Some Approaches to Modeling Wrong-Way Risk in Counterparty Credit Risk Management and CVA

Alex Levin, Leon Shegalov

alex.levin@rbccm.com, leon.shegalov@rbccm.com

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  - **Pricing of CCR: CVA with WWR and Conditional EPE (CEPE)**
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* The views expressed in the presentation are of the authors only and not necessarily of the Royal Bank of Canada

Pricing and Measurement of Counterparty Credit Risk (CCR) with Wrong-Way Risk (WWR) / Right-Way Risk (RWR)

The recent credit crisis has demonstrated the need to capture Wrong-Way Risk (WWR) in the Counterparty Credit Risk Management and pricing. One of the regulatory requirements in Basel II and Basel III concerning the counterparty credit risk is the ability of financial institutions to capture and manage WWR.

- **General Wrong-Way Risk** is defined in BIS (2006) as the risk when "the probability of default of counterparties is positively correlated with general market risk factors"; or in BIS (2010) as the risk "where the exposure increases when the credit quality of the counterparty deteriorates". A so-called Right-Way Risk (RWR) is opposite to the WWR. RWR represents the case when the exposure to the counterparty is negatively correlated with the counterparty's default probability.
- **Specific Wrong-Way Risk** is defined in BIS (2006) as the risk "when the exposure to a particular counterparty is positively correlated with the probability of default of the counterparty due to the nature of the transactions with the counterparty". An example of Specific WWR is a put option on the counterparty's own stock.

Naturally, financial institutions should be rewarded (in terms of CVA, CCR measures and counterparty credit capital) for doing Right-Way Risk business, and penalized for doing Wrong-Way Risk business.

Pricing of the counterparty credit risk, i.e., calculation of the Credit Value Adjustment (CVA) and Bilateral CVA (BCVA) should be performed in the risk neutral measure.

The counterparty credit risk measures (for example, Potential Future Exposure (PFE)) are usually calculated by Risk Management in the historical measure based on the parameters estimated from the historical data. As usually, change of measure from risk neutral to historical can be performed within presented here reduced form framework by the corresponding change of drifts in stochastic processes for market variables and hazard rates. Historical parameter estimation for CCR requires very long time series for the risk factors, cannot account for possible future economic regime changes, and usually has insufficient accuracy, especially in the long-term drift prediction. On the other hand, regulators allow for calculation of the CCR exposures in both risk-neutral measure (i.e., based on the market implied data) and historical measure (i.e., based on the historical data including the data for stress periods) (see BIS (2010), paragraph 98).

For simplicity of exposition and possibility to compare CVA numbers with the CCR measures, we consider all stochastic processes for both CVA and CCR under the risk-neutral measure.

Pricing of CCR: CVA with WWR and Conditional EPE (CEPE)

We refer to the investor and counterparty by index "0" and "1". Let T be the maturity of the portfolio. We denote by τ_0 and τ_1 the default times of the investor and counterparty, and by $D(t,s)$ the discount factor at time t for maturity s. Stochastic dynamics of all processes is considered in the risk-neutral measure assuming standard no-arbitrage conditions. As we are interested in the default and market factor simulation model, for simplicity, we will consider only the case of uncollateralized counterparties.

The price of credit risk with WWR/RWR is defined by the following quantities:

- **Credit Valuation Adjustment (CVA)**

(1) $$CVA(t_0) = LGD_1 \mathrm{E}_{t_0}\left\{ \mathbf{1}_{t_0<\tau_1\le T} D(t_0,\tau_1) NPV^+(\tau_1) \right\}$$

- **Debit Value Adjustment (DVA)**

(2) $$DVA(t_0) = LGD_0 \mathrm{E}_{t_0}\left\{ \mathbf{1}_{t_0<\tau_0\le T} D(t_0,\tau_0) (-NPV)^+(\tau_0) \right\}$$

- **Bilateral CVA**

(3) $$\begin{aligned} BCVA(t_0) = {} & LGD_1 \mathrm{E}_{t_0}\left\{ \mathbf{1}_{t_0<\tau_1\le T} \mathbf{1}_{\tau_1<\tau_0} D(t_0,\tau_1) NPV^+(\tau_1) \right\} \\ & - LGD_0 \mathrm{E}_{t_0}\left\{ \mathbf{1}_{t_0<\tau_0\le T} \mathbf{1}_{\tau_0<\tau_1} D(t_0,\tau_0) (-NPV)^+(\tau_0) \right\} \end{aligned}$$

The expectations in the above expressions are taken over the joint distribution of the correlated market and credit factors. This allows for modeling WWR/RWR.

A default risk measure closely related to CVA is **Expected Positive Exposure (EPE)**.

In the case of independent market and credit factors, the EPE at time t_0 for the tenor t is defined as:

$$EPE_0(t) = \mathrm{E}_{t_0}\left\{NPV^+(t)\right\} = \int_{\bar{X}^M} NPV^+(\bar{X}^M(t))\, g_1(\bar{X}^M(t))\, d\bar{X}^M \tag{4}$$

where the expectation is taken over the distribution of the market factors $\bar{X}^M$ only.

For independent market and credit factors, CVA can be expressed via EPE as:

$$CVA = LGD_1 \int_{t_0}^{T} D(t_0, s)\, EPE_0(s)\, f_1(s) ds \tag{5}$$

where counterparty's default probability density $f_1(t)$ is calculated from the survival probability $S_1(t)$ as $f_1(t) = -S_1'(t)$. Survival probability $S_1(t)$ is usually bootstrapped from the CDS spread term structure by a standard procedure (see JP Morgan (2001)).

Standard EPE (4) and the corresponding CVA (5) do not require a joint simulation of the market factors and hazard rates, but they do not capture WWR/RWR.

An extension of the standard EPE (4) that accounts for the WWR is a so-called **Conditional EPE (CEPE)**. CEPE was considered in Redon (2006) in regards to modeling of WWR for Sovereign Risk (see also earlier works of Levy (1999) and Finger (2000)). Merton's framework was utilized in these papers for modeling CEPE.

In the most general case, when the market factors (including the discount factor through the interest rate factors) and credit factors are dependent, the CEPE at time t_0 for the tenor t can be defined as the expected exposure conditional on the counterparty's default:

$$CEPE_0(t) = D_0(t_0,t)^{-1} E_{t_0}\left\{\left(D(t_0,\tau_1)\, NPV^+(\tau_1)\right) \middle| \, \tau_1 = t\right\}$$

(6) $$= D_0(t_0,t)^{-1} \iint_{\overline{X}^M \times \overline{X}^h} \left\{D(\overline{X}^M(t)) NPV^+(\overline{X}^M(t))\right\} \mathbf{1}_{\tau=t}\; g_1(\overline{X}^M(t), \overline{X}^h(t))\, d\overline{X}^M d\overline{X}^h$$

Here, $g_1(\overline{X}^M(t), \overline{X}^h(t))$ is a joint PDF of the market factors $\overline{X}^M$ and credit factors $\overline{X}^h$, and $D_0(t_0,t)$ is a given initial discount factor term structure.

The CVA with WWR/RWR (1) can be expressed via CEPE as:

(7) $$CVA = LGD_1 \int_{t_0}^{T} D_0(t_0,s)\; CEPE_0(s)\, f_1(s) ds$$

We propose the use of the Conditional EPE and the corresponding Effective Conditional EPE (that naturally include WWR/RWR) instead of standard EPE and Effective EPE in calculation of the Basel III Counterparty Credit Risk capital. This will reward RWR business and penalize WWR business.

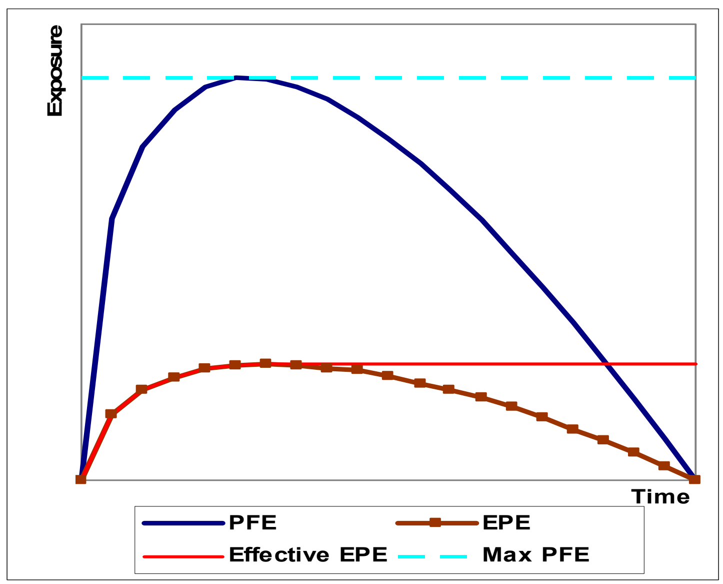
CCR measure: Conditional Potential Future Exposure (CPFE) with WWR

The most popular risk measure in Financial Industry for estimating the Counterparty Credit Risk and monitoring credit limits is **Potential Future Exposure (PFE)**.

The Potential Future Exposure profile $PFE(t)$ is the maximum amount of exposure $NPV^{+}(t)$ expected to occur on the future date t with a given degree of statistical confidence α (usually, $\alpha = 95\%$). In other words, $PFE(t)$ is a α-percentile of the exposure distribution:

(8) $$PFE(t) := q\left(\alpha, NPV^{+}(t)\right)$$

The **Maximum (Peak) Potential Future Exposure** is the maximum of the $PFE(t)$ over the life of the portfolio.



The standard PFE (8) requires simulation only of the market factors, but it does not capture WWR/RWR.

Similarly to extension of EPE to Conditional EPE, we propose to extend a standard PFE (8) to a new CCR measure that accounts for the WWR/RWR, which we call a **Conditional PFE (CPFE)**.

A CPFE(t) profile at time t_0 for the tenor t is a α-percentile of the exposure distribution conditional on the counterparty's default:

$$CPFE(t) = q\left(\alpha,\, NPV^{+}(\tau_1) \middle|\, \tau_1 = t\right) \tag{9}$$

Calculation of CPFE requires joint simulation of the correlated market and credit factors.

