Unified Credit-Equity Modeling: From Single-Name to Multi-Name

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Outline

- **Single-name unified credit-equity modeling:** literature review
- From **single name to multi-name**: a class of **multi-name defaultable stock models** with **stock price-dependent jumps, stock return correlations, and default correlations**

From Structural to Reduced-Form

- **Structural models:** all corporate securities (debt, equity, CB) and derivatives (stock options, credit derivatives) are contingent claims on the **value of the firm.**
- **Drawback: not directly observable.**
- **Reduced-form credit models:** take **default intensity** as the fundamental variable.
- **Equity derivatives models:** take **positive stock price** as the fundamental variable.
- **Disconnect between equity and credit models.**

Single-Name Credit-Equity Modeling – CB Origins

- Disconnect is particularly acute for **convertible bonds (CB).**

- Reduced-form **unified credit-equity models** originated in CB literature (Davis-Lischka (2002), etc.)

- **Defaultable stock** is the fundamental state variable. Stock price evolves as diffusion. A **doubly-stochastic Poisson process with intensity a decreasing function of the stock price** is running in the background. When it jumps, the **firm defaults on its debt, and its stock price drops to zero (jump-to-default).**

- The defaultable stock framework was originally primarily confined to the CB valuation literature.

Single-Name Unified Credit-Equity Modeling

- Linetsky (2006) "Pricing Equity Derivatives subject to Bankruptcy", Math Fin:

  1) An **OTM put** has an embedded **credit derivative** (pays strike if the stock drops to zero)

  2) **OTM puts** can be used to gauge **market implied default probability.**

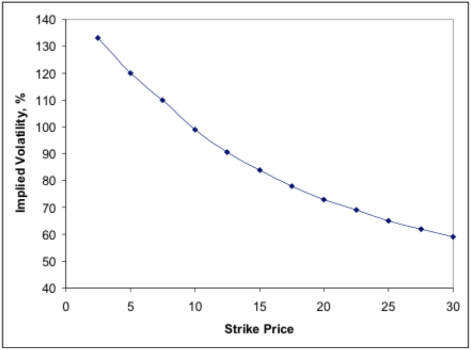
  3) Positive probability of **jump to default** substantially contributes to the **implied volatility skew.**

  4) Solved a **jump-to-default extended Black-Scholes** model with **default intensity a negative power of the stock price.**

Example

- GM price on 02.22.2006 was \$21.19.
- Historical Vol. of GM stock price over the previous 12 months $\approx 46\%$

Implied Vol. Skew for GM Jan 2007 Puts on 2.22.2006



- Total outstanding notional for Jan 07 and Jan 08 Puts with strikes \$2.50-\$10 was 130 million shares.

Single-Name Credit-Equity Modeling with Diffusions

- Carr & Linetsky (2006) *Fin & Stoch* solved **jump-to-default extended CEV (JDCEV)** with the **CEV volatility** and the **default intensity an affine function of the CEV variance.**

- Carr & Madan (2010) *SIAM J of Fin Math* estimated **jump-to-default extended local volatility diffusions** from CDS and stock options data

- Carr & Wu (2010) *J of Fin Econometrics* introduced and estimated **jump-to-default extended Heston SV model**

- Bayraktar & Yang (2011) *Math Fin* introduced and estimated **more general jump-to-default extended SV model**

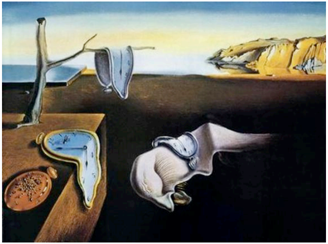
Single-Name Credit-Equity Modeling with Jumps

- Mendoza-Arriaga, Carr & Linetsky (2010) *Math Fin* introduced **pure jump** and **jump-diffusion models with jump-to-default**
- **Levy measure with the leverage effect** (**more frequent arrival of larger jumps as the stock price falls**)
- Constructed by **time changes of jump-to-default extended diffusions (JDCEV) by Levy subordinators**

From Single Name to Multi-Name

- Mendoza-Arriaga & Linetsky "**Constructing Markov Processes with Dependent Jumps by Multivariate Subordination: Applications to Multi-name Credit-Equity modeling**"

- A class of **multi-name defaultable stock models** with **stock price-dependent jumps, stock return correlations, and default correlations**



Model Architecture

- Joint risk-neutral dynamics of stock prices S_t^i of n firms under the EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}.$$

