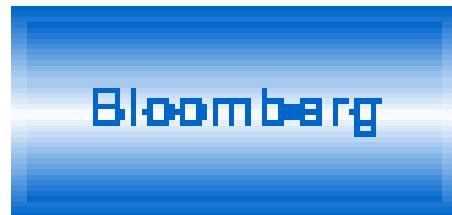


Functional Itô Calculus and Volatility Risk Management

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Outline

1) Functional Itô Calculus

- Functional Itô formula
- Functional Feynman-Kac
- PDE for path dependent options

2) Volatility Hedge

- Local Volatility Model
- Volatility expansion
- Vega decomposition
- Robust hedge with Vanillas
- Examples

1) Functional Itô Calculus

Why?

Most often, the impact of uncertainty is cumulative. The link

Cause → Consequence

$$X_t \xrightarrow{f} y_t = f(X_t)$$

depends on certain patterns :

temperature → water level

price history → path dependent payoff

exposure to antigen → viral reaction

interest rate path → MBS prepayment

Itô calculus deals with functions of processes.

We extend it to functions of the current path,
or history, of the process.

Review of Itô Calculus

- 1D

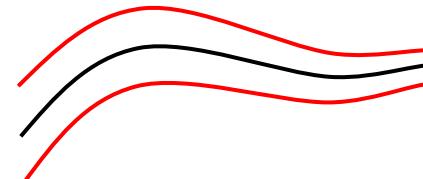


- current value
- possible evolutions

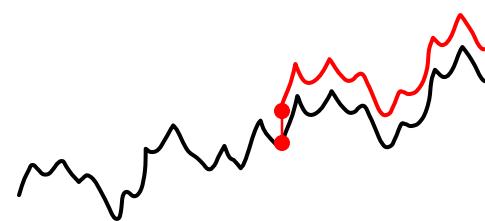
- nD



- infiniteD



- Malliavin Calculus



- Functional Itô Calculus



Functionals of running paths

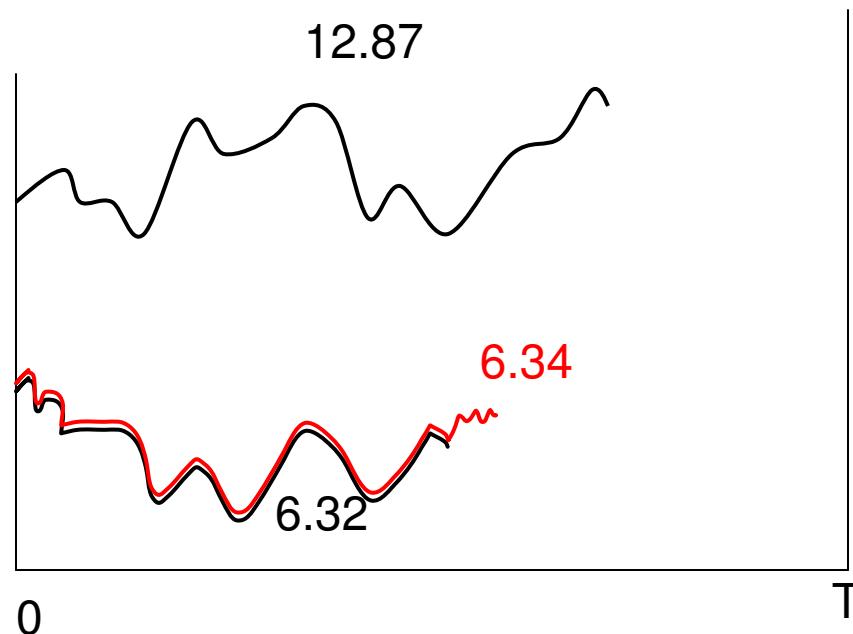
Functionals defined not only on the whole path $[0, T]$ but on all starting sections $[0, t]$

$$\Lambda_t \equiv \{\text{RCLL functions } [0, t] \rightarrow \mathfrak{R}\}$$

$$\Lambda \equiv \bigcup_{t \in [0, \bar{T}]} \Lambda_t$$

f functional : $f : \Lambda \rightarrow \mathfrak{R}$, for $X_t \in \Lambda_t$, $f(X_t) \in \mathfrak{R}$

The value of X_t at $s \leq t$ is $X_t(s)$ and $x_t \equiv X_t(t)$



Examples of Functionals

- Current average
- Current drawdown
- Conditional expectation of final value
- Super - replicating price

The last one covers the important case of path dependent option price

Finite number of state variables (excluding time) :

- European (1)
- Asian (2)
- Option on range (3)

Infinite number of state variables

- Max of rolling average
- First time last value is hit

Derivatives

$$\Lambda_t \equiv \{\text{RCLL functions } [0, t] \rightarrow \mathfrak{R}\}, \quad \Lambda \equiv \bigcup_{t \in [0, \bar{T}]} \Lambda_t$$

For $X_t \in \Lambda_t, f : \Lambda \rightarrow \mathfrak{R}$,

$$X_t^h(s) \equiv X_t(s) \text{ for } s < t \quad X_t^h(t) = X_t(t) + h$$



$$X_{t,\delta t}(s) = X_t(s) \text{ for } s \leq t \quad X_{t,\delta t}(s) = X_t(t) \text{ for } s \in [t, t + \delta t]$$



Space derivative

$$\Delta_x f(X_t) \equiv \lim_{h \rightarrow 0} \frac{f(X_t^h) - f(X_t)}{h}$$

Time derivative

$$\Delta_t f(X_t) \equiv \lim_{\delta t \rightarrow 0} \frac{f(X_{t,\delta t}) - f(X_t)}{\delta t}$$

Examples

	f	x_t	$\int_0^t x_u du$	QV_t
	$\Delta_x f$	1	0	$2(x_t - x_{t^-})$
	$\Delta_t f$	0	x_t	0

If $f(X_t) = h(x_t, t)$, then

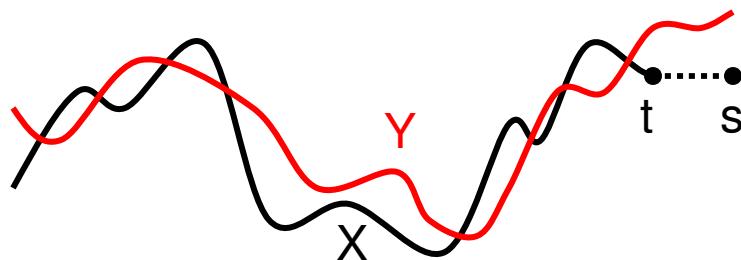
$$\begin{cases} \Delta_x f = \frac{\partial h}{\partial x} \\ \Delta_t f = \frac{\partial h}{\partial t} \end{cases}$$

Topology and Continuity

Λ - distance :

For all X_t, Y_s in Λ , (we can assume $t \leq s$)

$$d_\Lambda(X_t, Y_s) \equiv \|X_{t,s-t} - Y_s\|_\infty + s - t$$



Λ - continuity :

A functional $f : \Lambda \rightarrow \mathfrak{R}$ is Λ - continuous at $X_t \in \Lambda$ if :

$$\forall \varepsilon > 0, \exists \alpha > 0 : \forall Y_s \in \Lambda,$$

$$d_\Lambda(X_t, Y_s) < \alpha \Rightarrow |f(Y_s) - f(X_t)| < \varepsilon$$

Functional Itô Formula

Definition : a functional $f : \Lambda \rightarrow \Re$ is *smooth* if it is Λ - continuous, C^2 in x and C^1 in t , with these derivatives themselves Λ - continuous.

Theorem

If x is a continuous semi - martingale and X_t denotes its path over $[0, t]$, then, for all smooth functional f and for all $T \geq 0$,

$$f(X_T) = f(X_0) + \int_0^T \Delta_x f(X_t) dx_t + \int_0^T \Delta_t f(X_t) dt + \frac{1}{2} \int_0^T \Delta_{xx} f(X_t) d\langle x \rangle_t$$

or, in more concise notation,

$$df = \Delta_x f \, dx + \Delta_t f \, dt + \frac{1}{2} \Delta_{xx} f \, d\langle x \rangle$$

Fragment of proof

$$\begin{aligned}
 df &= f(\text{wavy line}) - f(\text{dot}) \\
 &= (f(\text{wavy line}) - f(\text{dot connected})) \\
 &\quad + (f(\text{dot connected}) - f(\text{dot})) \\
 &\quad + (f(\text{dot}) - f(\text{dot connected}))
 \end{aligned}$$

From Taylor expansion of $f(Y_i) - f(Z_i)$ we get,
with $\delta x_i = X_i(\tau_i) - X_i(\tau_{-i})$ and $\delta t_i = \tau_i - \tau_{-i}$,

$$\begin{aligned}
 B &= \sum_i^N (f(Y_i) - f(Z_i)) = \sum_i^N \Delta_{\text{in}} f(Z_i) \delta x_i & B_1 \\
 &\quad + \frac{1}{2} \sum_i^N \Delta_{\text{in}} f(Z_i^{\theta, \bar{\theta}}) (\delta x_i)^2 & B_2
 \end{aligned}$$

with $\theta_i \in (0,1)$

$$\begin{aligned}
 B_1 &= \sum_i^N \Delta_{\text{in}} f(Z_i) \delta x_i = \sum_i^N \Delta_{\text{in}} f(X_{-i}) \delta x_i & B_{11} \\
 &\quad + \sum_i^N (\Delta_{\text{in}} f(U_i) - \Delta_{\text{in}} f(X_{-i})) \delta x_i & B_{12} \\
 &\quad + \sum_i^N (\Delta_{\text{in}} f(Z_i) - \Delta_{\text{in}} f(U_i)) \delta x_i & B_{13}
 \end{aligned}$$

$$\begin{aligned}
 B_{11} &= \sum_i^N \Delta_{\text{in}} f(X_{-i}) \delta x_i \xrightarrow{L_i} \int_0^T \Delta_{\text{in}} f(X_i) dx_i \\
 \text{As } \Delta_{\text{in}} f(U_i) - \Delta_{\text{in}} f(X_{-i}) &= \Delta_{\text{in}} f(X_{-i, \theta_i, \bar{\theta}_i}) \delta t_i \leq \beta \max \Delta_{\text{in}} f, B_{12} \xrightarrow{L_i} 0 \\
 \text{As } \Delta_{\text{in}} f(Z_i) - \Delta_{\text{in}} f(U_i) &\leq k_{\text{in}} \alpha, B_{13} \xrightarrow{L_i} 0
 \end{aligned}$$

$$\begin{aligned}
 2B_2 &= \sum_i^N \Delta_{\text{in}} f(Z_i^{\theta, \bar{\theta}}) (\delta x_i)^2 = \sum_i^N \Delta_{\text{in}} f(X_{-i}) (\delta x_i)^2 & B_{21} \\
 &\quad + \sum_i^N (\Delta_{\text{in}} f(Z_i^{\theta, \bar{\theta}}) - \Delta_{\text{in}} f(Z_i)) (\delta x_i)^2 & B_{22} \\
 &\quad + \sum_i^N (\Delta_{\text{in}} f(Z_i) - \Delta_{\text{in}} f(X_{-i})) (\delta x_i)^2 & B_{23}
 \end{aligned}$$

$$\begin{aligned}
 B_{21} &= \sum_i^N \Delta_{\text{in}} f(X_{-i}) (\delta x_i)^2 \Rightarrow L_2 - \lim_{\alpha, \bar{\theta} \rightarrow 0} B_{21} = \int_0^T \Delta_{\text{in}} f(X_i) d\langle x \rangle_i \\
 \text{As } |\Delta_{\text{in}} f(Z_i^{\theta, \bar{\theta}}) - \Delta_{\text{in}} f(Z_i)| &\leq k_{\text{in}} \alpha, L_2 - \lim_{\alpha, \bar{\theta} \rightarrow 0} B_{22} = 0 \\
 \text{As } |\Delta_{\text{in}} f(Z_i) - \Delta_{\text{in}} f(X_{-i})| &= |\Delta_{\text{in}} f(X_{-i, \theta_i, \bar{\theta}_i}) \delta t_i| \leq \beta \max \Delta_{\text{in}} f, L_2 - \lim_{\alpha, \bar{\theta} \rightarrow 0} B_{23} = 0
 \end{aligned}$$

$$\begin{aligned}
 L_2 - \lim_{\alpha \rightarrow 0} B_2 &= \int_0^T \Delta_{\text{in}} f(X_i) dx_i \\
 L_2 - \lim_{\alpha \rightarrow 0} B_2 &= \frac{1}{2} \int_0^T \Delta_{\text{in}} f(X_i) d\langle x \rangle_i
 \end{aligned}$$