It is the usual practice of Credit Risk Departments not to include discounting in the PFE profile (as in (8) and (9)). However, to be more consistent with the definition of CVA (1) and CEPE (6), we modify the formula (9) for CPFE as†:

$$CPFE(t) = q\left(\alpha,\, D_0(t_0,t)^{-1}\left(D(t_0,\tau_1)\, NPV^{+}(\tau_1)\right) \middle|\, \tau_1 = t\right) \tag{10}$$

where $D_0(t_0,t)$ is the initial discount factor term structure.

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†Authors thank Terry Demopoulos of RBC QRA for this idea

Counterparty Credit Economic Capital (CCEC)-like measure with WWR

A drawback of the Standard PFE and introduced Conditional PFE is absence of the counterparty's credit quality in their definitions. Current practice of Credit Risk Departments is to factor in the counterparty's credit rating in addition to the PFE profile. This approach does not properly relate the full term structure of the counterparty's default probability with the PFE profile. To the contrary, CVA accounts for the full default probability term structure.

For this reason, we propose to complement the CPFE for each counterparty with a new CCR measure - **Counterparty Credit Economic Capital** (**CCEC**). The proposed CCEC is a quantile of a full counterparty's credit loss distribution (while CVA is the expected value of the discounted counterparty's credit loss distribution). CCEC is similar to a Portfolio Credit Economic Capital, but it is calculated separately for each counterparty:

$$CCEC := q_{t\in[t_0,T]}\left(\alpha, \mathbf{1}_{t=\tau_1} NPV^{+}(t)\right) \tag{11}$$

CCEC (11) includes WWR/RWR for correlated market and credit factors and a full default probability term structure of the counterparty.

Unified multifactor Gaussian with jumps default intensity framework for CVA, CPFE and CCEC with WWR and credit rating transitions

In the reduced form (intensity) framework, time of the counterparty's default τ is modeled by the first jump of the Cox process (see Lando (1998), Duffie and Singleton (1999)). Equivalently, the default occurs when the integrated default intensity hits for the first time the independent exponential random boundary. Default intensity $\lambda(t)$ is usually chosen as positive stochastic process (e.g., CIR process or CIR + Exponential jumps, see Brigo and Pallavicini (2008)). For **positive affine jump- diffusion default intensities**, the survival probability of the counterparty can be found in a closed form (see Duffie and Singleton (1999), Duffie, Pan and Singleton (2000)).

Gaussian intensity approach was considered in Schönbucher (2003) and other publications. However, **negative** default intensities lead to non-monotonic integrated intensities and cause the lack of affinity and analytical tractability. Practically, this approach was abandoned by researchers. In this presentation, we consider Monte Carlo simulation approach for **Gaussian mean-reverting Ornstein-Uhlenbeck (OU) intensity with Poisson jumps of arbitrary sign** framework with Wrong-Way Risk, develop an effective numerical calibration procedure for fitting the drift of $\lambda(t)$ into the observed CDS spreads, and extend the model by consistent with the hazard rate dynamics simple (CreditMetrics-type) simulation of the credit rating transitions for modeling credit triggers in the case of collaterized counterparties (see Yi (2011) for credit triggers).

Gaussian hazard rate model for one name

Similar to the Hull-White short rate model (see Brigo and Mercurio (2006)), we consider an "additive" form of the OU process for possibly negative default intensity $\lambda(t)$:

(12) $$\lambda(t) = \varphi(t) + X^h(t), \qquad t \in [t_0, T]$$

where $\varphi(t)$ is a deterministic function subject to fitting into the initial term structure of the survival probability bootstrapped from the CDS spreads at time zero, and $X^h(t)$ is a homogeneous OU process

(13) $$\mathrm{d}X^h(t) = -\kappa X^h(t) + \sigma dW, \qquad X^h(t_0) = 0$$

Here, $W(t)$ is a standard Wiener process in the risk neutral measure that can be correlated with market variables in the case of Wrong/Right-Way Risk (e.g., with Hull-White interest rates and log-normal FX rates in Amin and Jarrow (1991), equity indices, commodity prices; etc.) and default intensities of other names.

We define the Gaussian integrated stochastic intensity $\Lambda(t)$ and its maximum $M(t)$ as:

(14) $$\Lambda(t) = \int_0^t \lambda(s)ds = \int_0^t \varphi(s)ds + \int_0^t X^h(s)ds = \Phi(t) + I^h(t), \qquad M(t) = \max_{s\in[t_0,t]} \{\Lambda(s)\}$$

If the default intensity was a positive stochastic process, then the counterparty's survival (default) probability $S(t)$ ($P(t)$) would be expressed by a well-known Lando's formula:

(15) $$S(t) = 1 - P(t) = \mathrm{E}_{t_0}\left\{e^{-\Lambda(t)}\right\}$$

In the case of non-positive intensity (e.g., Gaussian), the survival probability is expressed via $M(t) = \max\{\Lambda(s)\}$ rather than $\Lambda(t)$ (Jeanblanc et. al. (2009), p. 420):

(16) $$S(t) = \mathrm{E}_{t_0}\left\{e^{-\max\limits_{s\in[t_0,t]}\{\Lambda(s)\}}\right\} = \mathrm{E}_{t_0}\left\{e^{-M(t)}\right\} = \int_0^{+\infty} e^{-y} f_M(t,y)\,dy$$

where $f_M(t,y)$ is a density of $M(t)$ at time t.

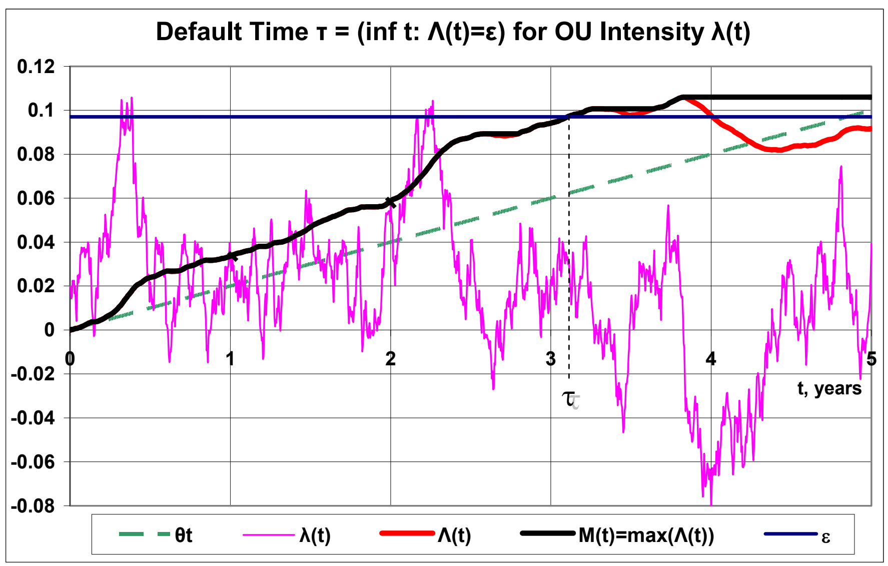


Fig. 1. Paths of OU and Integrated OU intensity hitting the exponential barrier

In our methodology for modeling WWR, we will strictly use Lando's approach, i.e., directly simulate the default times as hitting times of the exponential barrier y by $\Lambda(t)$. We will use the following connection between the default and hitting time densities.

<u>Lemma 1</u>. Let T_y be the hitting time of the boundary $y > 0$ by $\Lambda(t)$ and $f_{T_y}(t, y)$ be its density. Under some regularity conditions, the default time density

$$f_d(t) = P^{'}(t) = -S^{'}(t) \tag{17}$$

is given by the Laplace transform evaluated at 1 of the hitting time density $f_{T_y}(t, y)$ with respect to the barrier level y:

$$f_d(t) = \int_0^{+\infty} e^{-y} f_{T_y}(t, y)\, dy \tag{18}$$

■

Unfortunately, a closed analytical formula for the hitting time density for the non-Markovian Integrated OU process is unknown. A solution of such problem is even unknown in the case of an arbitrary drift $\varphi(t)$ for much simpler (Markovian) drifted Wiener process!

Therefore, there is a need for developing an effective numerical method for calculation of the drift $\varphi(t)$ in default intensity from the initial term structure of CDS spreads.

A model problem with "white noise" default intensity

As illustration of the problem, let us consider a model Cox process with a drifted "white noise" default intensity $\lambda(t) = dW_t + \varphi(t)$, i.e.

(19) $$\Lambda(t) = \int_0^t \lambda(s)ds = W_t + \Phi(t), \qquad \Phi(0) = 0$$

As we see later, the behavior for small time t of the Integrated OU intensity (14) and the corresponding function $\varphi(t)$ is very different from the behavior of Wiener integrated intensity (19) and its drift $\varphi(t)$. However, for large t, both integrated intensities have variances proportional to t, and the shapes of both drifts $\varphi(t)$ are similar.

If $\Phi(t) = \nu t$ is linear, the hitting time density is given by the Bachelier-Lévy formula:

(20) $$f_{T_y}(t, y) = \frac{y}{\sqrt{2\pi t^3}} \exp\left[-\frac{(y - \nu t)^2}{2t} \right], \qquad y > 0$$

From Lemma 1, the corresponding default time density $f_d(t)$ is:

(21) $$f_d(t, y) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{\nu^2 t}{2} \right] + (\nu - 1)\exp\left[\frac{1 - 2\nu}{2} t \right] \mathsf{N}\left((\nu - 1)\sqrt{t} \right)$$

This density can also be directly calculated from the well-known CDF of the maximum of a Wiener process with linear drift (see, for example, Jeanblanc et. al. (2009)).

In general case, when the survival probability term structure $S(t)$ is given from the market (i.e., the default time density $f_d(t)$ is known), the corresponding non-linear function $\Phi(t)$ can be found by the following effective numerical method.