- Ingredients:
  - X_t^i: n independent 1D diffusions, such as JDCEV diffusions
  - $\mathcal{T}_t^i$: a multivariate time change (an n-dimensional Lévy subordinator) independent of X^i
  - τ_i: default times of firm i
  - ρ_i: parameters to ensure that discounted stock prices are martingales

Ingredients: n 1D Diffusions X^i

- X^i are n 1D diffusions:

$$dX_t^i = [\mu_i + k_i(X_t^i)]X_t^i\,dt + \sigma_i(X_t^i)X_t^i\,dB_t^i.$$

- Local volatilities $\sigma_i(x)$. For JDCEV:

$$\sigma_i(x) = a_i x^{\beta_i},$$

  where $a_i > 0$ are the volatility scale parameters and $\beta_i < 0$ are the volatility elasticity parameters.

- Killing rates $k_i(x)$ enter the drift to compensate for jump-to-default. For JDCEV:

$$k_i(x) = b_i + c_i\sigma_i^2(x) = b_i + c_i a_i^2 x^{2\beta_i},$$

  $b_i \geq 0$, $c_i \geq 0$.

- Constant parameters μ_i.
- Initial stock prices $X_0^i = S_0^i > 0$.

Ingredients: n–dimensional Subordinator

- An ***n*-dimensional subordinator** $\mathcal{T}_t$ is a Lévy process in $\mathbb{R}_+^n$ increasing in each of its coordinates. Let $\pi_t(d\mathbf{s}) = \mathbb{Q}(\mathcal{T}_t \in d\mathbf{s})$ denote its transition function.
- The Laplace transform:

$$\mathbb{E}[e^{-\langle \mathbf{u}, \mathcal{T}_t \rangle}] = \int_{\mathbb{R}_+^n} e^{-\langle \mathbf{u}, \mathbf{s} \rangle} \pi_t(d\mathbf{s}) = e^{-t\phi(\mathbf{u})}.$$

- The **Laplace exponent**:

$$\phi(\mathbf{u}) = \langle \gamma, \mathbf{u} \rangle + \int_{\mathbb{R}_+^n} (1 - e^{-\langle \mathbf{u}, \mathbf{s} \rangle}) \nu(d\mathbf{s}),$$

$\gamma \in \mathbb{R}_+^n$ is the drift vector with non-negative coordinates and the **Lévy measure** ν on $\mathbb{R}_+^n \backslash \{0\}$ satisfies

$$\int_{\mathbb{R}_+^n \backslash \{0\}} (\|\mathbf{s}\| \wedge 1) \nu(d\mathbf{s}) < \infty.$$

Example: Linear Factor Model

- Linear transformations of m independent 1D subordinators $\mathcal{S}_t^a$:

$$\mathcal{T}_t^i = \sum_{a=1}^{m} A_{i,a} \mathcal{S}_t^a, \quad i = 1, ..., n.$$

  Each $\mathcal{S}^a$ contributes to each $\mathcal{T}^i$ with the factor loading $A_{i,a} \geq 0$.
- Important three-parameter family of 1D Lévy measures:

$$\nu(ds) = Cs^{-\alpha-1} e^{-\eta s} ds$$

  with $C > 0$, $\eta \geq 0$, and $\alpha < 1$.
- $\alpha \in (0,1)$: tempered stable, $\alpha = 1/2$: inverse Gaussian, $\alpha = 0$: Gamma, $\alpha < 0$: compound Poisson with gamma distributed jump sizes (exponential when $\alpha = -1$).
- The Laplace exponent: $\phi(u) = -C\Gamma(-\alpha)\left((u+\eta)^\alpha - \eta^\alpha\right)$, $\alpha \neq 0$, $\phi(u) = C\ln(1 + u/\eta)$, $\alpha = 0$.

Ingredients: Default Times τ_i

- Define

$$\zeta_i := \inf\{t \geq 0 : \int_0^t k_i(X_u^i)du \geq E_i\},$$

where E_i are n independent exponential r.v. with unit mean and independent of X^i and $\mathcal{T}^i$.

- The time of default of the ith firm is defined by applying the time change $\mathcal{T}^i$:

$$\tau_i := \inf\{t \geq 0 : \mathcal{T}_t^i \geq \zeta_i\}.$$

At the time of default of the ith firm, its stock price jumps to zero and stays at zero for all subsequent times.