Functional Feynman-Kac Formula

Generalisation of FK to non Markov dynamics and path dependent payoff.

$$dx_t = a(X_t) dt + b(X_t) dW_t$$

For g suitably integrable, $g : \Lambda_T \rightarrow \mathfrak{R}$, $r : \Lambda \rightarrow \mathfrak{R}$. We define $f : \Lambda \rightarrow \mathfrak{R}$ by

$$f(Y_t) \equiv E[e^{-\int_t^T r(Z_u) du} g(Z_T) | Z_t = Y_t]$$

where $\begin{cases} \text{for } u \in [0, t], Z_T(u) = Y_t(u) \\ \text{for } u \in [t, T], dz_u = a(Z_u) du + b(Z_u) dW_u \end{cases}$

Then, if f is smooth, it satisfies

$$\Delta_t f(X_t) + a(X_t) \Delta_x f(X_t) - r(X_t) f(X_t) + \frac{b^2(X_t)}{2} \Delta_{xx} f(X_t) = 0$$

(apply functional Itô formula to the martingale $e^{-\int_0^t r(X_u) du} f(X_t)$)

Delta Hedge/Clark-Ocone

If f defined by $f(Y_t) \equiv E[g(Z_T) | Z_t = Y_t]$ is smooth,
then we have the explicit Martingale Representation :

$$g(X_T) = E[g(X_T) | X_0] + \int_0^T b(X_t) \Delta_x f(X_t) dW_t$$

From functional Itô and Feynman - Kac with $r = 0$,

$$df(X_t) = b(X_t) \Delta_x f(X_t) dW_t$$

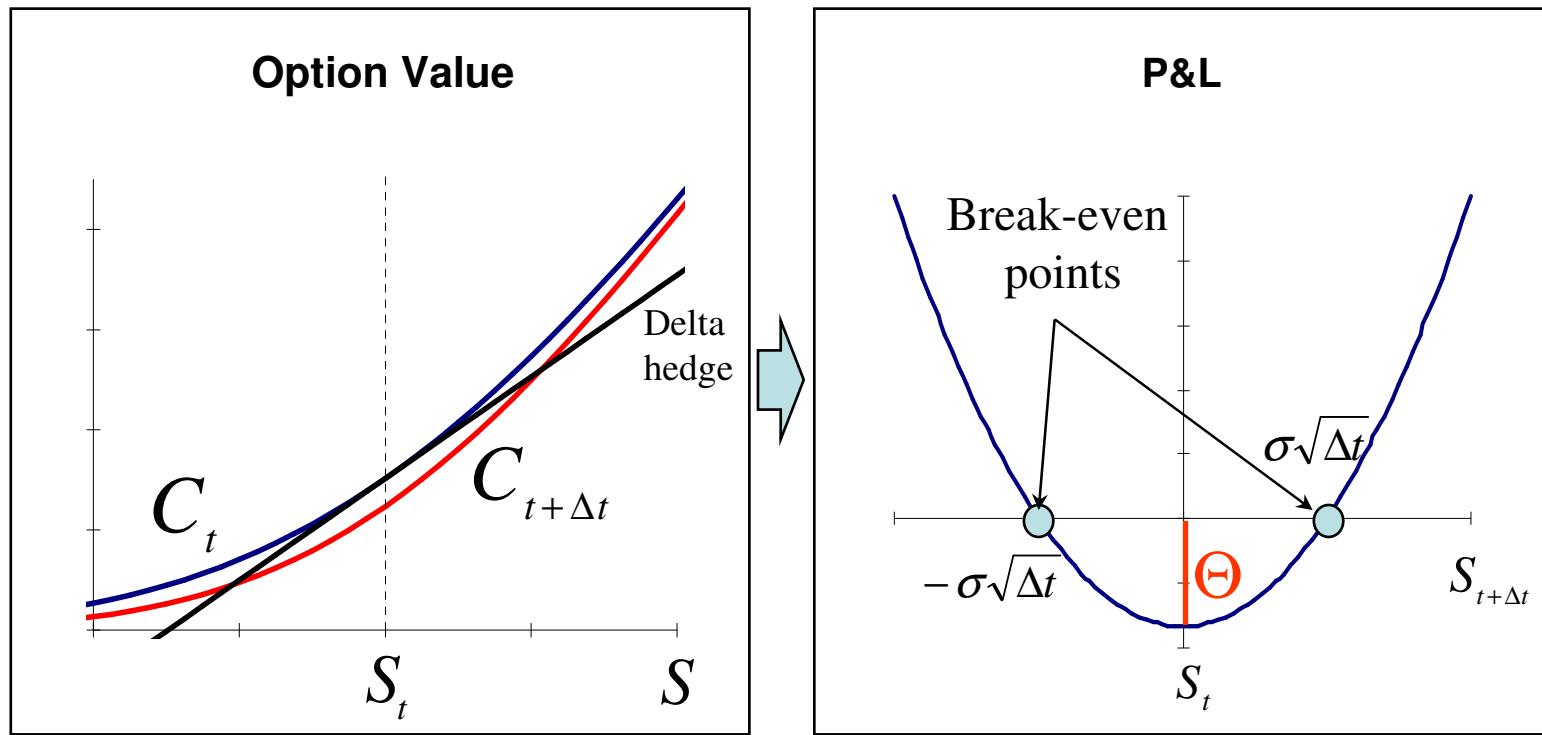
and

$$g(X_T) = f(X_T) = f(X_0) + \int_0^T b(X_t) \Delta_x f(X_t) dW_t$$

It can be compared to the Clark - Ocone formula
from Malliavin Calculus :

$$g(X_T) = E[g(X_T) | X_0] + \int_0^T b(X_t) E[D_t g(X_T) | X_t] dW_t$$

P&L of a delta hedged Vanilla



Functional PDE for Exotics

If f given by $f(Y_t) \equiv E[e^{-\int_t^T r(Z_u) du} g(Z_T) | Z_t = Y_t]$ is smooth, then it satisfies

$$\Delta_t f(X_t) + \frac{1}{2} b^2 \Delta_{xx} f(X_t) + r(X_t) (\Delta_x f(X_t) x_t - f(X_t)) = 0$$

The Γ/Θ trade-off for European options also holds for path dependent options, even with an infinite number of state variables. However, in general Γ and Θ will be path dependent.

Classical PDE for Asian

Payoff of Asian Call: $g(X_T) = (\int_0^T x_u du - K)^+$

Assume $dx_t = b(x_t, t) dW_t$ Define $I_t \equiv \int_0^t x_u du, f(X_t) \equiv l(x_t, I_t, t)$


$$\Delta_t I = x_t \Rightarrow \Delta_t f = x \frac{\partial l}{\partial I} + \frac{\partial l}{\partial t}$$


$$\Delta_x I = 0 \Rightarrow \Delta_x f = \frac{\partial l}{\partial x}, \Delta_{xx} f = \frac{\partial^2 l}{\partial x^2}$$

$$\Delta_t f + \frac{1}{2} b^2 \Delta_{xx} f = 0 \Rightarrow \frac{\partial l}{\partial t} + x \frac{\partial l}{\partial I} + \frac{1}{2} b^2 \frac{\partial^2 l}{\partial x^2} = 0$$

$x \frac{\partial l}{\partial I}$ is a bothering convection term.

Better Asian PDE

Define $J_t \equiv E_t[\int_0^T x_u du] = \int_0^t x_u du + (T-t)x_t$, $f(X_t) \equiv h(x_t, J_t, t)$



$$\Delta_t J = 0 \Rightarrow \Delta_t f = \frac{\partial h}{\partial t}$$



$$\Delta_x J = (T-t) \Rightarrow \begin{cases} \Delta_x f = \frac{\partial h}{\partial x} + (T-t) \frac{\partial h}{\partial J} \\ \Delta_{xx} f = \frac{\partial^2 h}{\partial x^2} + 2(T-t) \frac{\partial^2 h}{\partial x \partial J} + (T-t)^2 \frac{\partial^2 h}{\partial J^2} \end{cases}$$

$$\Delta_t f + \frac{1}{2} b^2 \Delta_{xx} f = 0 \Rightarrow \frac{\partial h}{\partial t} + \frac{1}{2} b^2 \left(\frac{\partial^2 h}{\partial x^2} + 2(T-t) \frac{\partial^2 h}{\partial x \partial J} + (T-t)^2 \frac{\partial^2 h}{\partial J^2} \right) = 0$$

2) Robust Volatility Hedge

Local Volatility Model

- Simplest model to fit a full surface
- Forward volatilities that can be locked

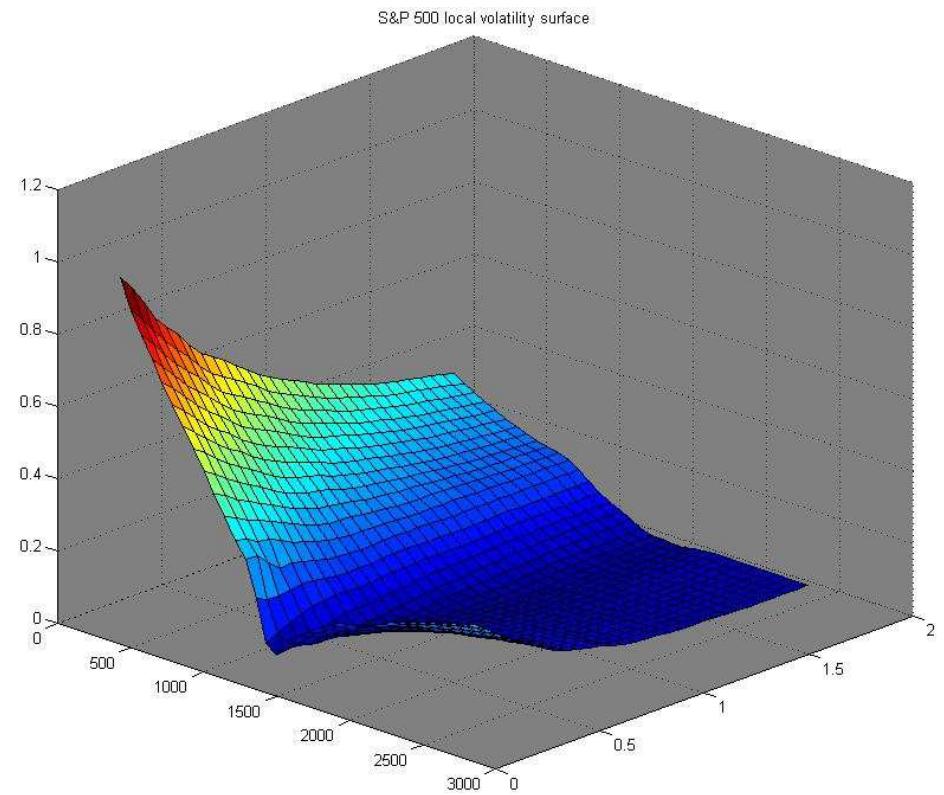
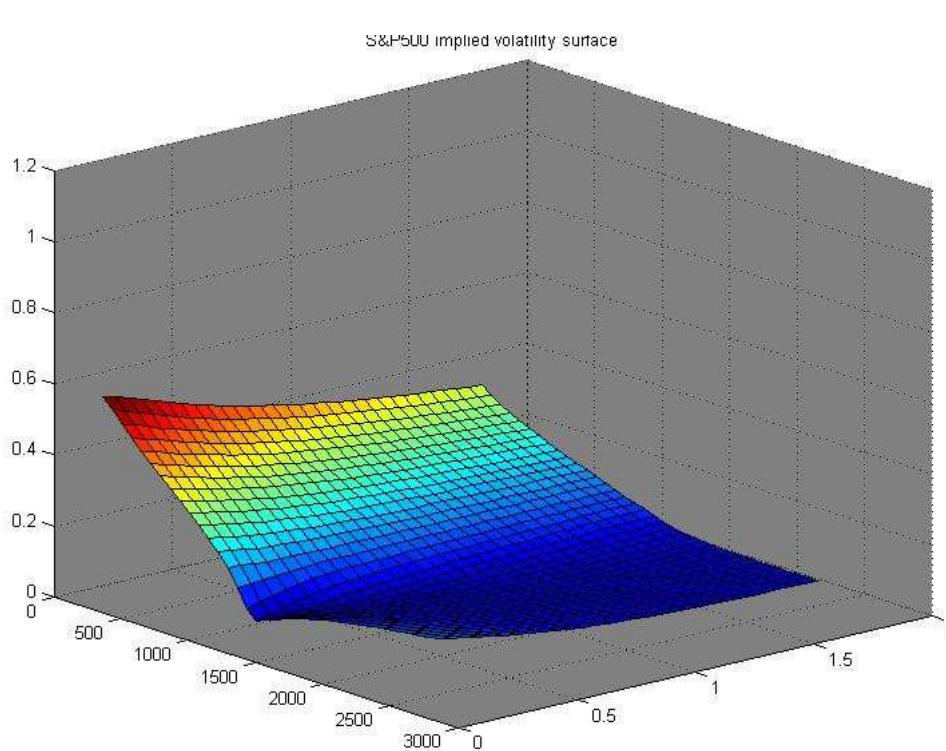
$$\frac{dS}{S} = (r - q) dt + \sigma(S, t) dW$$