<u>**Proposition 1**</u>. The function $\Phi(t)$ solves the following Volterra-type equation:

$$f_d(t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{\Phi^2(t)}{2t}} + (\Phi'(t)-1)e^{\frac{t}{2}-\Phi(t)} \mathrm{N}\left(\frac{\Phi(t)-t}{\sqrt{t}}\right)$$

(22)
$$+\int_0^t \left[\frac{\Phi(t)-\Phi(s)}{t-s} - \Phi'(t)\right] \frac{f_d(s)}{\sqrt{2\pi(t-s)}} e^{-\frac{(\Phi(t)-\Phi(s))^2}{2(t-s)}} ds, \qquad \Phi(0)=0$$

Proof: A Volterra integral equation for the hitting time density of a non-linear boundary $a(t)$ by a Gaussian Markovian stochastic process was derived in Durbin (1985). Specifically, given the value $y>0$ of the exponential random variable ε, the hitting time density $f_{T_y}(t,y)$ of the boundary $a(t)=y-\Phi(t)$ by the driftless Wiener process $W(t)$ is the solution of the following Volterra integral equation:

(23)
$$f_{T_y}(t,y) = \frac{1}{\sqrt{2\pi t^3}} e^{-\frac{(\Phi(t)-y)^2}{2t}} \left(y - \Phi(t) + t\Phi'(t)\right)$$
$$+ \int_0^t \left[\frac{\Phi(t)-\Phi(s)}{t-s} - \Phi'(t) \right] \frac{f_{T_y}(s,y)}{\sqrt{2\pi(t-s)}} e^{-\frac{(\Phi(t)-\Phi(s))^2}{2(t-s)}} ds$$

From Lemma 1, by taking the Laplace transform with respect to y of the both sides of this equation (23), one obtains the required equation (22).

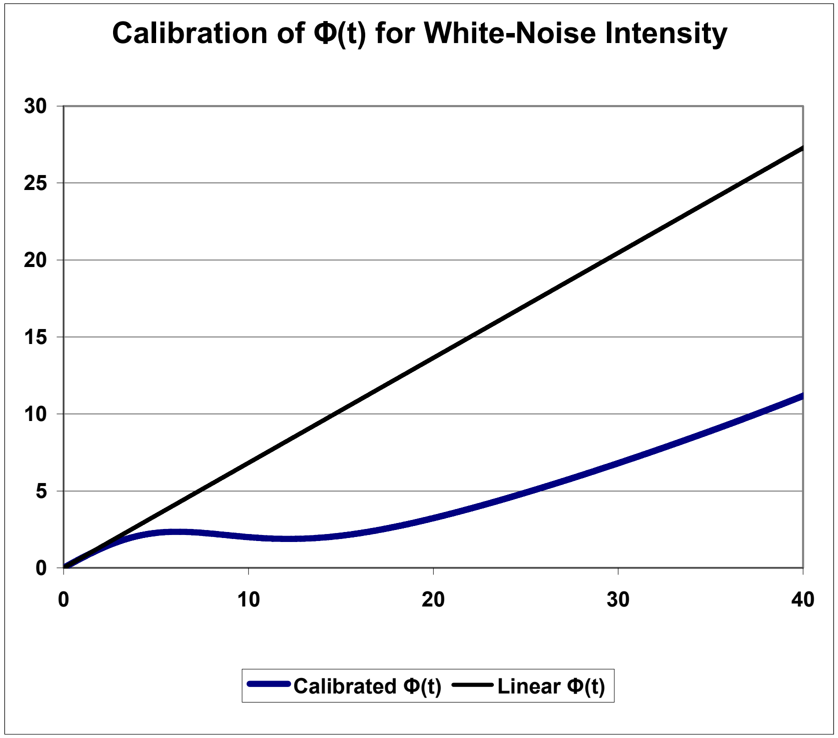
■

The Volterra integral equation (22) gives a very effective numerical algorithm for finding the drift $\Phi(t)$ from a given default time density $f_d(t)$ by the discretization method with sequential solution of the corresponding non-linear equation for $\Phi(t_i)$ given the previously found values $\Phi(t_0), \Phi(t_1), \ldots, \Phi(t_{i-1})$.

Testing results for this numerical method and direct Monte Carlo simulation of the default times are presented in Fig. 2 for the following testing default time density

(24)
$$\widetilde{f}_d(t) = \frac{1}{\sqrt{2\pi t}} e^{-0.5\nu^2 t} + (1-\nu)e^{-\alpha t}\left[\beta\sqrt{\frac{\alpha t}{\pi}} - 0.5\right]$$

($\nu = 0.682, \quad \beta = 0.671, \quad \alpha = 0.112$) and linear-drift default time density (20).



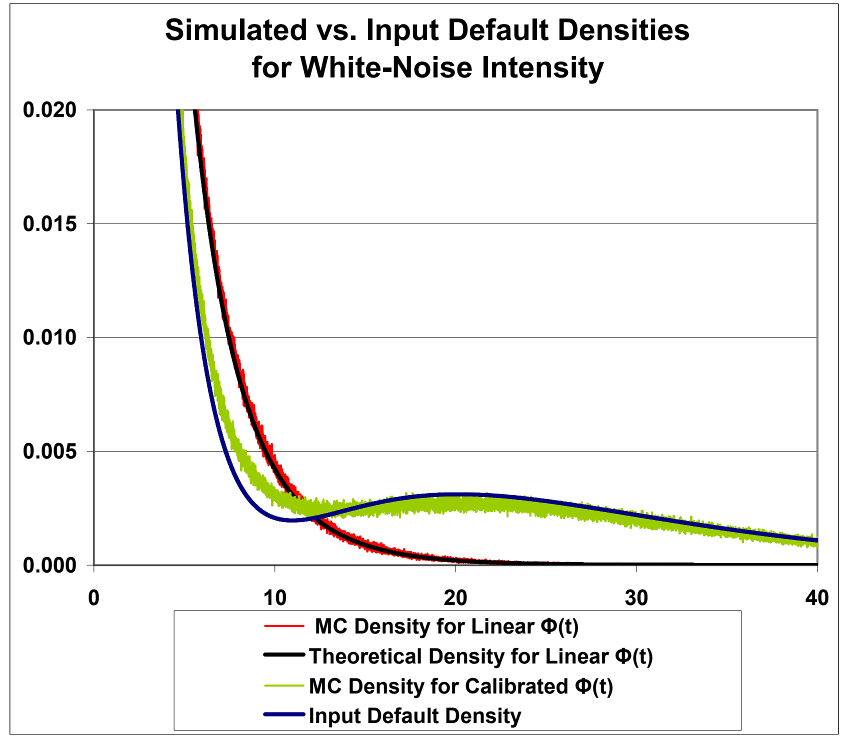


Fig. 2. Example of calibration of $\Phi(t)$ for "white noise" intensity model from Volterra integral equation and its verification by Monte Carlo simulation of hitting times.

Monte Carlo based calibration for OU default intensity

There is no known integral equation for function $\Phi(t)$ for Integrated OU process (14) (things are complicated by the fact that Integrated OU process is not a Markovian process). We develop a Monte-Carlo based fitting procedure using the idea of sequential calculation of the values of $\Phi(t)$ similar to solution of the Volterra integral equation.

We assume the survival probability $S(t)$ is calculated from the observed CDS spreads using a standard bootstrap method (see JP Morgan (2001)). We also assume the mean reversion speed κ and volatility σ for the OU hazard rate process (14) are known. We simulate a significant number *N* of paths for the integrated OU state variable $I^h(t)$ with fine time steps. Assuming piece-wise linear $\Phi(t)$, we sequentially calculate $\Phi(t_i)$ from non-linear equations using the previously found values $\Phi(t_0),\ \Phi(t_1),\dots,\Phi(t_{i-1})$ directly from the sampling mean in the Lando's formula for the survival probability:

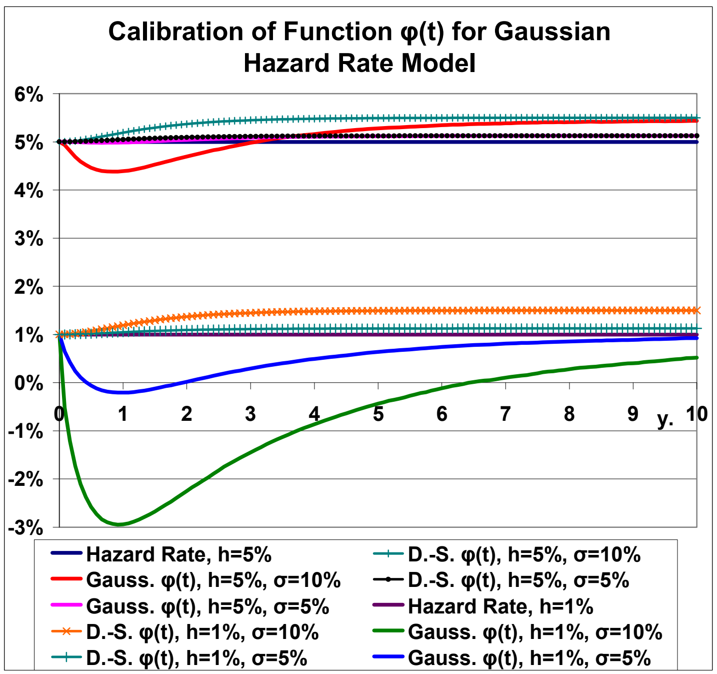
(25) $$S(t_i) = \frac{1}{N}\left\{\sum_1^N \exp\left[-\max_{t=t_0,\dots,t_i}\left\{\Phi(t) + \int_{t_0}^{t} \mathrm{X}^{\mathrm{h}}(s)ds\right\}\right]\right\}$$

The advantage of Gaussian framework is the highest performance for joint simulation of the Wiener, OU and Integrated OU processes in closed form for significant number of paths required for achieving the sufficient accuracy (see Glasserman (2003)).