Ingredients: Martingale Conditions

- Each single-name stock price S^i with the dividends reinvested and discounted at the risk-free interest rate is a non-negative martingale under $\mathbb{Q}$ if and only if:
- (i) μ_i in the drift of X^i satisfies

$$\int_{[1,\infty)} e^{\mu_i s}\nu_i(ds) < \infty,$$

  where ν_i is the Lévy measure of the one-dimensional subordinator $\mathcal{T}^i$
- (ii) the constant ρ_i is:

$$\rho_i = r - q_i + \phi_i(-\mu_i),$$

  where $\phi_i(u)$ is the Laplace exponent of $\mathcal{T}^i$, $\phi_i(u) = \phi(0, ..., 0, u, 0, ..., 0)$ (u is in the ith place), $r \geq 0$ is the risk-free interest rate, and $q_i \geq 0$ is the dividend yield of the ith stock.
- Then each

$$e^{-(r-q_i)t}S_t^i = e^{\phi_i(-\mu_i)t}\mathbf{1}_{\{t<\tau_i\}}X^i_{\mathcal{T}^i_t}$$

  is a non-negative martingale under $\mathbb{Q}$.

Solution Strategy

- We are interested in pricing claims with payoffs on n firms:

$$e^{-rt}\mathbb{E}[f(S_t^1, ..., S_t^n)].$$

- This includes both multi-name equity derivatives (e.g., basket options) and credit (e.g., Nth-to-default).
- Each of the n firms may default by time t. For each stock either $S_t^i > 0$ or $S_t^i = 0$.
- By combinatorics (inclusion-exclusion principle type calculation), the expectation can be decomposed into expectations of the form

$$\mathbb{E}[\mathbf{1}_{\{\tau_\Xi > t\}} f_\Xi(S_t^{i_1}, ..., S_t^{i_k})] = \mathbb{E}[\mathbf{1}_{\{\tau_\Xi > t\}} f_\Xi(e^{\rho_{i_1} t} X^{i_1}(\mathcal{T}_t^{i_1}), ..., e^{\rho_{i_k} t} X^{i_k}(\mathcal{T}_t^{i_k})],$$

where $\Xi = \{i_1, ..., i_k\}$ is a subset of k firms, $1 \le k \le n$, τ_Ξ is the time of the first default in the subset, and f_Ξ is constructed from the payoff f.

Semigroup Structure

- The process $(X^{i_1}(\mathcal{T}_t^{i_1}), ..., X^{i_k}(\mathcal{T}_t^{i_k}))$ killed at the first time of default τ_Ξ is a k-dimensional, m-symmetric Markov process on $\mathbb{R}_+^k$ with lifetime τ_Ξ and the symmetry measure m given by the product measure of k speed measures of the underlying diffusions X^i.
- Its transition semigroup, infinitesimal generator, and their spectral representation in $L^2(\mathbb{R}_+^k, m)$ can be fully characterized.
- Rich mathematical structure involves:

1) **a multivariate extension of Phillip's Theorem on subordination of operator semigroups to multivariate subordination of multi-parameter semigroups in Banach spaces (Baeumer et al. (2008))**,
2) **a multivariate extension of the result by Okura (2002) on subordinate symmetric Markov processes and Dirichlet forms,**
3) **the Spectral Theorem for commuting families of self-adjoint operators in Hilbert spaces.**

Properties of the Subordinate Process

- If the drift vector of the subordinator is zero, the process is a **pure jump process** taking values in $\mathbb{R}_+^k$ with dependent jumps and killing.

- If the drift of the subordinator is not zero, the process is a **jump-diffusion process** on $\mathbb{R}_+^k$ with independent diffusions, dependent jumps, and killing.

Explicit Solution for JDCEV

- JDCEV generator acting on $f \in C_c^2(\mathbb{R}_+)$:

$$\mathcal{G}f(x) = \frac{1}{2}a^2 x^{2+2\beta} f''(x) + (\mu + b + ca^2 x^{2\beta})xf'(x) - (b + ca^2 x^{2\beta})f(x)$$

$$= \frac{1}{s(x)}\left(\frac{f'(x)}{m(x)}\right)' - (b + ca^2 x^{2\beta})f(x)$$

with scale and speed densities $s(x)$ and $m(x)$ (for $mu + b < 0$)

$$m(x) = \frac{2}{a^2} x^{2c-2-2\beta} e^{-Ax^{-2\beta}}, \quad s(x) = x^{-2c} e^{Ax^{-2\beta}}, \quad A = |\mu + b|/(a^2|\beta|),$$