$$\frac{\partial C}{\partial T} = \frac{\sigma^2(K, T)}{2} K^2 \frac{\partial^2 C}{\partial K^2} - (r - q)K \frac{\partial C}{\partial K} - q \cdot C$$

Summary of LVM

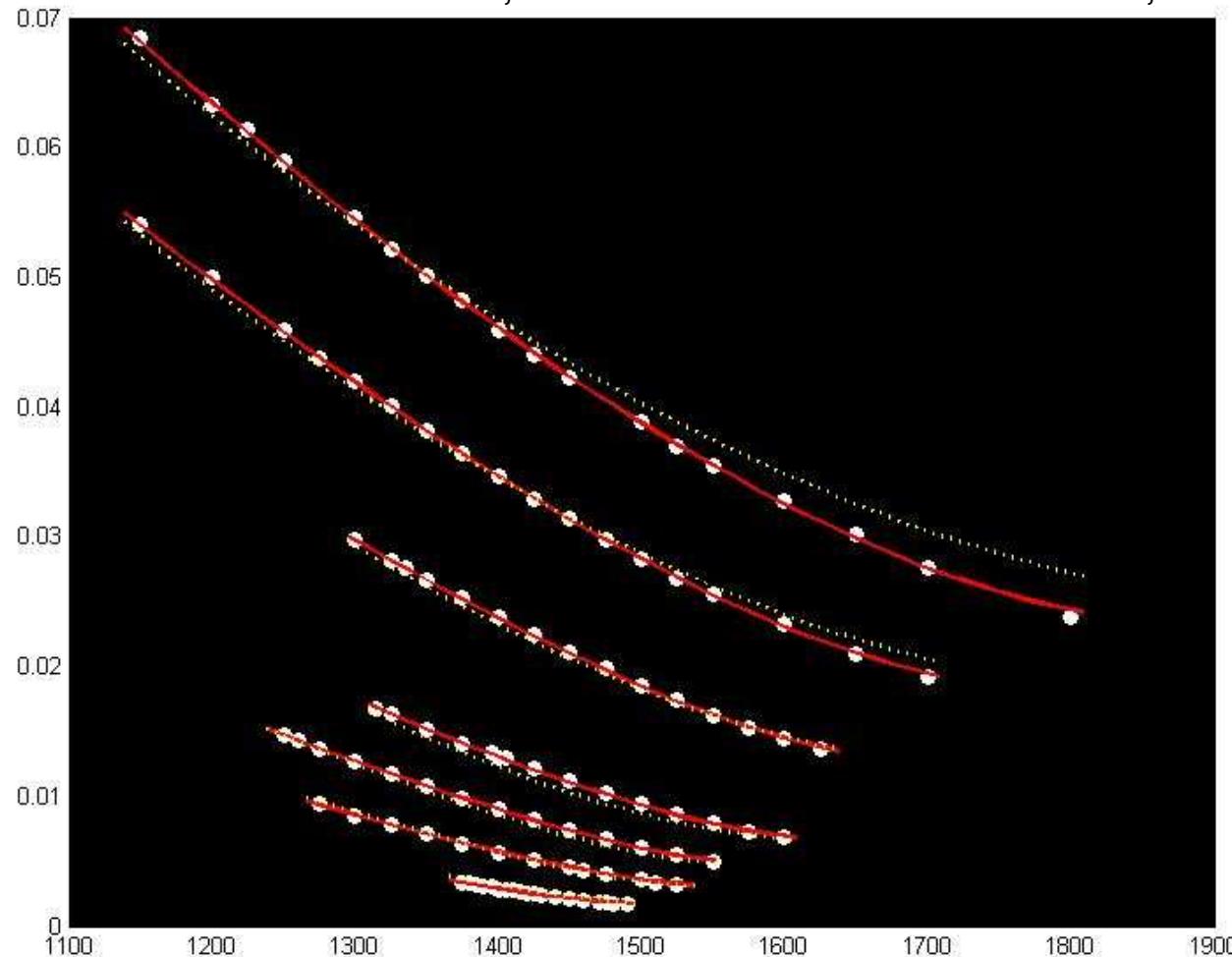
- Simplest model that fits vanillas
- In Europe, second most used model (after Black-Scholes) in Equity Derivatives
- Local volatilities: fwd vols that can be locked by a vanilla PF
- Stoch vol model calibrated $E[\sigma_t^2 | S_t = S] = \sigma_{loc}^2(S, t)$
- If no jumps, deterministic implied vols => LVM

S&P500 implied and local vols



S&P 500 Fit

Cumulative variance as a function of strike. One curve per maturity.
Dotted line: Heston, Red line: Heston + residuals, bubbles: market

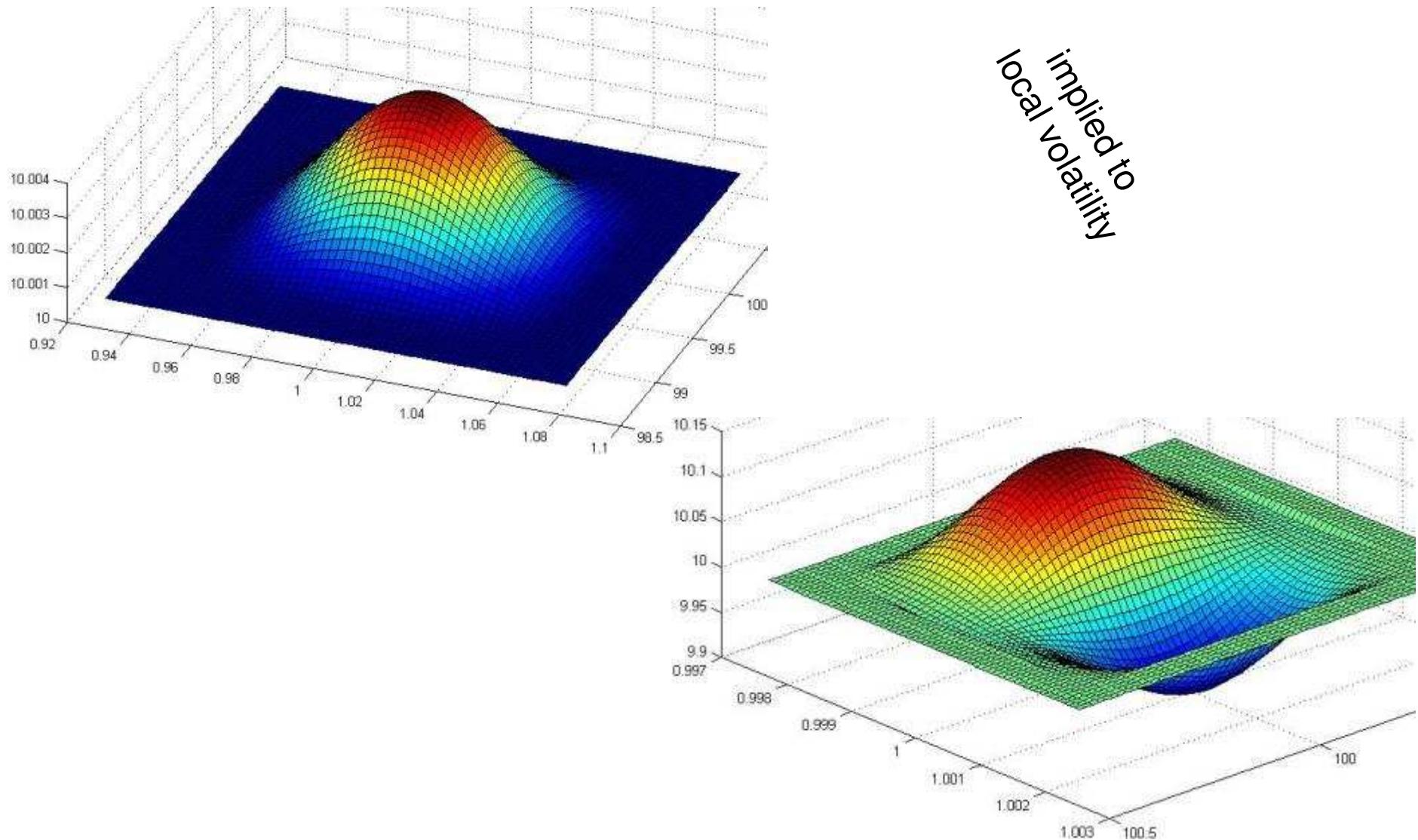


RMS in bps
BS: 305
Heston: 47
H+residuals: 7

Hedge within/outside LVM

- 1 Brownian driver => complete model
- Within the model, perfect replication by Delta hedge
- Hedge outside of (or against) the model: hedge against volatility perturbations
- Leads to a decomposition of Vega across strikes and maturities

Implied and Local Volatility Bumps



P&L from Delta hedging

Assume $r = q = 0$, $dx_t = \sqrt{v_0(x_t, t)} dW_t$.