We present in Fig. 3 two examples for Monte Carlo based fitting of $\Phi(t)$ into the piece-wise constant initial hazard rate term structures $h_0(t)$: first for flat hazard rates, and second for actual hazard rates of the BB and A rated companies. The calculated $\varphi(t)$ is compared with the approximate Duffie-Singleton drift of the form

(26) $$\varphi^{D-S}(t) = \frac{\sigma^2}{\kappa^2}\left\{t + \frac{1-\exp(-2\kappa t)}{2\kappa} - 2\frac{1-\exp(-\kappa t)}{\kappa}\right\} + h_0(t)$$

that corresponds to the affine survival probability formula if one replaces the maximum of the Integrated OU intensity by the Gaussian Integrated OU intensity itself.



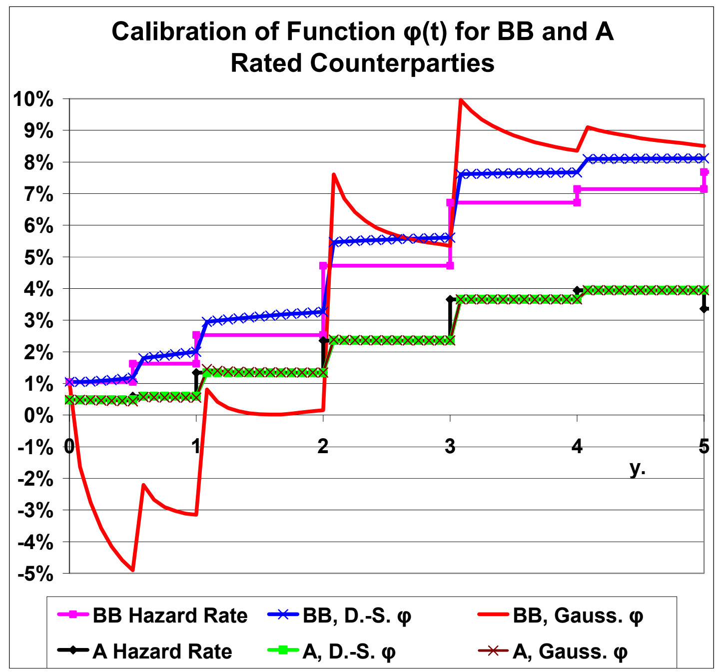
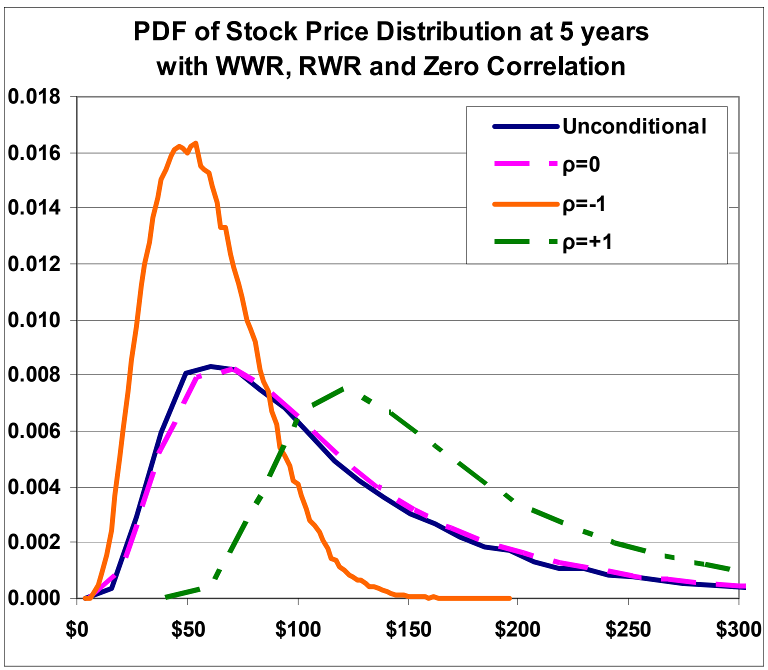


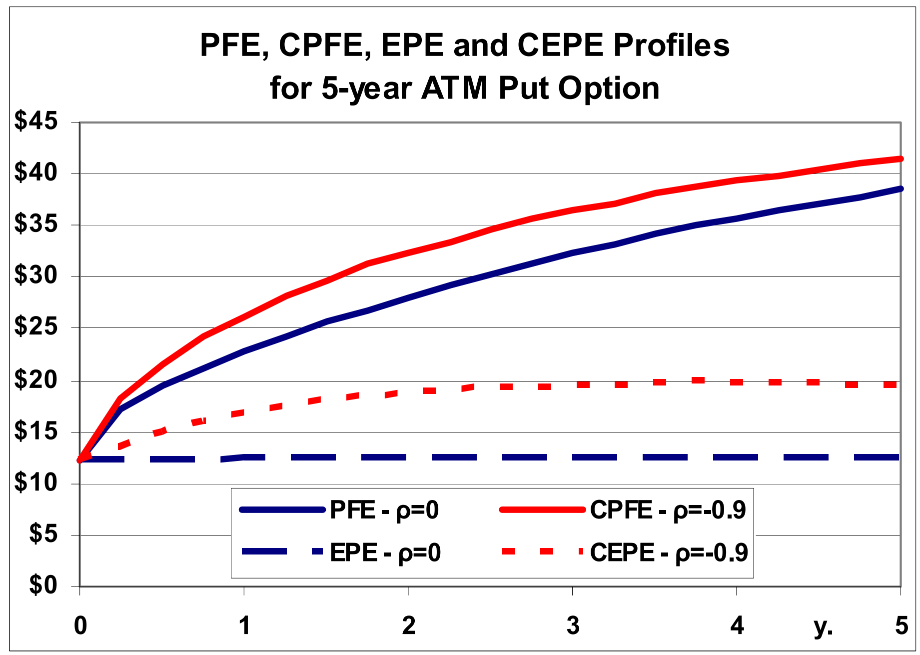
Fig. 3.Calibration of $\Phi(t)$ using Monte Carlo method

Monte Carlo simulation of EPE and CPFE profiles with WWR

For calculation of CVA, CPFE and CCEC with WWR, we use brute force joint Monte Carlo simulation of the correlated Gaussian integrated default intensities and relevant to the counterparty's portfolio market variables (e.g., Hull-White interest rates) with explicit calculation of the default times as hitting times of the exponential random boundary by Integrated OU intensities and revaluation of the portfolio MtM at the default time.

Example 1.1. Stock price distributions conditional on default and classical example of Specific WWR transaction – put option on the counterparty's own stock

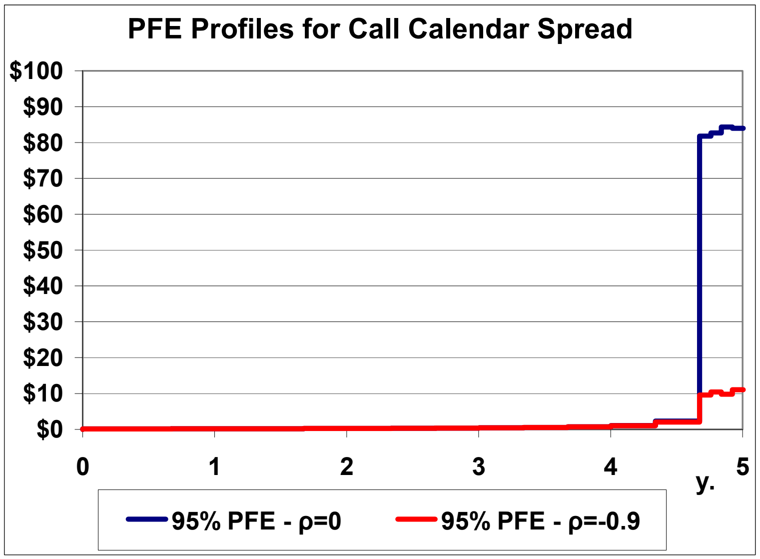


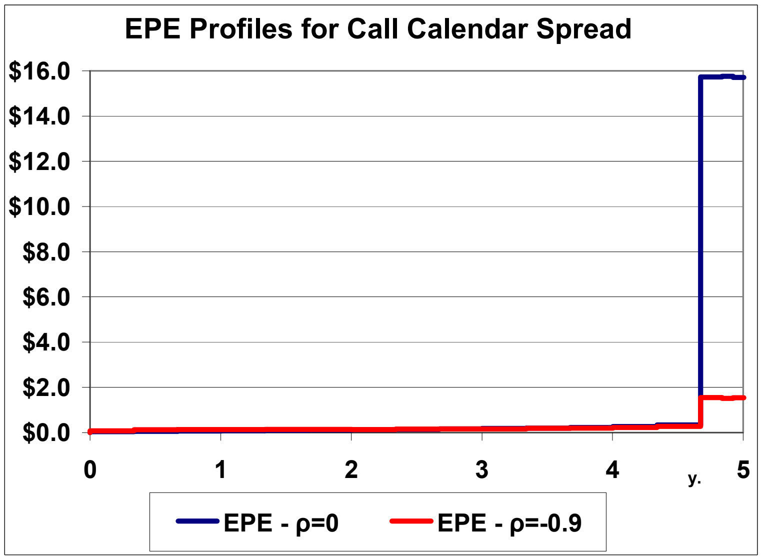


	ρ=0	ρ=-0.9	WWR Ratio
CPFE 95%	$38.26	$41.52	1.09
CCEC 95%	$18.34	$25.23	1.38
CCEC 99%	$31.78	$36.45	1.15
Unilateral CVA	$2.72	$4.05	1.49
Bilateral CVA (BCVA)	$2.53	$3.78	1.49

Table 1. CPFE, CCEC, CVA and BCVA for the 5-year put option (LGD=60%)

