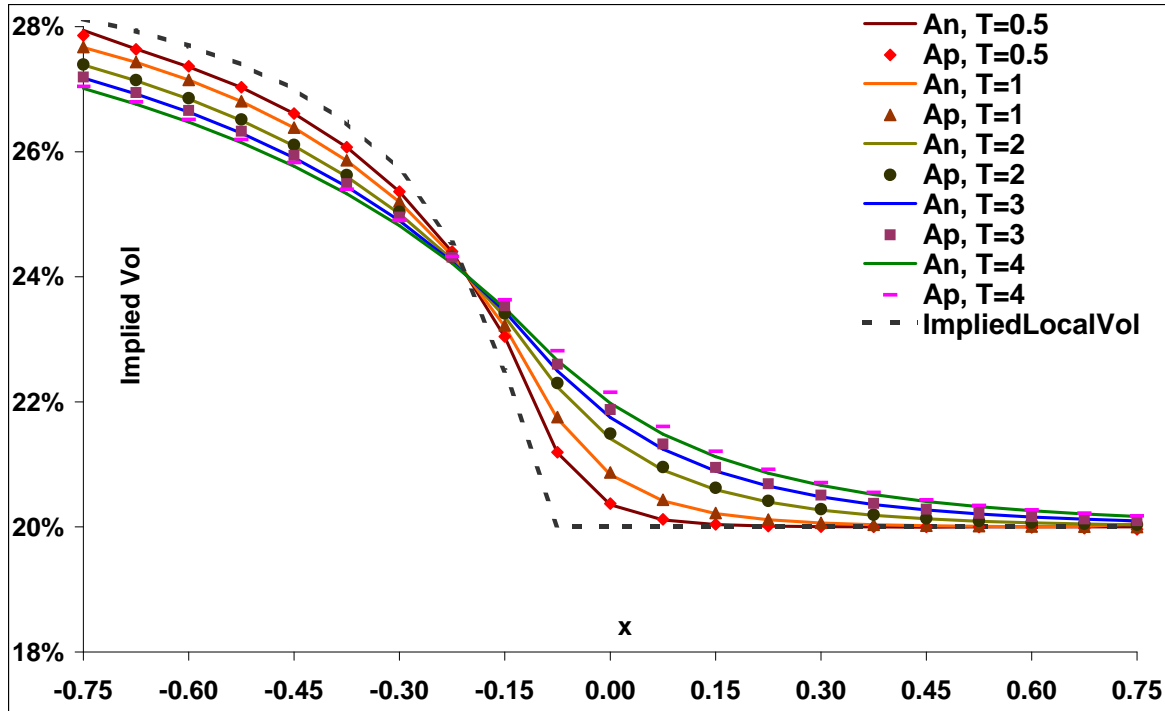is extremely inaccurate in the case under consideration.

Figure 7. Two-tile case with $\sigma_0 = 30\%$, $\sigma_1 = 20\%$, $\bar{X}_0 = -0.1$. Implied volatility computed using equation (19) vs. implied volatility computed via the exact Carson-Laplace using equation (18); for five representative maturities $T = 1, ..., 5$. For comparison, implied volatility computed via equation (20) is shown as well.

Conclusions

We have proposed a robust and exact algorithm for calibration of a tiled local volatility model to sparse market data

The efficacy of the algorithm is illustrated by showing how to apply it for calibration of a tiled local volatility model to a particular set of sparse market data

It is shown that the algorithm generates a non-arbitrageable and well-behaved implied volatility surface for options on SX5E