For $g \in \Lambda_T$, define $f \in \Lambda$ by $f(X_t) \equiv E^{\mathcal{Q}_{v_0}}[g(X_T) | X_t]$

By functional PDE, $\Delta_t f(X_t) + \frac{1}{2} v_0(x_t, t) \Delta_{xx} f(X_t) = 0$

If y follows $dy_t = \sqrt{v_t} dW_t$ with $y_0 = x_0$, by the functional Itô formula,

$$\begin{aligned} g(Y_T) &= f(Y_T) = f(Y_0) + \int_0^T \Delta_x f(Y_t) dy_t + \int_0^T \Delta_t f(Y_t) dt + \frac{1}{2} \int_0^T v_t \Delta_{xx} f(Y_t) dt \\ &= f(X_0) + \int_0^T \Delta_x f(Y_t) dy_t + \frac{1}{2} \int_0^T (v_t - v_0(y_t, t)) \Delta_{xx} f(Y_t) dt \end{aligned}$$

Model Impact

Recall

Comparing calibrated models

Recall

$$\Pi_g(v) = \Pi_g(v_0) + \frac{1}{2} \int_0^T \int \phi^v(x, t) E^{\mathcal{Q}_v} [(v_t - v_0(x, t)) \Delta_{xx} f(X_t) \mid x_t = x] dx dt$$

For a knock - out Barrier option,

$$\begin{aligned} \Pi_g(v) - \Pi_g(v_0) &= \frac{1}{2} \int_0^T \int \phi^v(x, t) E^{\mathcal{Q}_v} [(v_t - v_0(x, t)) \Delta_{xx} f(X_t) \mid x_t = x] dx dt \\ &= \frac{1}{2} \int_0^T \int \phi^v(x, t) P_{alive}(x, t) E^{\mathcal{Q}_v} [(v_t - v_0(x, t)) \Delta_{xx} f(X_t) \mid x_t = x, alive] dx dt \\ &= \frac{1}{2} \int_0^T \int \phi_{alive}^v(x, t) \frac{\partial^2 f_{alive}(x, t)}{\partial x^2} E^{\mathcal{Q}_v} [(v_t - v_0(x, t)) \mid x_t = x, alive] dx dt \end{aligned}$$

If v and v_0 are calibrated on the same vanilla options, $E^{\mathcal{Q}_v}[v_t \mid x_t = x] = v_0(x, t)$

It amounts to evaluate the impact of conditioning by "alive":

$$E^{\mathcal{Q}_v}[v_t \mid x_t = x, alive] - E^{\mathcal{Q}_v}[v_t \mid x_t = x]$$

Volatility Expansion in LVM

In general,

$$\Pi_g(v) = \Pi_g(v_0) + \frac{1}{2} \int_0^T \int \phi^v(x, t) E^{Q_v} [(v_t - v_0(x, t)) \Delta_{xx} f(X_t) \mid x_t = x] dx dt$$

In the case where v is a LVM of the form $v_0 + u : dx_t = \sqrt{v_0(x_t, t) + u(x_t, t)} dW_t$

$$\begin{aligned} \Pi_g(v_0 + u) &= \Pi_g(v_0) + \frac{1}{2} \int_0^T \int \phi^{v_0+u}(x, t) u(x, t) E^{Q_{v_0+u}} [\Delta_{xx} f(X_t) \mid x_t = x] dx dt \\ &= \Pi_g(v_0) + \int_0^T \int m(x, t) u(x, t) dx dt \end{aligned}$$

where $m(x, t) \equiv \frac{1}{2} \phi^{v_0+u}(x, t) E^{Q_{v_0+u}} [\Delta_{xx} f(X_t) \mid x_t = x]$

Fréchet Derivative in LVM

In particular,

$$\Pi_g(v_0 + \varepsilon u) = \Pi_g(v_0) + \frac{\varepsilon}{2} \int_0^T \int \phi^{v_0 + \varepsilon u}(x, t) u(x, t) E^{\mathcal{Q}_{v_0 + \varepsilon u}} [\Delta_{xx} f(X_t) \mid x_t = x] dx dt$$

The Fréchet derivative in the direction of u satisfies :

$$\begin{aligned} < \nabla_v \Pi_g, u > &\equiv \lim_{\varepsilon \rightarrow 0} \frac{\Pi_g(v_0 + \varepsilon u) - \Pi_g(v_0)}{\varepsilon} \\ &= \frac{1}{2} \int_0^T \int \phi^{v_0}(x, t) u(x, t) E^{\mathcal{Q}_{v_0}} [\Delta_{xx} f(X_t) \mid x_t = x] dx dt \\ &= \int_0^T \int m(x, t) u(x, t) dx dt \end{aligned}$$

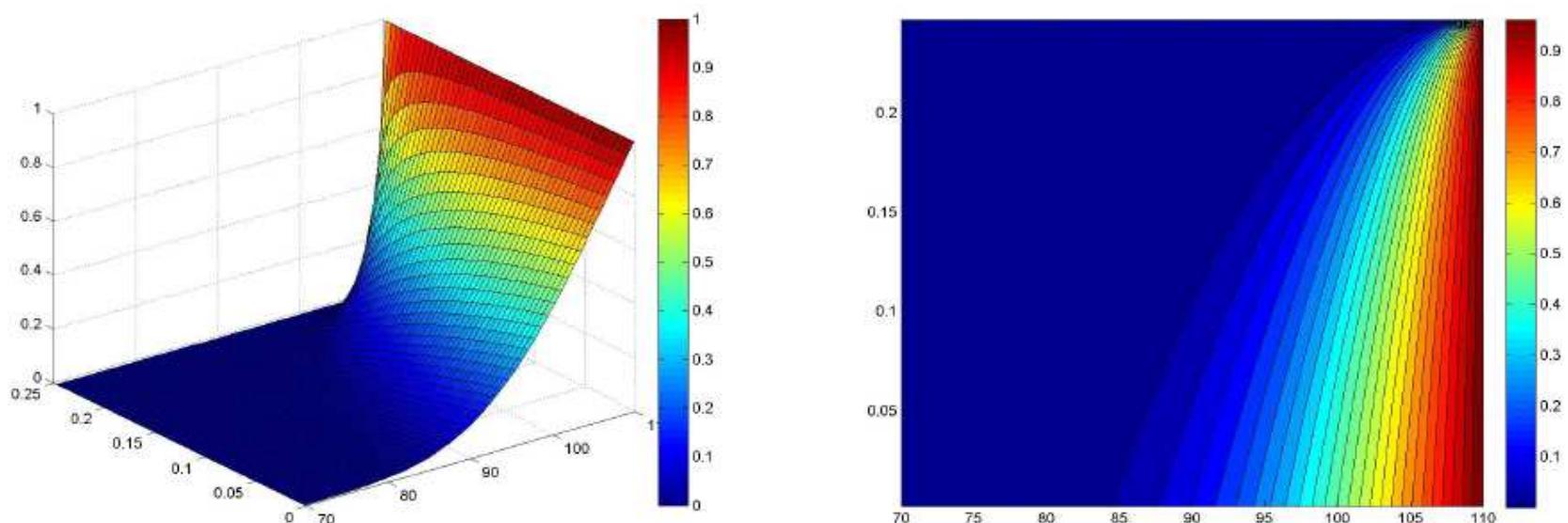
where

$$m(x, t) \equiv \frac{1}{2} \phi^{v_0}(x, t) E^{\mathcal{Q}_{v_0}} [\Delta_{xx} f(X_t) \mid x_t = x]$$

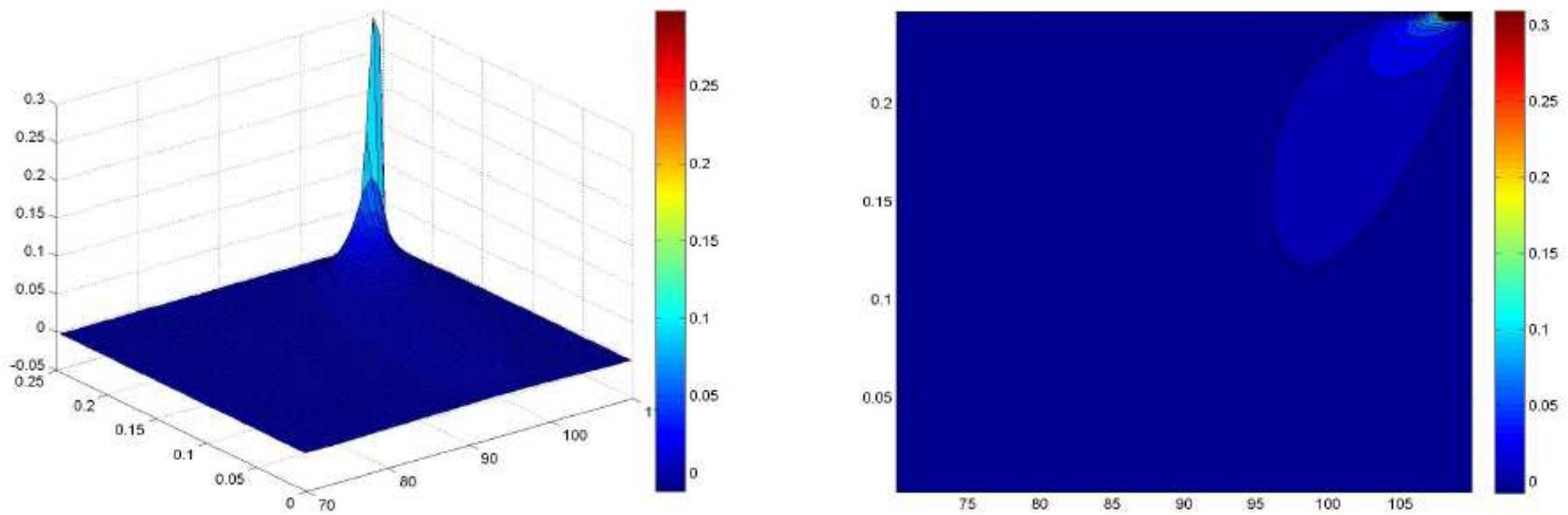
is the sensitivity of g to the local variance at (x, t) (Fréchet derivative)

One Touch Option - Price

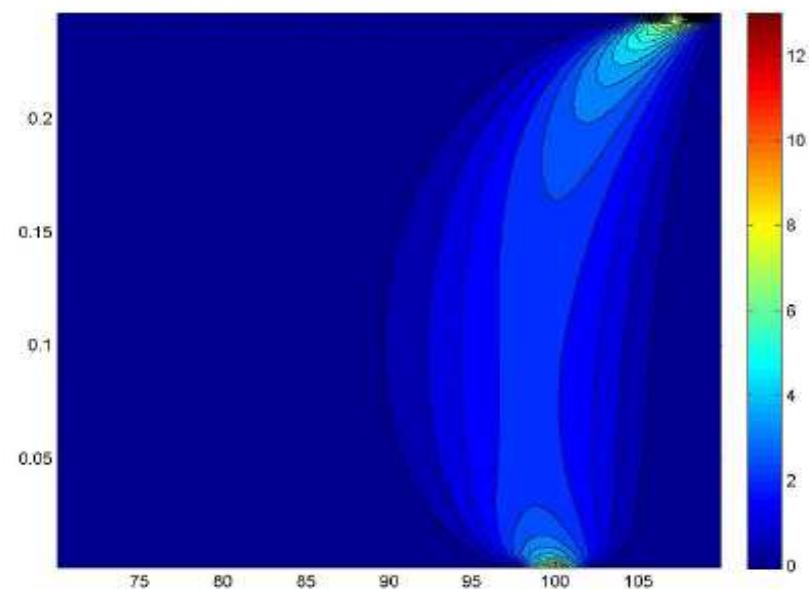
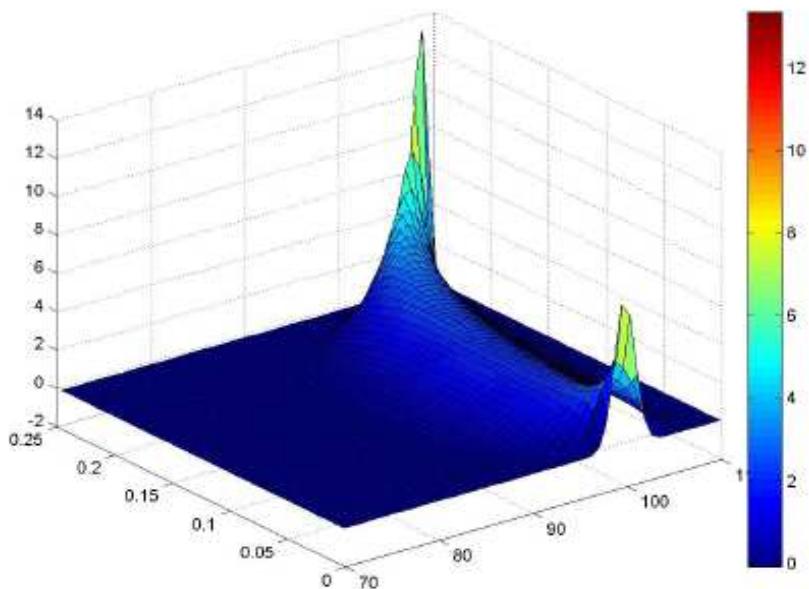
Black-Scholes model $S_0=100$, $H=110$, $\sigma=0.25$, $T=0.25$