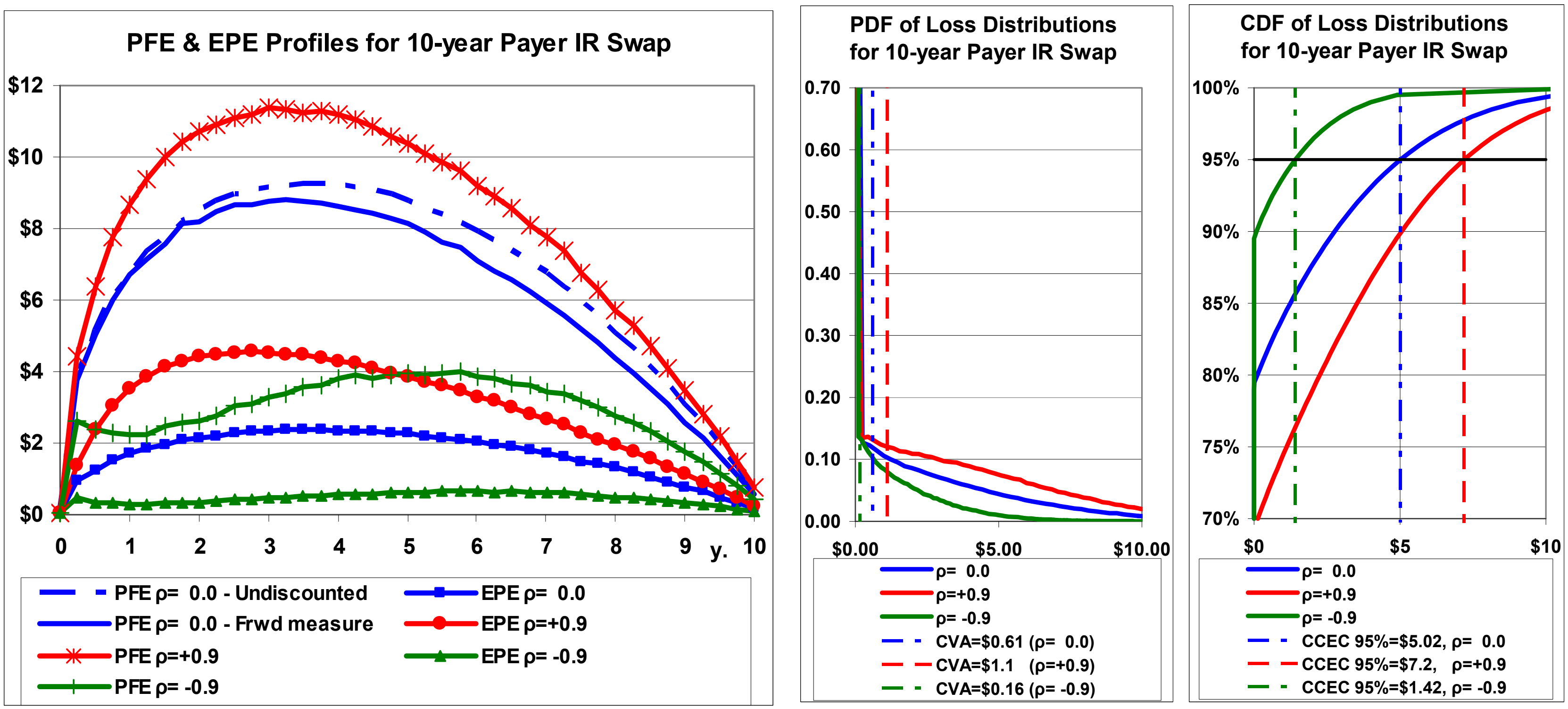
Example 1.2. CPFE/CEPE for RWR transaction –Call Calendar Spread on counterparty's stock - 5-yr. long call strike $130 and 56-mo. short call strike $126





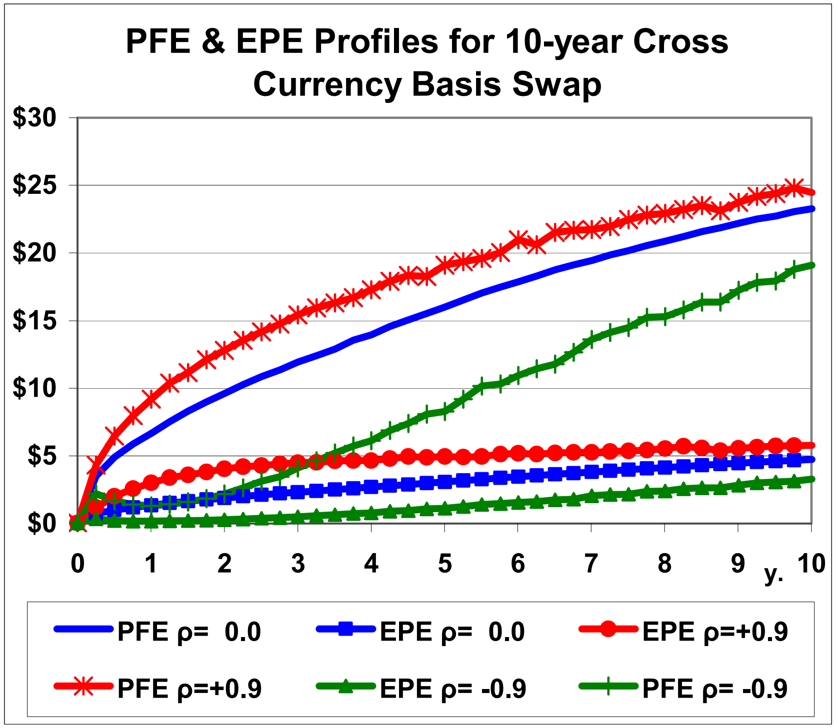
For this RWR transaction, the proposed Economic Capital-like measure – CCEC makes more sense than PFE, because the peak exposure is achieved during a very short period, which corresponds to a very low probability of default.

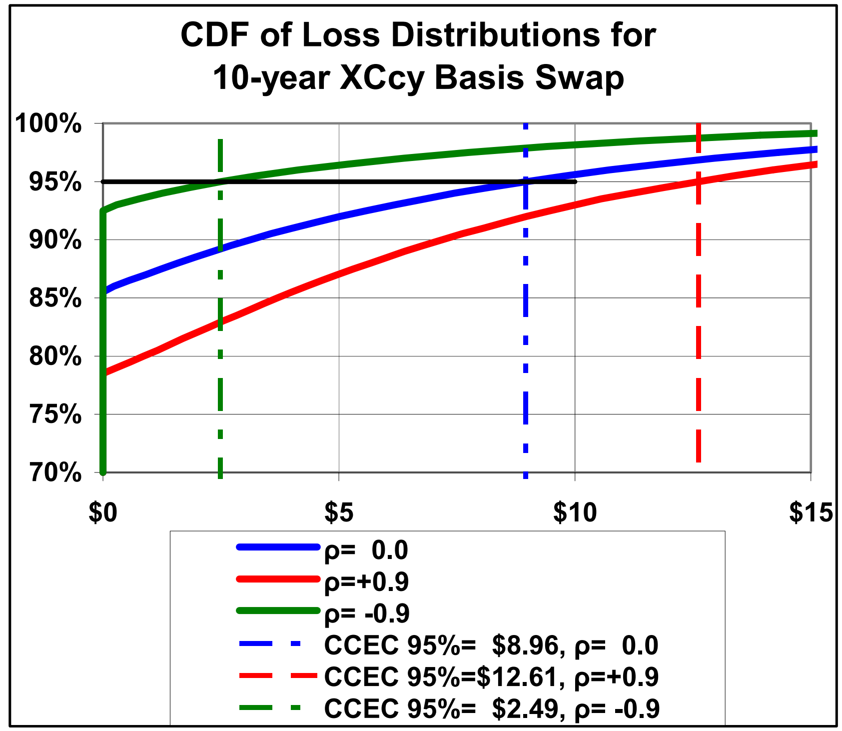
Example 2. CPFE/EPE profiles with the corresponding PDF/CDF of the conditional credit loss distribution and CVA/CCEC values with WWR/RWR for the 10-year payer interest rate swap



Note. "Undiscounted" PFE is based on distribution of $\mathrm{MtM}^{+}(t)$, "Forward measure" PFE is based on distribution of $D(t_0,t)^{-1}\left[D(t)\cdot \mathrm{MtM}^{+}(t)\right]$, where $D(t_0,t)$ is the initial discount factor term structure and $D(t)$ is a simulated stochastic discount factor. "Forward measure" PFE is more consistent with CVA formula.

Example 3. CPFE/EPE profiles with the corresponding CDF of the conditional credit loss distribution and CCEC values with WWR/RWR for the 10-year USD/EUR cross currency basis swap





UCVA=$0.74 (ρ=0), WWR CVA=$1.18 (ρ=0.9), RWR CVA=$0.32 (ρ=-0.9)
(for LGD=100%)

Consistent Simulation of Defaults and Credit Rating Transitions in Gaussian Hazard Rate Model

The considered path-wise Monte Carlo simulation framework allows for implementation of full collateral logic with thresholds for collateralized counterparties (see Brigo et. al. (2011), Gregory (2010), Pykhtin and Zhu (2007)). However, the majority of agreements with collaterized counterparties include credit triggers that depend on credit rating transitions. A default intensity framework determines only default times. Therefore, there is a need for consistent extension of the hazard rate framework by credit rating transition modeling. One of such approaches was considered in Lando (1998). However, that model includes a full matrix of stochastic intensities for defaults and credit rating transitions, and it is too complicated for calibration and practical use. On the other hand, practitioners widely use a simple CreditMetrics™ approach based on the Markov transition matrix model of Jarrow, Lando and Turnbull (1997).

A presented Gaussian hazard rate framework allows for a simple reasonable extension that consistently combines the OU default intensity model with the CreditMetrics™ credit rating transition approach.