- Self-adjoint extension in $L^2(\mathbb{R}_+, m)$ with Dirichlet boundary condition at zero has a purely discrete spectrum with eigenvalues and eigenfunctions (L_n^ν generalized Laguerre polynomials):

$$\mathcal{G}\varphi_n = -\lambda_n \varphi_n, \quad n = 1, 2, ...,$$

$$\lambda_n = 2|\beta(\mu + b)|(n-1) + |\mu|, \;\; \varphi_n(x) = A^{\frac{\nu}{2}} \sqrt{\frac{(n-1)!|\mu + b|}{\Gamma(\nu + n)}}\, x\, L_{n-1}^{\nu}(Ax^{-2\beta}).$$

Multi-variate Subordination

- The eigenfunction expansion for the JDCEV semigroup in $L^2(\mathbb{R}_+, m)$:

$$\mathcal{P}_t f(x) = \sum_{n=1}^{\infty} e^{-\lambda_n t} c_n \varphi_n(x), \quad c_n = (f, \varphi_n)_m.$$

- Now let X^i be k JDCEV diffusions. The subordinate semigroup $\mathcal{P}_t^\phi$ in $L^2(\mathbb{R}_+^k, m)$, where $m(d\mathbf{x}) = m_1(dx_1)...m_k(dx_k)$ is the product measure, has the eigenfunction expansion:

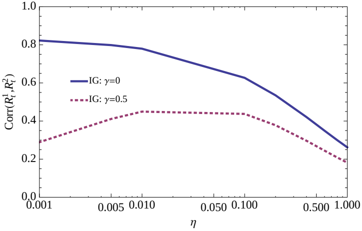
$$\mathcal{P}_t^\phi f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{N}^k} e^{-\phi(\lambda^1_{n_1},\dots,\lambda^k_{n_k})t} c^f_{\mathbf{n}} \varphi_{\mathbf{n}}(\mathbf{x}), \quad c^f_{\mathbf{n}} = (f, \varphi_{\mathbf{n}})_m,$$

  where $\mathbf{x} = (x_1, ..., x_k) \in \mathbb{R}_+^k$, $\mathbf{n} = (n_1, ..., n_k)$, $\varphi_{\mathbf{n}}(\mathbf{x}) = \varphi^1_{n_1}(x_1)...\varphi^k_{n_k}(x_k)$, λ^i_n and $\varphi^i_n(x)$ are the eigenvalues and eigenfunctions of the ith JDCEV, and $\phi(u_1, ..., u_k)$ is the Laplace exponent of the k-dimensional subordinator.
- Expansion coefficients can be calculated in closed form for equity basket options, joint survival probabilities, and Nth-to-default swaps.

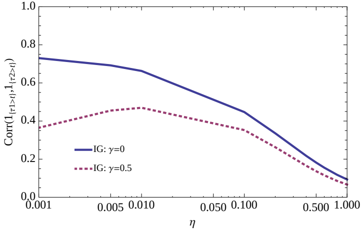
Simulation

- When the number of firms is large, the curse of dimensionality makes computation of the formulas infeasible, but the modeling architecture is *highly* amenable to Monte Carlo simulation.

- To simulate the distribution of the time changed process at time t, simulate the subordinator $(\mathcal{T}_t^1, ..., \mathcal{T}_t^n)$, and sample $(X^1(\mathcal{T}_t^1), ..., X^n(\mathcal{T}_t^n))$.



(a) Correlation of stock returns.

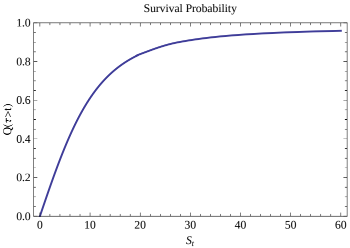


(b) Default correlation.

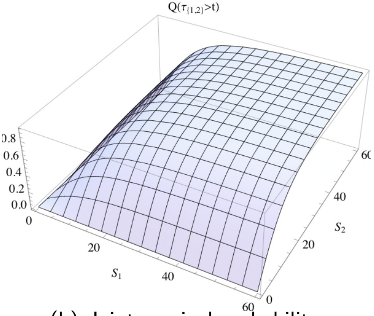
Figure: *One year correlation of stock returns,* $\mathrm{Corr}(R_t^1, R_t^2)$*, and default correlation,* $\mathrm{Corr}(\mathbf{1}_{\{\tau_1>t\}}, \mathbf{1}_{\{\tau_2>t\}})$*, levels induced by different specifications of a single common subordinator* $\mathcal{S}_t$*.*

The decay parameter η *varies according to* $\eta \in \{0.001, 0.005, 0.01, 0.1, 0.2, 0.4, 0.5, 0.6, 0.8, 1\}$*.*

The scale factor C *is chosen such that for each specification,* $\mathbb{E}[\mathcal{S}_1] = 1.5$*.*