One Touch Option - Γ

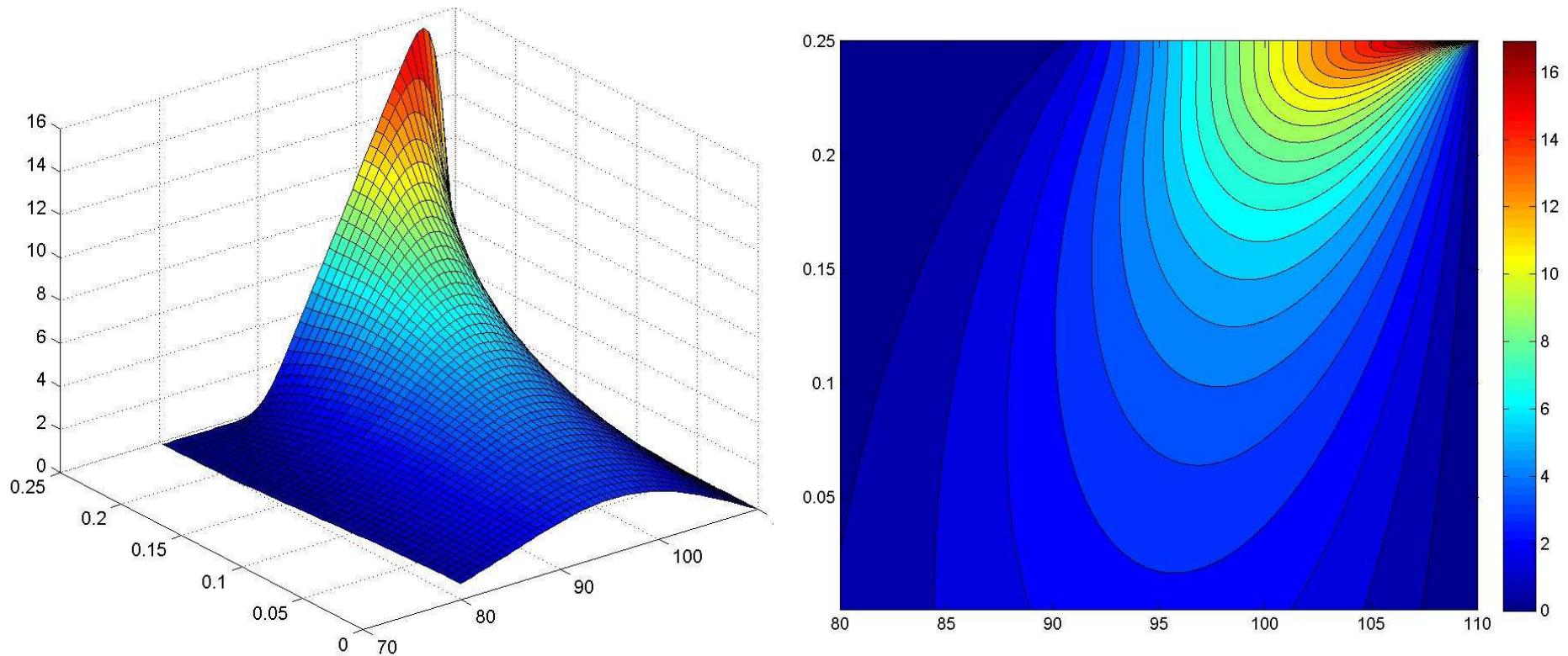


$$O.T.: m(S, t) = \frac{1}{2} \Gamma \cdot \varphi \cdot P$$

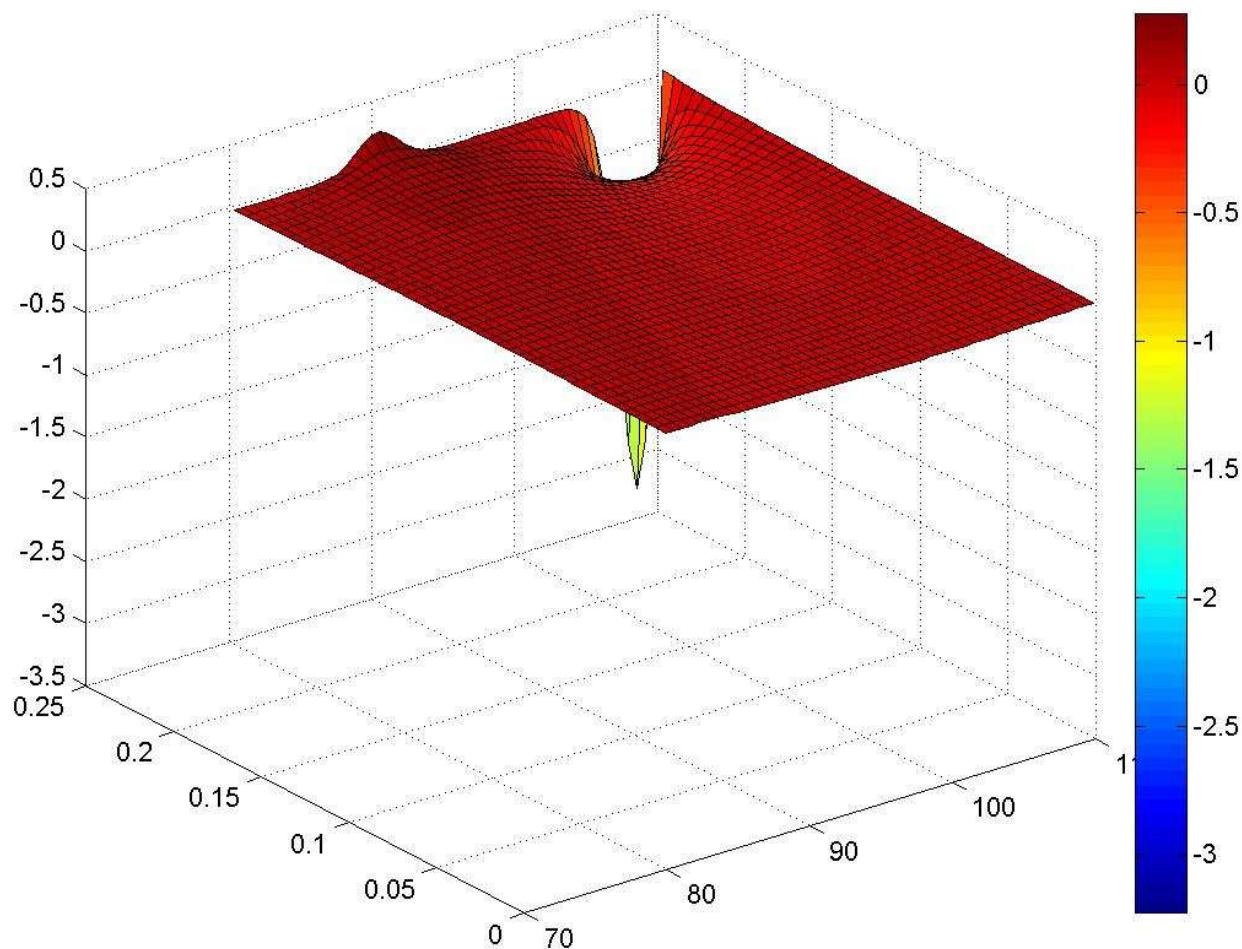


Up-Out Call - Price

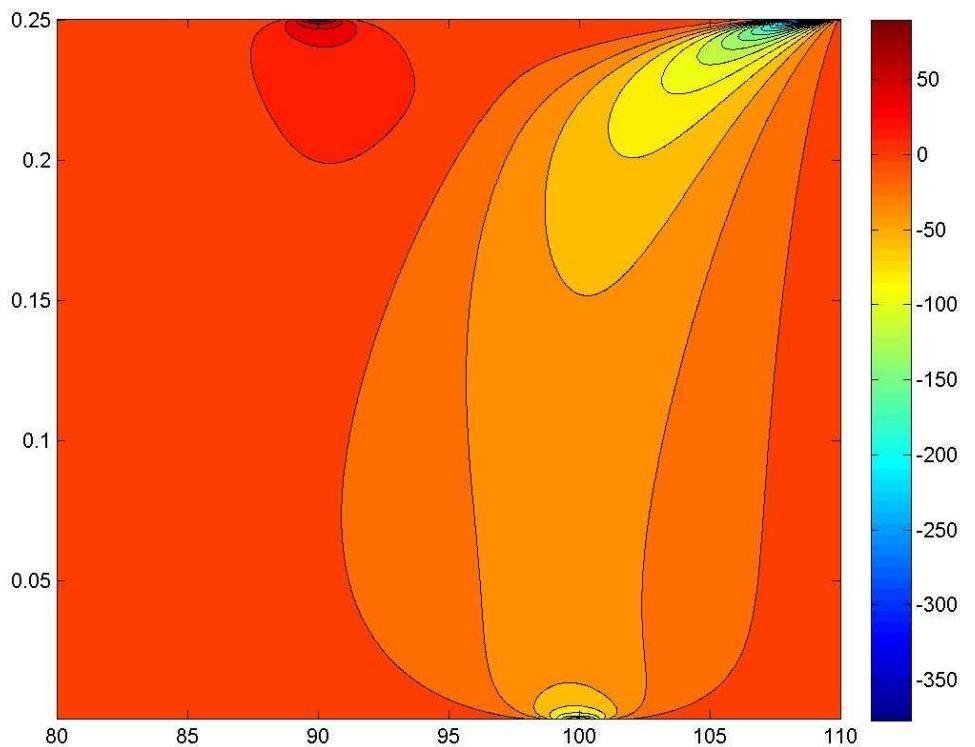
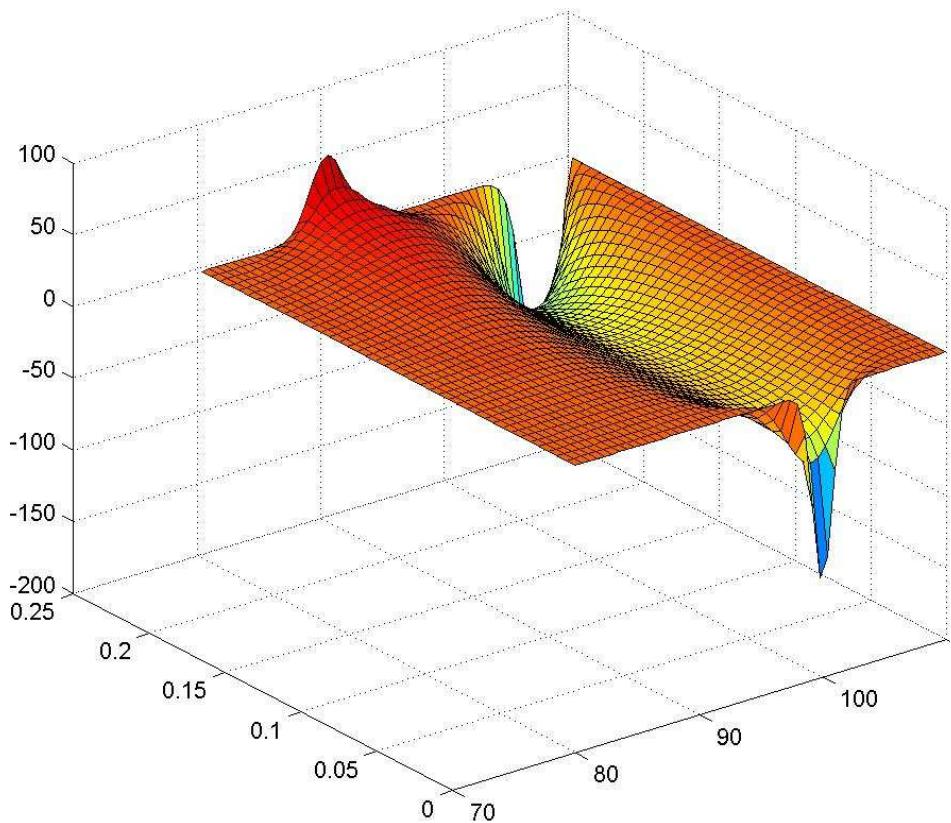
Black-Scholes model $S_0=100$, $H=110$, $K=90$, $\sigma=0.25$, $T=0.25$