Assume, the annual credit rating transition/default matrix $\mathbf{A}$ is given. A proposed joint OU default intensity/credit rating transition model is as follows:

- Because the default times in our model are fully determined by the default intensity, we recalculate a reduced credit rating transition matrix $\widetilde{\mathbf{A}}$ conditional on no-default from the initial full transition/default matrix $\mathbf{A}$

- We pre-calculate a sequence of the corresponding roots of the conditional on no-default credit rating transition matrix $\widetilde{\mathbf{A}}^i$ for each time step Δt_i using, for example, Markov generator approach (see Israel (2000)) and convert the transition probabilities into the normal quantiles
- In the original CreditMetrics™ method, a standard normal random variable (interpreted as "asset return") is generated for each time step and compared with the normal quantiles. A default or credit rating transition occurs when this normal random variable falls into the corresponding bucket. In our approach, we jointly simulate the Gaussian default intensity and negatively correlated with it "asset" Wiener process:

$$\begin{aligned} \mathrm{d}X^h(t) &= -\kappa X^h(t) + \sigma dW^h \\ dW^a &= \rho dW^h + \sqrt{1-\rho^2}\, dW^\xi \end{aligned} \tag{27}$$

(W^h and W^ξ are independent Wiener processes; the correlation ρ should be close to -1, because the credit spread returns are strongly negatively correlated with the asset returns)

- If default did not occur for a given time step Δt_i (i.e., the integrated default intensity $I^h(t)$ did not hit the exponential barrier), then the standard normal "asset return" variable $\Delta W_i^a \big/ \sqrt{\Delta t_i}$ is compared with the quantiles of the conditional on no-default credit rating transition matrix $\widetilde{\mathbf{A}}^i$ and the corresponding credit rating is assigned

Multifactor Gaussian and Jump-Diffusion frameworks

- Gaussian framework allows for easy and efficient implementation of multifactor models with thousands of correlated counterparties and market factors.
- In practice, all correlations between counterparty credit spreads are not available. A standard industry practice is to rely on the CAPM-like regression approach: each counterparty credit spread is regressed on a set of credit indices and market factors contributing to the Wrong-Way Risk for this counterparty. Gaussian framework is ideal for regressions.
- Proposed Gaussian default intensity framework is easily extendable to jump-diffusion model with no restriction on the sign of jumps (to the contrary, the affine jump diffusion framework requires positive jumps and positive diffusion processes, e.g., Brigo, Pallavicini and Papatheodorou (2011) use square-root jump-diffusion process with positive exponential jumps). In addition to Wiener processes, independent compound Poisson (or Lévy jump) processes J_i are introduced and default intensity X_i^h of the counterparty i is modeled as:

(28)
$$dX_i^h(t) = -\kappa_i X_i^h(t)dt + \sigma_i \left[\sum_j a_{i,j} dW_j + \sum_k b_{i,k} dJ_k \right]$$

where coefficients $a_{i,j}, b_{i,k}$ define the correlations.

The Monte Carlo based fitting procedure for $\Phi(t)$ stays the same, because integrated intensity is a continuous function of t and the roots for equation (25) can be found.

Comparison of Gaussian and Kou jump models

As an example of jump default intensity model with no restriction on the sign of jumps, we consider mean-reverting counterparty hazard rate $X_1^h(t)$ and logarithm of the FX rate $X^{FX}(t)$ driven by linear combination of independent Wiener and compound Poisson processes with double-exponential jump sizes, i.e. so-called Kou (2002) model:

$$dX^{FX}(t) = \left[r^d(t) - r^f(t) - \mu^J\right]dt + \sigma^{FX}\left[\omega dW_1 + \sqrt{1-\omega^2}\,dJ_1\right]$$

$$dX_1^h(t) = -\kappa_1^h X_1^h(t)dt + \sigma_1^h\left[\omega\left(\rho dW_1 + \sqrt{1-\rho^2}\,dW_2\right) + \sqrt{1-\omega^2}\left(\rho dJ_1 + \sqrt{1-\rho^2}\,dJ_2\right)\right]$$

$$\lambda_1^h(t) = \varphi_1^h(t) + X_1^h(t)$$

Here ω is a weight of the Gaussian component, and $J_{1,2}(t)$ are independent compound Poisson processes with constant intensity λ and exponential upward/downward jumps with means η^u, η^d and probability of upward jumps p (jump sizes are standardized to ensure unit volatilities for $J_{1,2}(t)$).

The density of jump sizes is: $f_Y(y) = \dfrac{p}{\eta^u} e^{-y/\eta^u} 1_{\{y \ge 0\}} + \dfrac{1-p}{\eta^d} e^{-y/\eta^d} 1_{\{y<0\}}$

The jump compensator is: $\mu^J = \lambda\left[\frac{1-p}{1+\tilde{\eta}^d} + \frac{p}{1-\tilde{\eta}^u} - 1\right], \quad \tilde{\eta}^{u,d} = \eta^{u,d}\sigma^s\sqrt{1-\omega^2}$

The coefficients with respect to common Wiener and compound Poisson processes define the FX/default intensity correlations and result in WWR/RWR. The drift $\varphi_1^h(t)$ in the default intensity is calculated from the CDS spreads by the Monte-Carlo based fitting procedure described earlier. Examples of $\varphi_1^h(t)$ are presented in Fig 4. It is well known that jumps have short-term impact compared to diffusions (e.g., Lando (2004)), which is confirmed by the blue and pink graphs in Fig. 4.

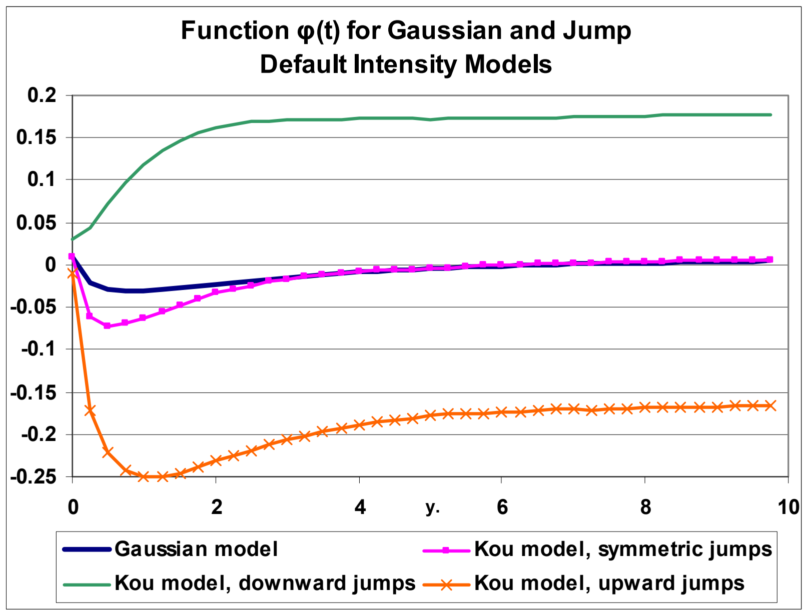


Fig. 4. Calibration of $\varphi(t)$ for jump model using Monte Carlo method

Figure 5 illustrates more significant Wrong/Right-Way Risk for jump model compared to Gaussian OU default intensity model (for short one-year horizon, where impact of jumps is significant).

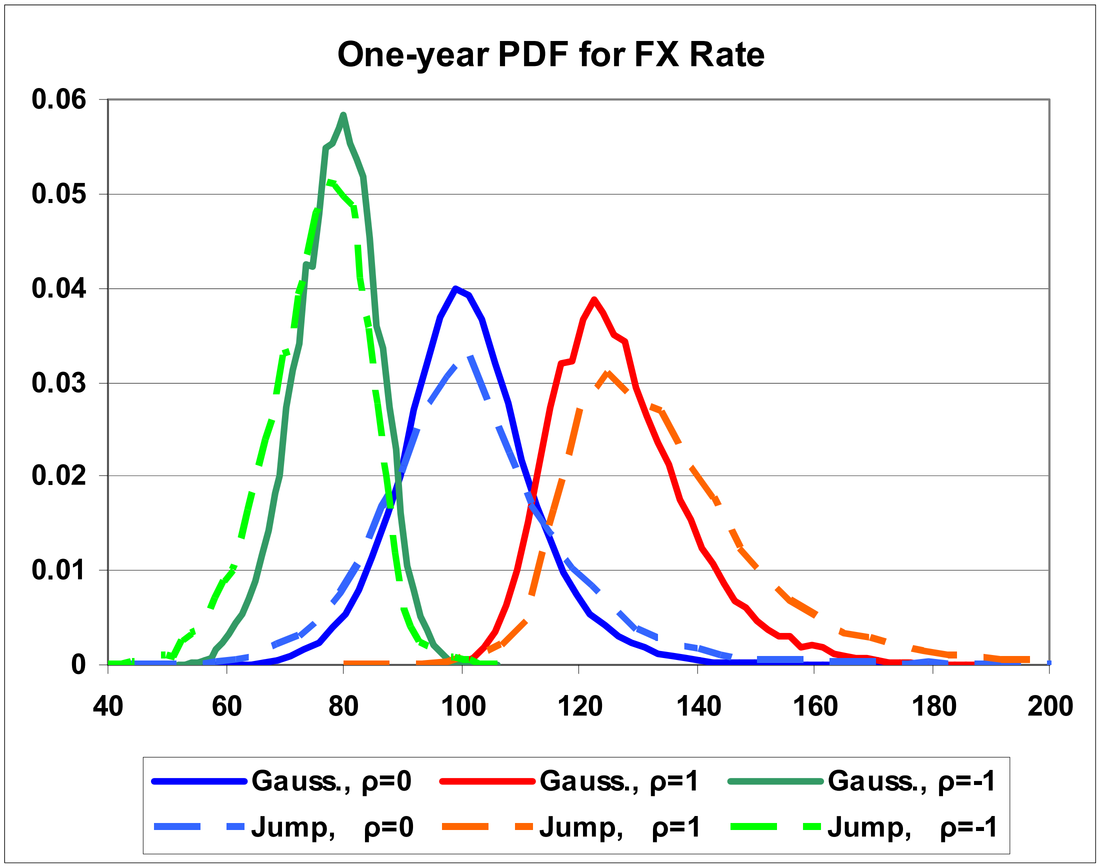


Fig. 5. Conditional FX Rate distributions for Gaussian and Kou default intensities

Figure 6 also confirms larger Wrong-Way Risk for jump model compared to Gaussian OU default intensity model for shorter horizons.