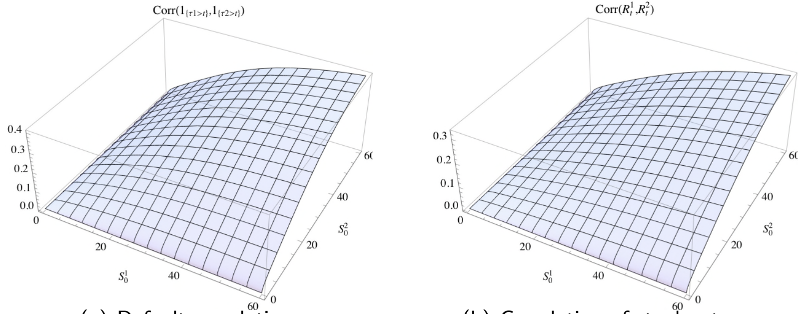


(a) Single-name survival probability.



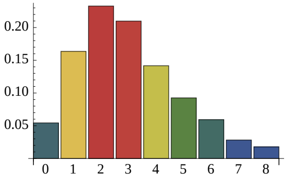
(b) Joint survival probability.

Figure: *Single-name survival probability $\mathbb{Q}(\tau > t)$, joint survival probability $\mathbb{Q}(\tau_{\{1,2\}} > t)$; for $t = 1$ year as functions of stock prices S_0^1 and S_0^2.*

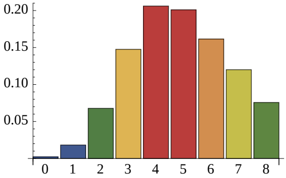


(a) Default correlation. (b) Correlation of stock returns.

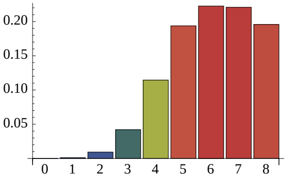
Figure: *Default correlation* $\mathrm{Corr}(\mathbf{1}_{\{\tau_1>t\}}, \mathbf{1}_{\{\tau_2>t\}})$*, and correlation of stock returns* $\mathrm{Corr}(R_t^1, R_t^2)$*; for* $t = 1$ *year as functions of stock prices* S_0^1 *and* S_0^2*.*



(a) $\mathbb{P}[N_t = N]$ by time $t = 3$.



(b) $\mathbb{P}[N_t = N]$ by time $t = 5$.



(c) $\mathbb{P}[N_t = N]$ by time $t = 7$.

Figure: *Probability of the number of defaults by time* $t = 3, 5, 7$.

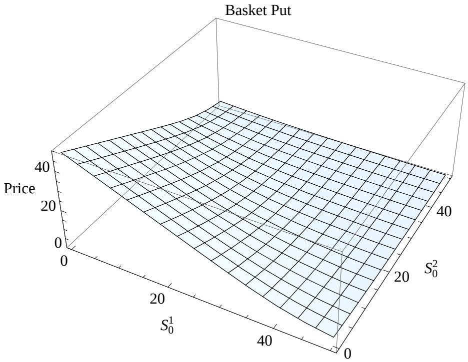


Figure: *Two-name basket put prices for the range of initial stock prices* S_0^1 *and* S_0^2 *from zero to \$50 for one year time to maturity and* $K = 50$.

n	$BPut(S_0^1, ..., S_0^d)$	Std. Error
2	6.402	0.0368
10	4.341	0.0236
20	3.574	0.0206

Table: *This is the result of MC simulation for a basket put option prices using* 10^5 *samples. When* $n = 2$*, the price corresponds to the basket put option with* $S_0^1 = S_0^2 = 25$*. The exact value for this option obtained by eigenfunction expansions is* $BPut(25, 25) = 6.4346$

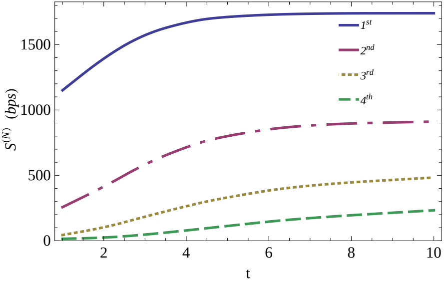


Figure: *Fair Nth-to-default swap rate $\mathcal{S}^{(N)}$ (bps) as a function of maturity time t (yrs).*

Conclusion

- Moving from **single-name** to **multi-name unified credit-equity modeling**, one can study
  - **Default correlation vs. equity correlation**
  - Consistent treatment of **multi-name equity derivatives** and **multi-name credit derivatives.**