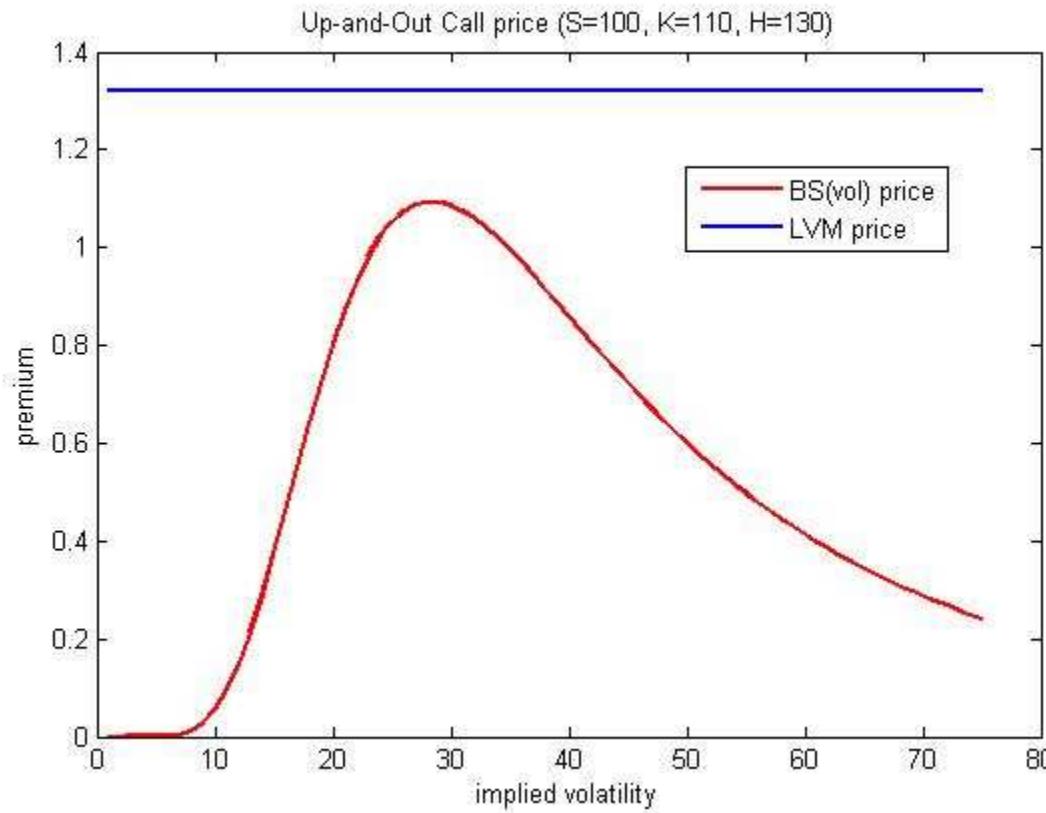
Up-Out Call - Γ



$$UOC : m(S, t) = \frac{1}{2} \Gamma \cdot \varphi \cdot P$$



Black-Scholes/LVM comparison



In this case, no volatility input of the Black - Scholes enables to reach the LVM price.

Vanilla hedging portfolio I

Recall $m(x, t) \equiv \frac{1}{2} \phi^{v_0}(x, t) E^{\mathcal{Q}_{v_0}} [\Delta_{xx} f(X_t) \mid x_t = x]$ is the sensitivity of Π_g to the local variance at (x, t) .

Portfolio $PF = \iint \alpha(K, T) C_{K,T} dK dT$ of vanillas hedges all small volatility moves if and only if for all (x, t)

$$\frac{\partial^2 PF(x, t)}{\partial x^2} = E^{\mathcal{Q}_{v_0}} [\Delta_{xx} \Pi_{PF}(X_t) \mid x_t = x] = E^{\mathcal{Q}_{v_0}} [\Delta_{xx} f(X_t) \mid x_t = x] \equiv h(x, t)$$

How can we get the function $\alpha(K, T)$?

Vanilla hedging portfolios II

a) For $k(x, t)$, we define $L(k) \equiv \frac{\partial k}{\partial t} + \frac{1}{2} \frac{\partial^2 (v_0 k)}{\partial x^2}$

For a call $C_{K,T}$, $L(\frac{\partial^2 C_{K,T}}{\partial x^2}) = 0$ with boundary condition $\frac{\partial^2 C_{K,T}}{\partial x^2}(x, T) = \delta_K(x)$

b) $PF \equiv \iint \alpha(K, T) C_{K,T} dK dT \Rightarrow L(\frac{\partial^2 PF}{\partial x^2})(x, t) = -\alpha(x, t)$

$k(x, t) \equiv E^{\mathcal{Q}_{v_0}} [\Delta_{xx} f(X_t) - \frac{\partial^2 PF}{\partial x^2}(x, t) \mid x_t = x] = h(x, t) - \frac{\partial^2 PF}{\partial x^2}(x, t)$ with $h(x, t) \equiv E^{\mathcal{Q}_{v_0}} [\Delta_{xx} f(X_t) \mid x_t = x]$

Thus, $L(k) = L(h) + \alpha$.

c) If we take $\alpha = -L(h) = -\frac{\partial h}{\partial t} - \frac{1}{2} \frac{\partial^2 (v_0 h)}{\partial x^2}$ then $L(k) = 0$ with $k(x, T) = 0 \Rightarrow k \equiv 0$

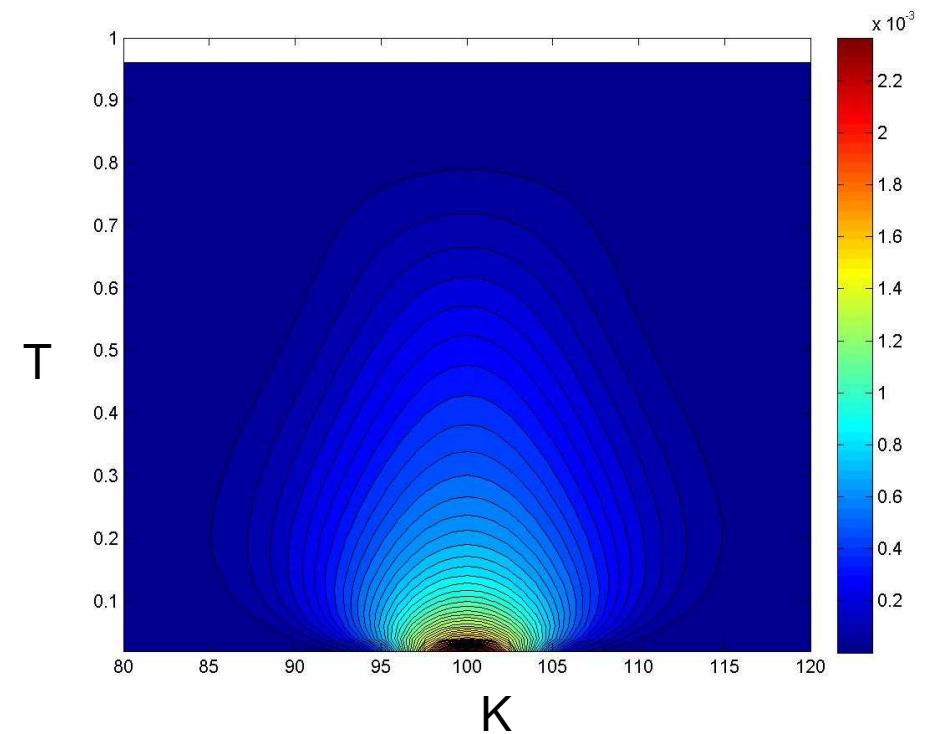
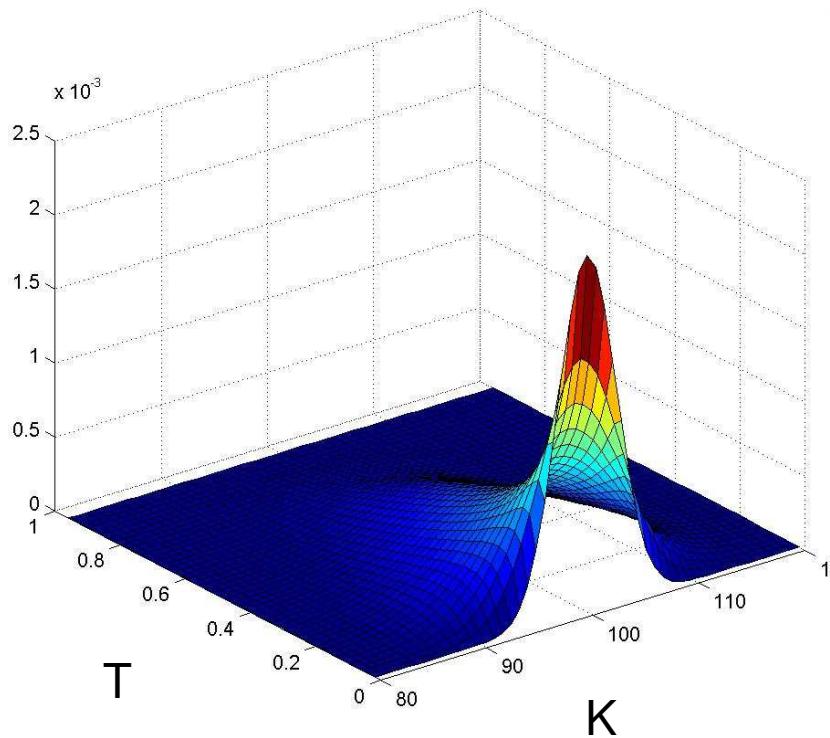
$\forall x, t$, $E^{\mathcal{Q}_{v_0}} [\Delta_{xx} f(X_t) - \frac{\partial^2 PF}{\partial x^2}(x, t) \mid x_t = x] = 0$ and f - PF has no sensitivity to local vol bumps

hence no sensitivity to implied vol bumps.