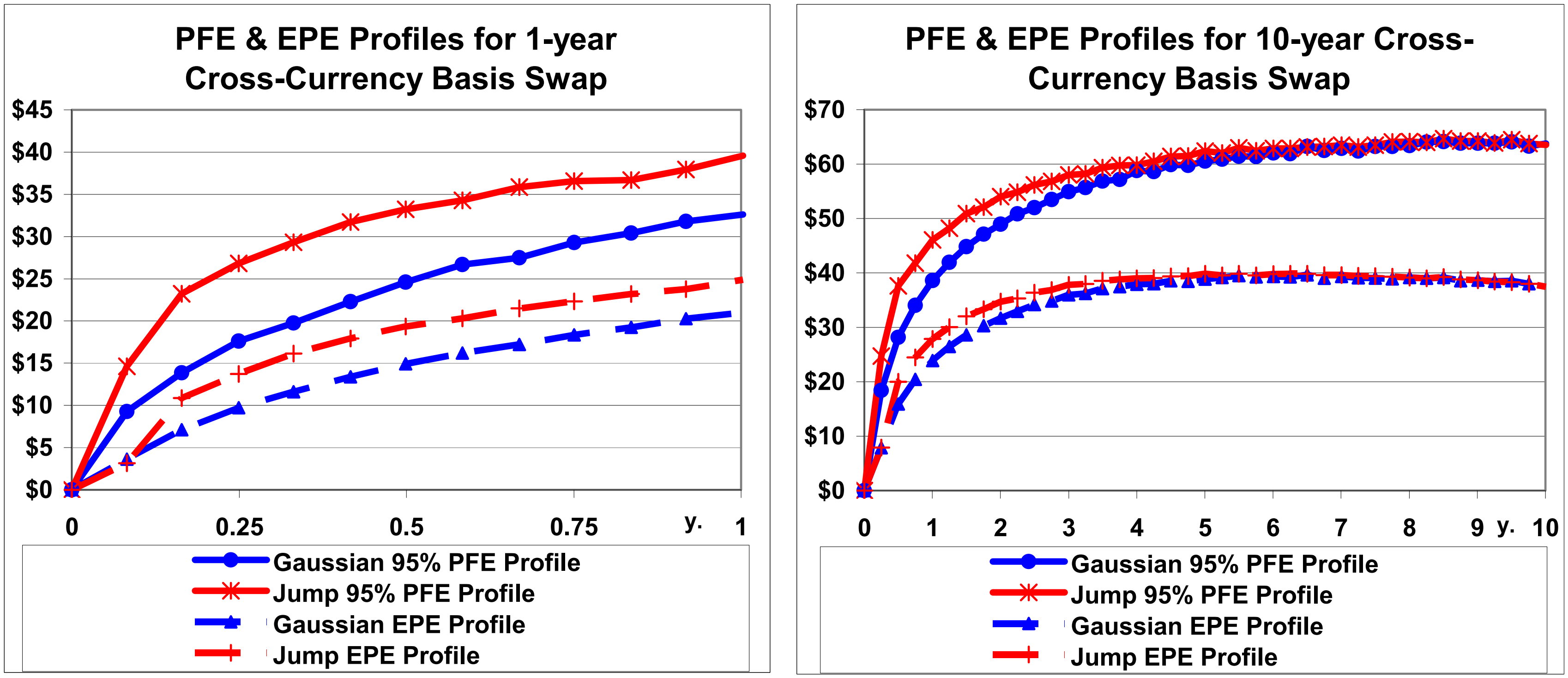


Fig. 6. WWR for Gaussian and Kou default intensities

The use of new "Gamma-Factor Copula" (GFC) for improving default correlations in the Portfolio Credit Economic Capital

- It is well documented in credit risk literature that traditional Gaussian correlations between hazard rate processes are not able to generate sufficient default correlations for Portfolio credit risk modeling. The application of copulas to relate marginal default probabilities of different names is considered as alternative approach to diffusion correlations. In general, there are three alternative approaches for modeling dependent defaults:
    - Correlated default intensity processes with independent exponential thresholds
    - Dependent exponential thresholds with independent intensity processes
    - Correlated default intensity processes with dependent exponential threshold

  Our preference is the third approach. Everyday correlations between credit spreads of the counterparties are observed in the market.

Gaussian copula and Student-t copula with $\nu > 0$ degrees of freedom are the most popular choices for a copula. However, the corresponding correlated multivariate Gaussian and t distributions are defined on a whole space, while a desired multivariate exponential distribution is defined on the positive n – dimensional octant, i.e., these non-linear copula transformations are not natural for the problem at hand. Gaussian copula has zero tail-dependence (see McNeil(2005), p. 211) and poorly models extreme low-probability joint defaults observed in the market during credit. Though a t-copula does have tail-dependence, it requires a very low number of degrees of freedom and high correlation coefficients to provide sufficient tail-dependence

But, because a multivariate t distribution is a normal-mixture distribution by the common χ_ν^2-distributed variance, the components of a t-distributed vector are dependent even for zero correlation coefficients. This means a t-copula is unable to model highly correlated exponential thresholds with tail-dependence for a group of required names and simultaneously independent exponential thresholds for another group of required names.

We propose a new construction natural for correlated exponential random variables through the decomposition of the exponential random variables into the sums of some independent Gamma random variables (“Gamma-factors").We call the copula that is implicitly defined by this construction a “**Gamma-factor**" copula. Let $\xi_1, \xi_2, \ldots, \xi_m$ denote m independent gamma random variables $\xi_i \sim \Gamma(\alpha,1), \quad \alpha \in [0,1]$, with the same scaling parameter 1 and probability density functions

(29) $$f_{\alpha_i}(y) = \frac{1}{\Gamma(\alpha_i)} y^{\alpha_i - 1} e^{y}, \qquad y > 0$$

We define $n\,(n \le m)$ dependent unit exponential random variables $\mathrm{Y}_1, \ldots, \mathrm{Y}_n$ as follows:

$$\vec{\mathrm{Y}} = A\vec{\gamma}, \qquad \vec{\mathrm{Y}} = [\mathrm{Y}_1, \ldots, \mathrm{Y}_n]^{\mathrm{T}}, \qquad \vec{\gamma} = [\gamma_1, \ldots, \gamma_n]^{\mathrm{T}}, \qquad \vec{\alpha} = [\alpha_1, \ldots, \alpha_n]^{\mathrm{T}}$$

$$\mathbf{A} = [a_{i,j}]_{(n \times m)}, \quad \mathbf{A}\vec{\alpha} = [1, \ldots, 1]^{\mathrm{T}}{}_{(n \times 1)}$$

The entries of the $n \times m$ load matrix $\mathbf{A}$ are either zero or one. Within these m Gamma factors, there are $m-n$ common factors (i.e., the corresponding columns of the matrix A have more than one entries equal to 1) and n idiosyncratic factors with all zero entries except one equal to 1 in the corresponding columns of the matrix $\mathbf{A}$. The example of matrix $\mathbf{A}$ is shown in Table 2 . The common Gamma random variables can be associated with certain economic factors such as sectors, industries, regions, credit ratings, etc. This idea is similar to Moody's Analytics Global Correlation Factor and CreditMetrics™ approaches for modeling asset correlations using some global and idiosyncratic factors.

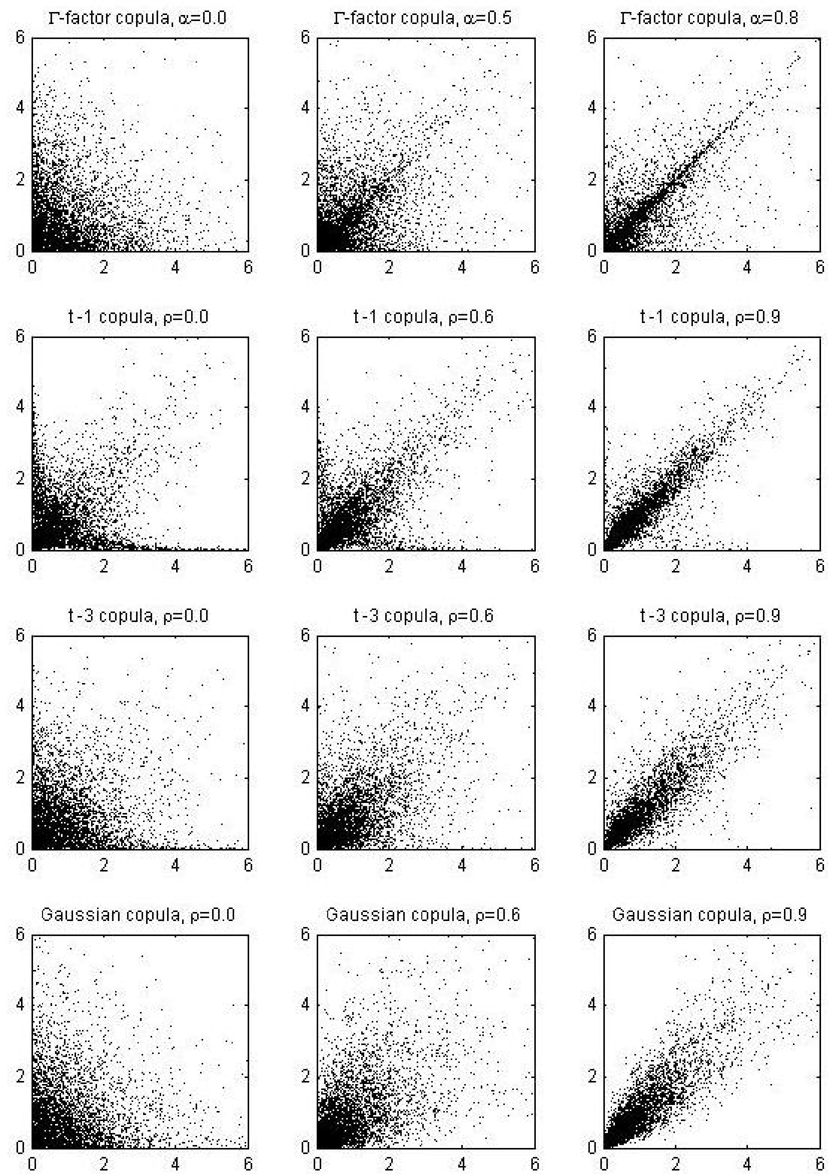
	Market	Ind.A	Ind.B	Ind.C	Cntry 1	Cntry 2	Idio1	Idio2	Idio3	Idio4
Name1	1	0	1	0	1	0	1	0	0	0
Name2	1	1	0	0	1	0	0	1	0	0
Name3	1	0	0	1	0	1	0	0	1	0
Name4	1	0	1	0	0	1	0	0	0	1