Example : Asian option

$dx_t = \sqrt{v_0} dW_t$ Pay-off : $g(X_t) = \left(\int_0^T x_t dt - K \right)^+$, $S_0 = K = 100$ volatility $\sqrt{v_0} = 20$ maturity $T = 1$

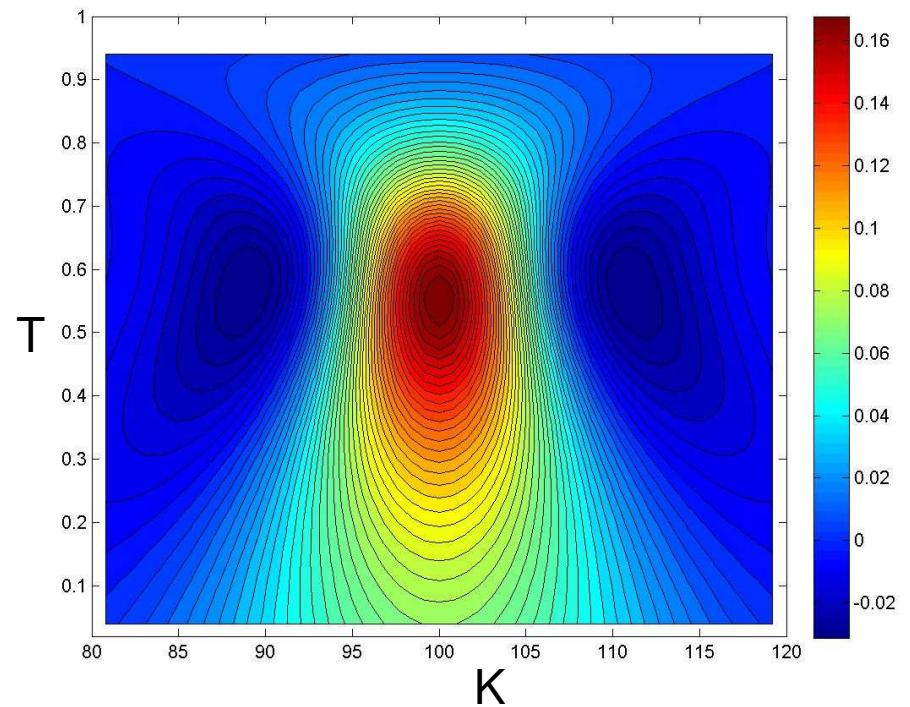
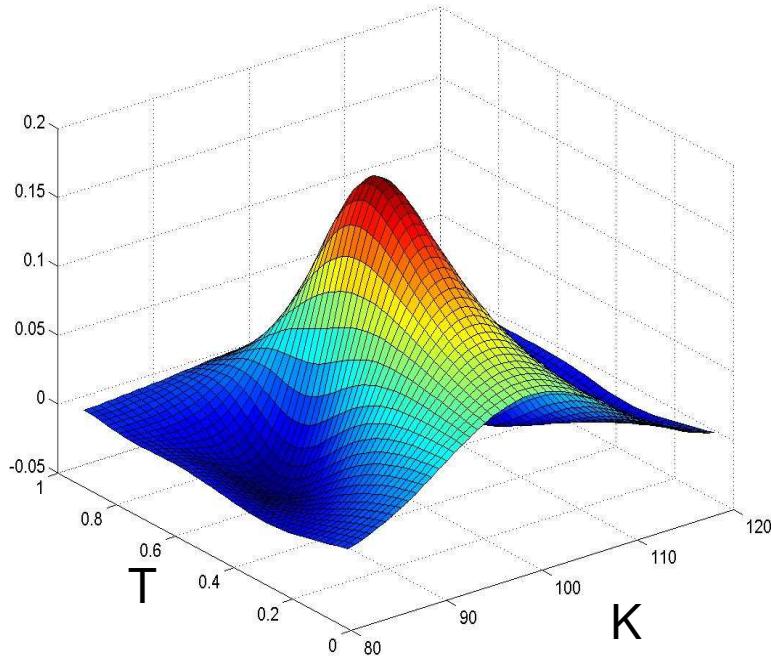
$m(x, t) \equiv \frac{1}{2} \phi^{v_0}(x, t) E^{Q_{v_0}} [\Delta_{xx} f(X_t) \mid x_t = x]$ is the sensitivity of Π_g to the local variance at (x, t) .



Asian Option Hedge

Robust volatility hedge with $PF \equiv \iint \alpha(K, T) C_{K,T} dK dT$

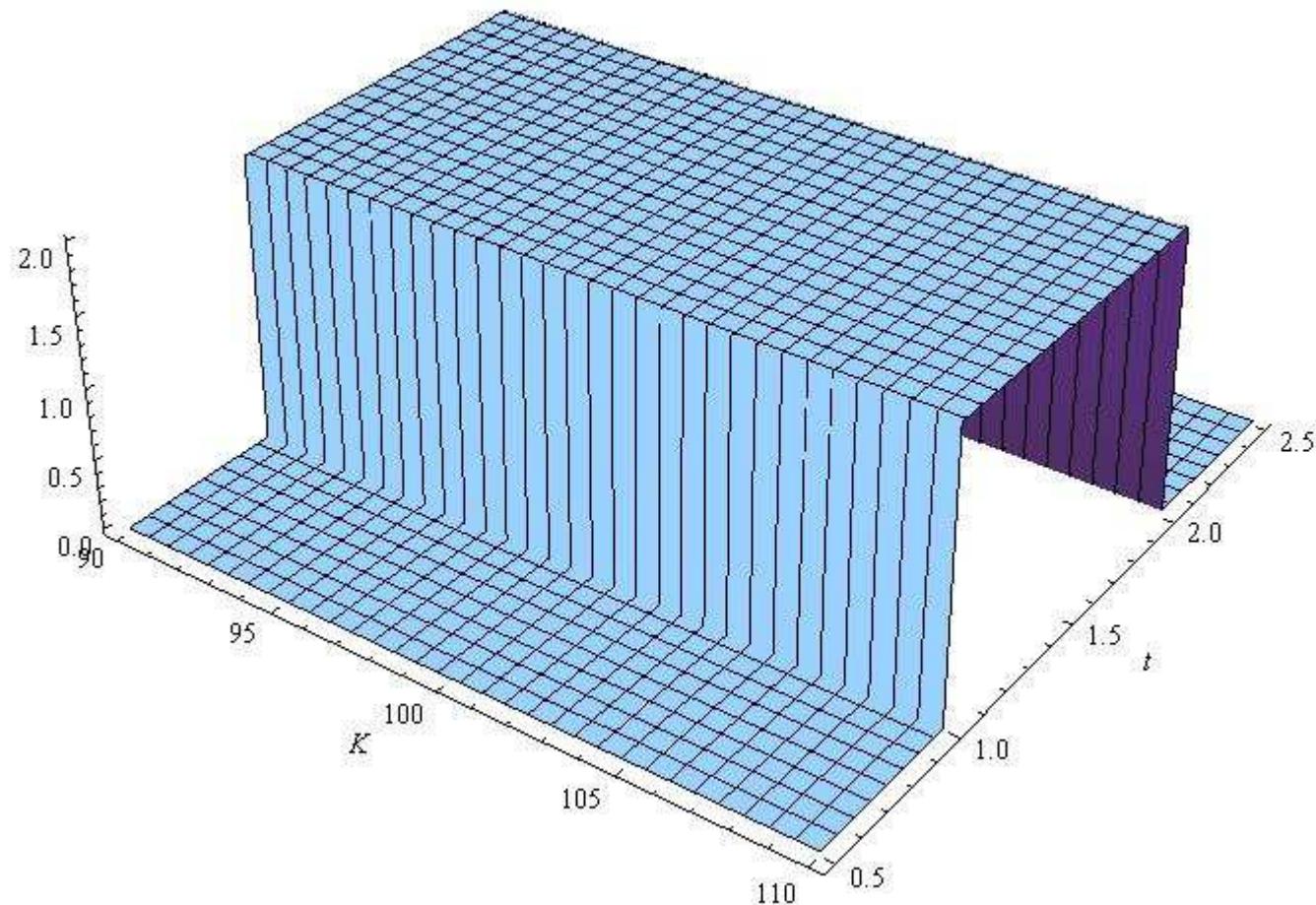
$$\alpha(K, T) = -\left(\frac{\partial h(K, T)}{\partial t} + \frac{1}{2} \frac{\partial^2 (v_0(K, T) h(K, T))}{\partial x^2} \right)$$