Table 2. Example of Gamma-Factor Copula load matrix $\mathbf{A}$

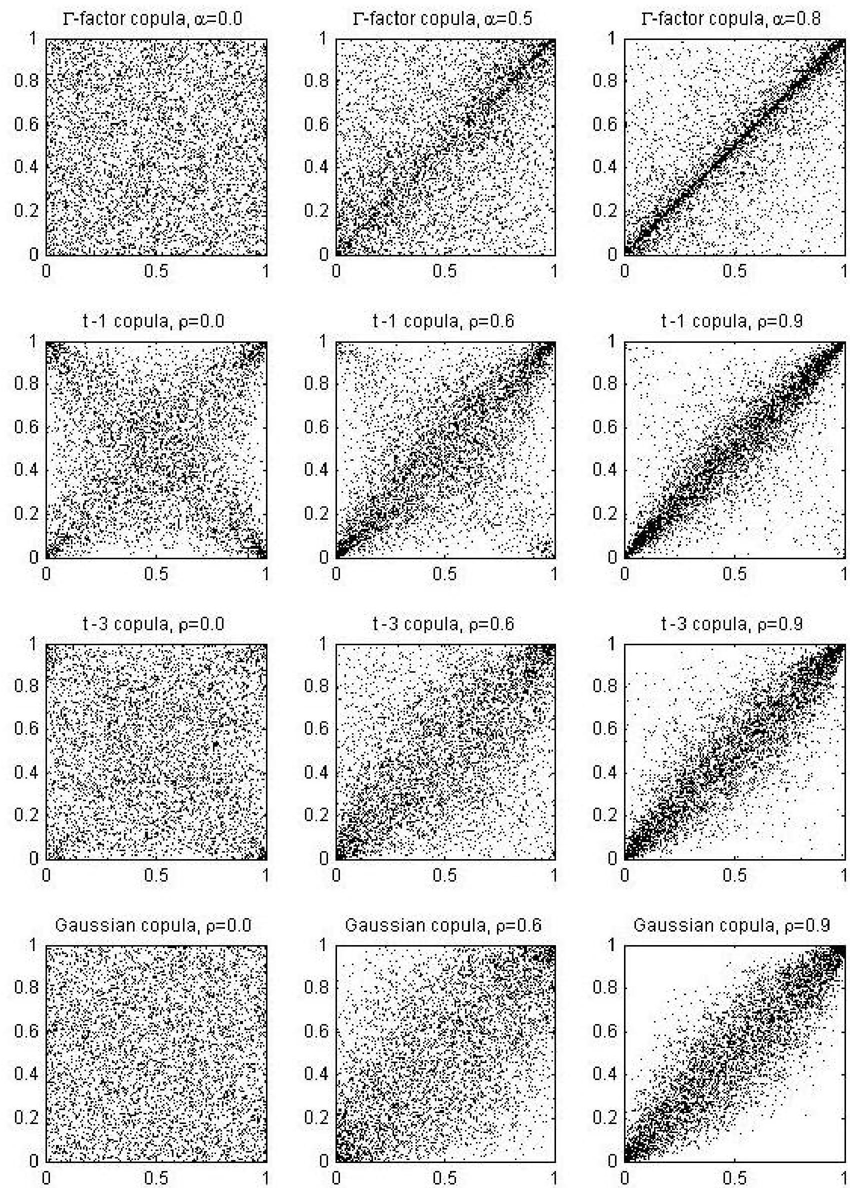
Let us consider two names for calculation of the default correlation. Let $\hat{\xi}, \xi_1, \xi_2$ be three independent gamma random variables – common gamma-factor $\hat{\xi} \sim \Gamma(\alpha,1)$, and two idiosyncratic gamma-factors $\xi_1 \sim \Gamma(1-\alpha,1)$, $\xi_2 \sim \Gamma(1-\alpha,1)$, all with the same scaling parameter $\beta = 1$, where the "dependency" parameter $\alpha \in [0,1]$. The correlated exponential random thresholds Y_1 and Y_2 for two given names are defined as

(30) $$\mathrm{Y}_1 = \hat{\xi} + \xi_1, \qquad \mathrm{Y}_2 = \hat{\xi} + \xi_2$$

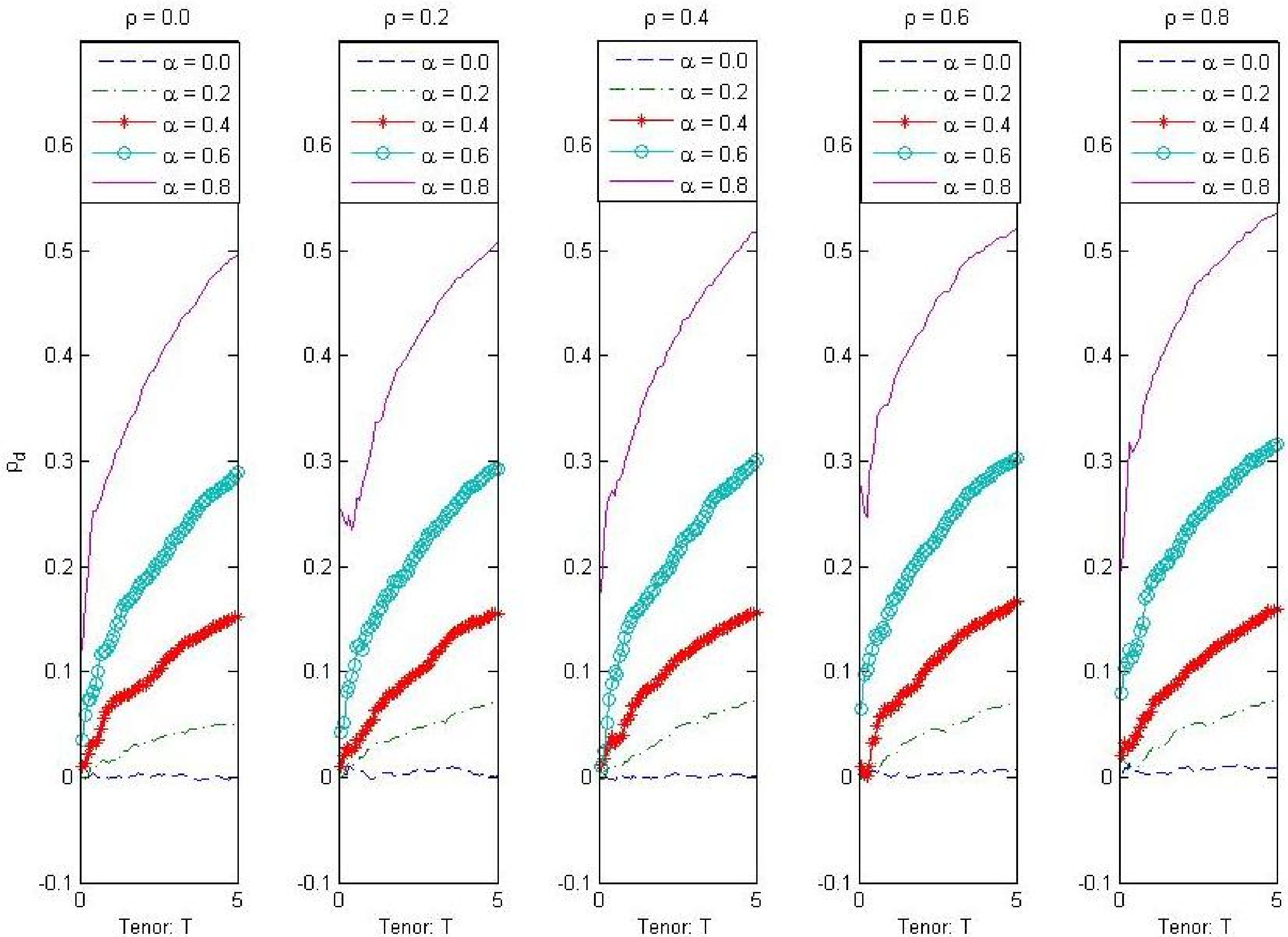
When $\alpha = 0$, Y_1 and Y_2 are independent; when $\alpha = 1$ they become perfectly correlated.



Bivariate exponential distribution



Bivariate copula



The impact of the Gaussian correlation and parameter α of Gamma-Factor Copula on the default correlation term structures for two counterparties ‡

‡ Authors thank Chuang Yi, formerly of RBC Risk Methodology, for performing this Monte Carlo investigation

Comparison of impact on Bilateral CVA of the Gamma-Factor Copula and Gaussian default intensity correlation

We compare the impact of the Gaussian correlation $\rho_{\lambda 0,\lambda 1}$ between the investor's (BAC) and counterparty's (F) default intensities and the "dependency" parameter α of the GFC on the Bilateral CVA of a 10-year Interest Rate Swap with notional \$100. Market data is as of Oct. 20, 2011, the Hull-White risk-neutral parameters for the USD interest rate were calibrated by a standard procedure, mean-reversion parameters and volatilities of the hazard rates were estimated from the historical data, recovery rates are 40%.

BCVA

α\ρ	0.0	0.5	1.0
0.0	\$1.15	\$1.13	\$1.12
0.5	\$1.10	\$1.08	\$1.05
1.0	\$1.01	\$0.97	\$0.90

Relative Impact w.r.t. α=0, ρ=0

α\ρ	0.0	0.5	1.0
0.0	100%	99%	97%
0.5	95%	94%	92%
1.0	88%	85%	78%

(CVA = \$1.77, DVA = \$0.48)

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