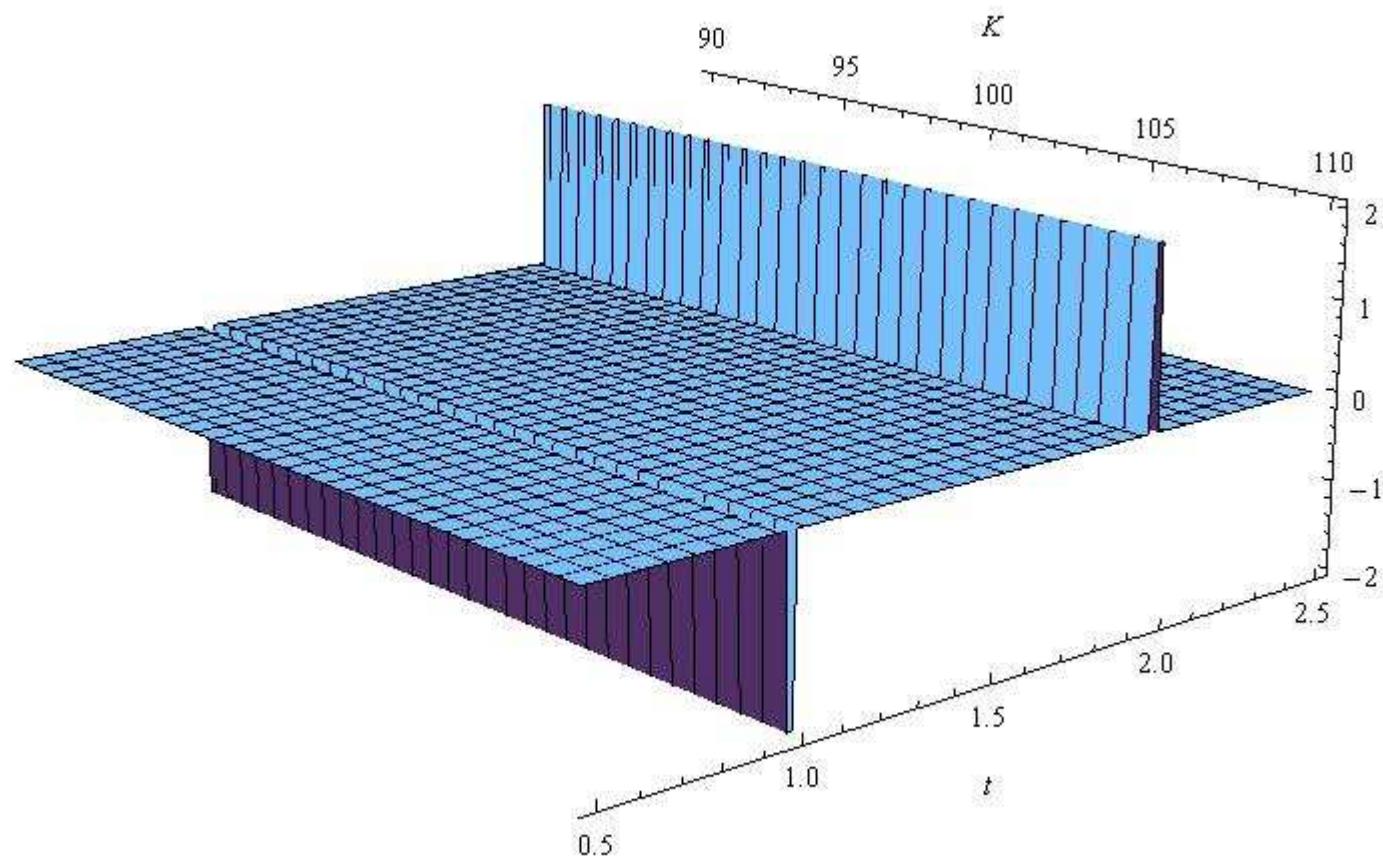
Forward Parabola - Γ

Payoff: $g(X_T) = (S_{T_2} - S_{T_1})^2$

$T_1 = 1, T_2 = 2, S_0 = 100, \sigma = 10$



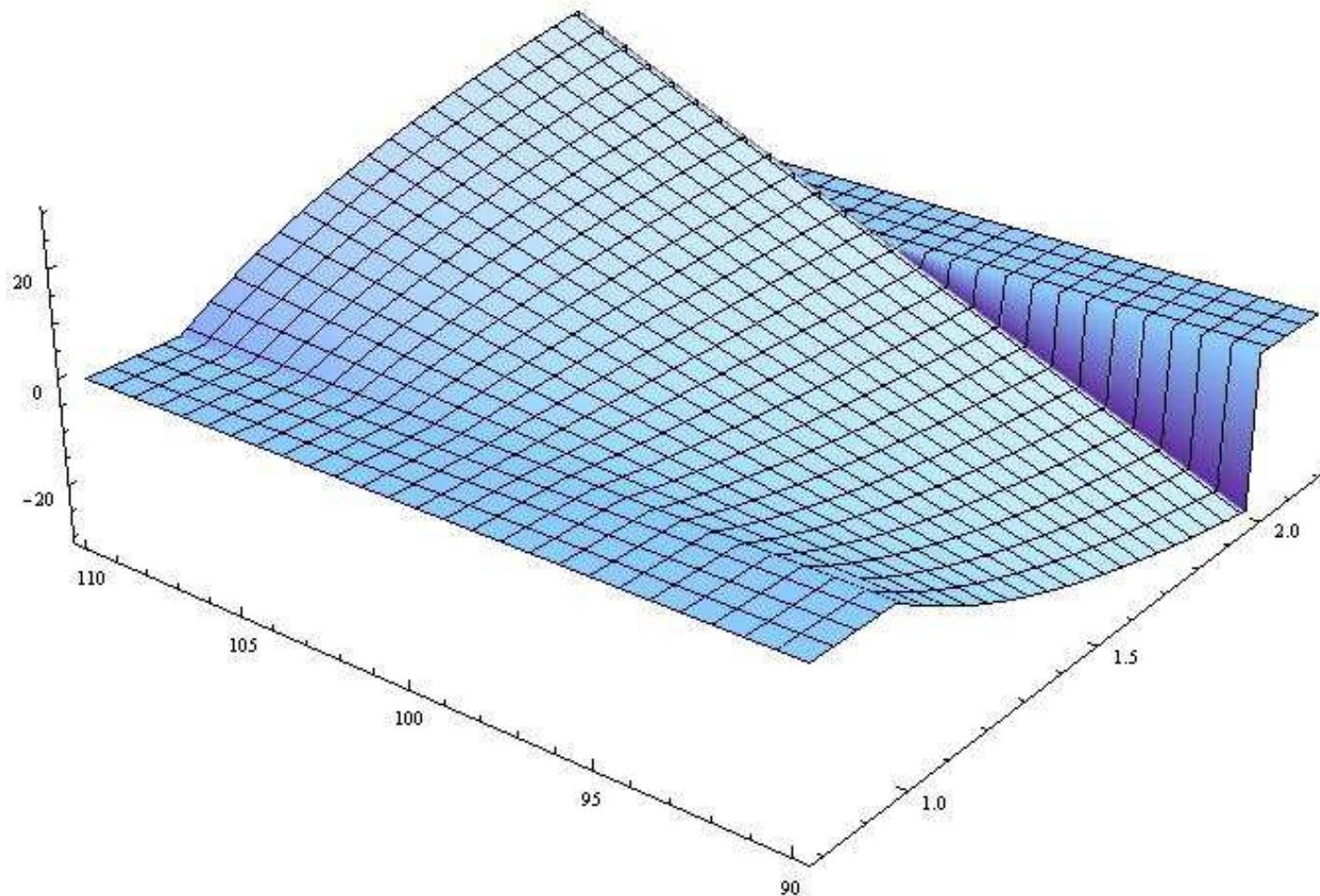
Forward Parabola Hedge



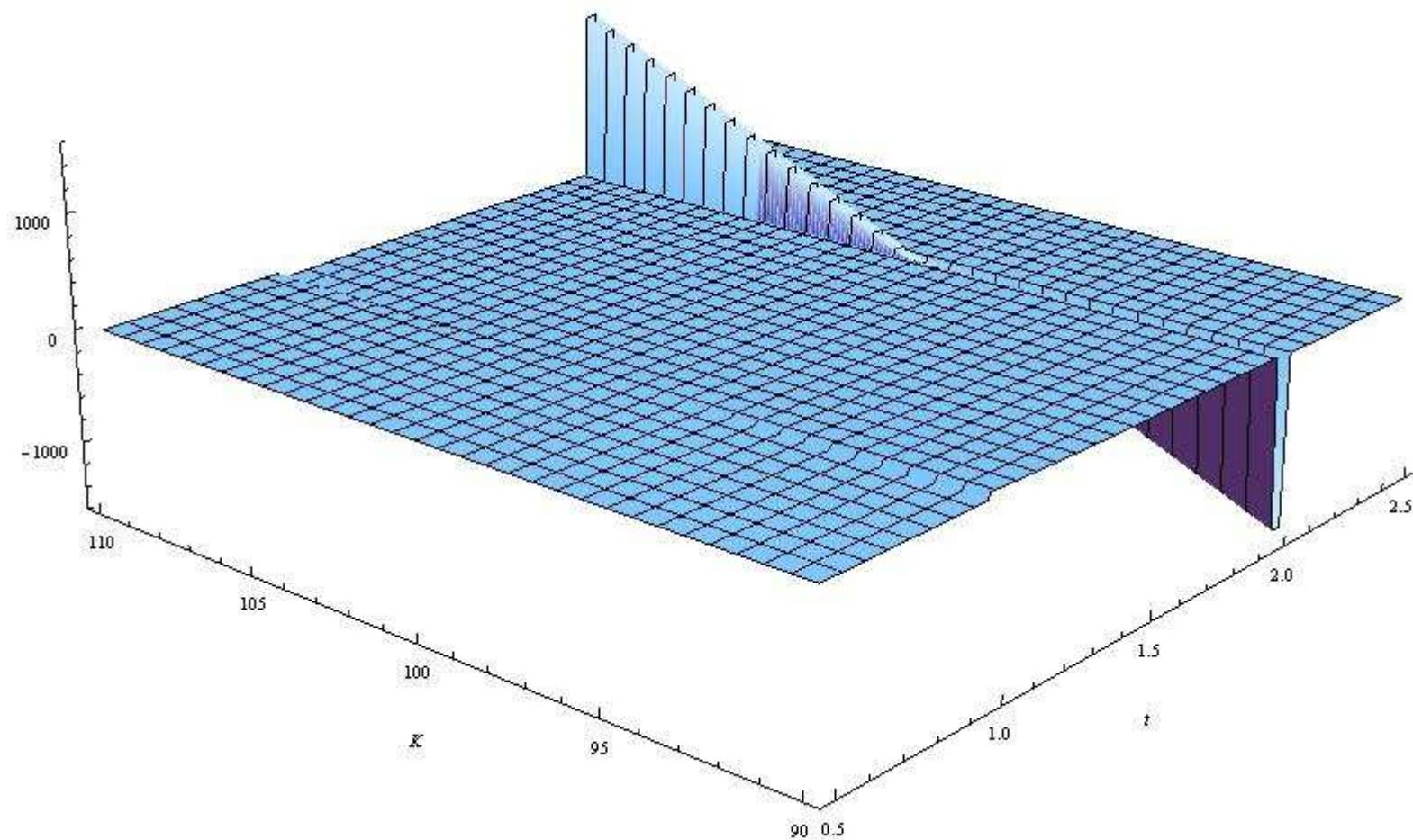
Forward Cube - Γ

Payoff: $g(X_T) = (S_{T_2} - S_{T_1})^3$

$T_1 = 1, T_2 = 2, S_0 = 100, \sigma = 10$



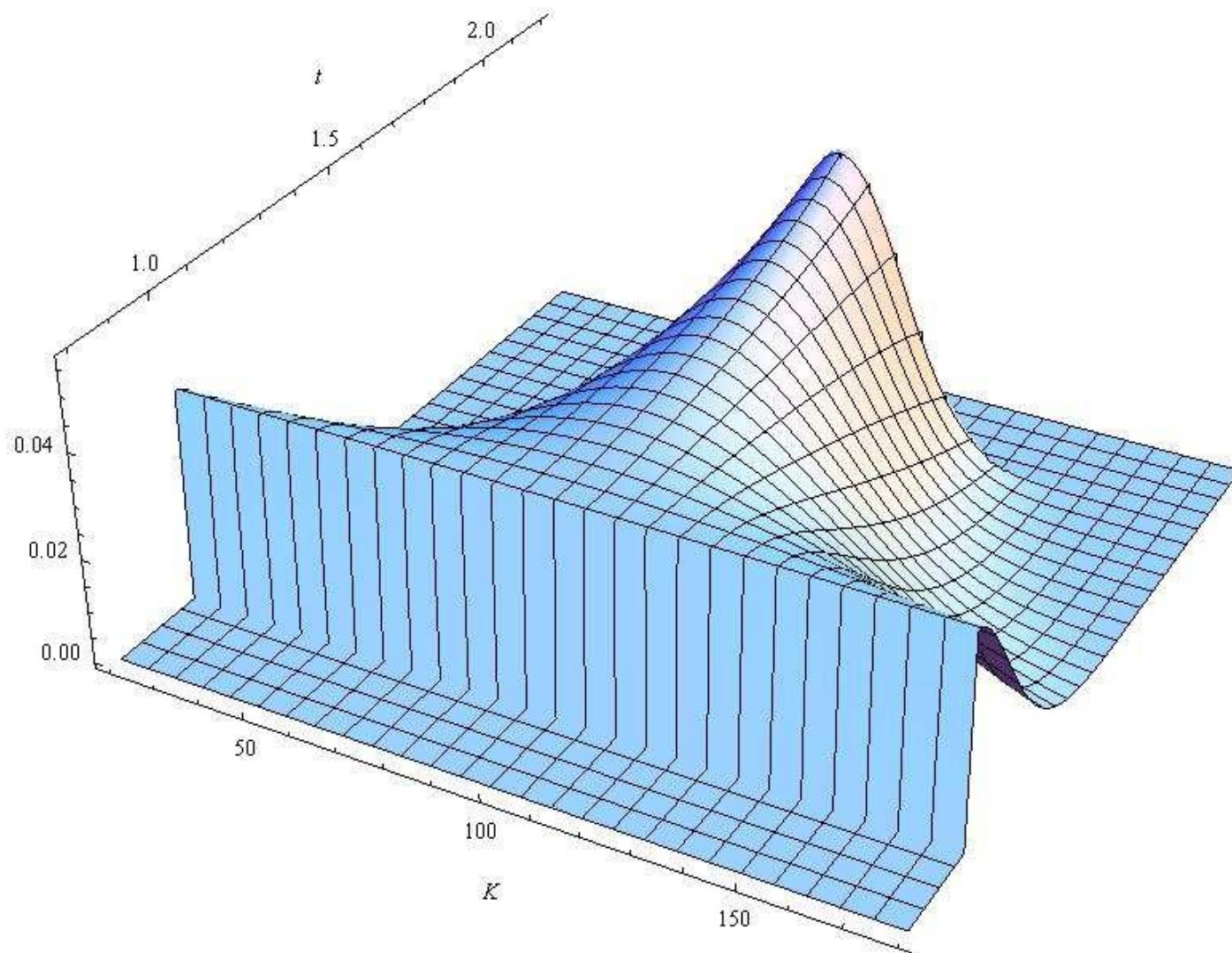
Forward Cube Hedge



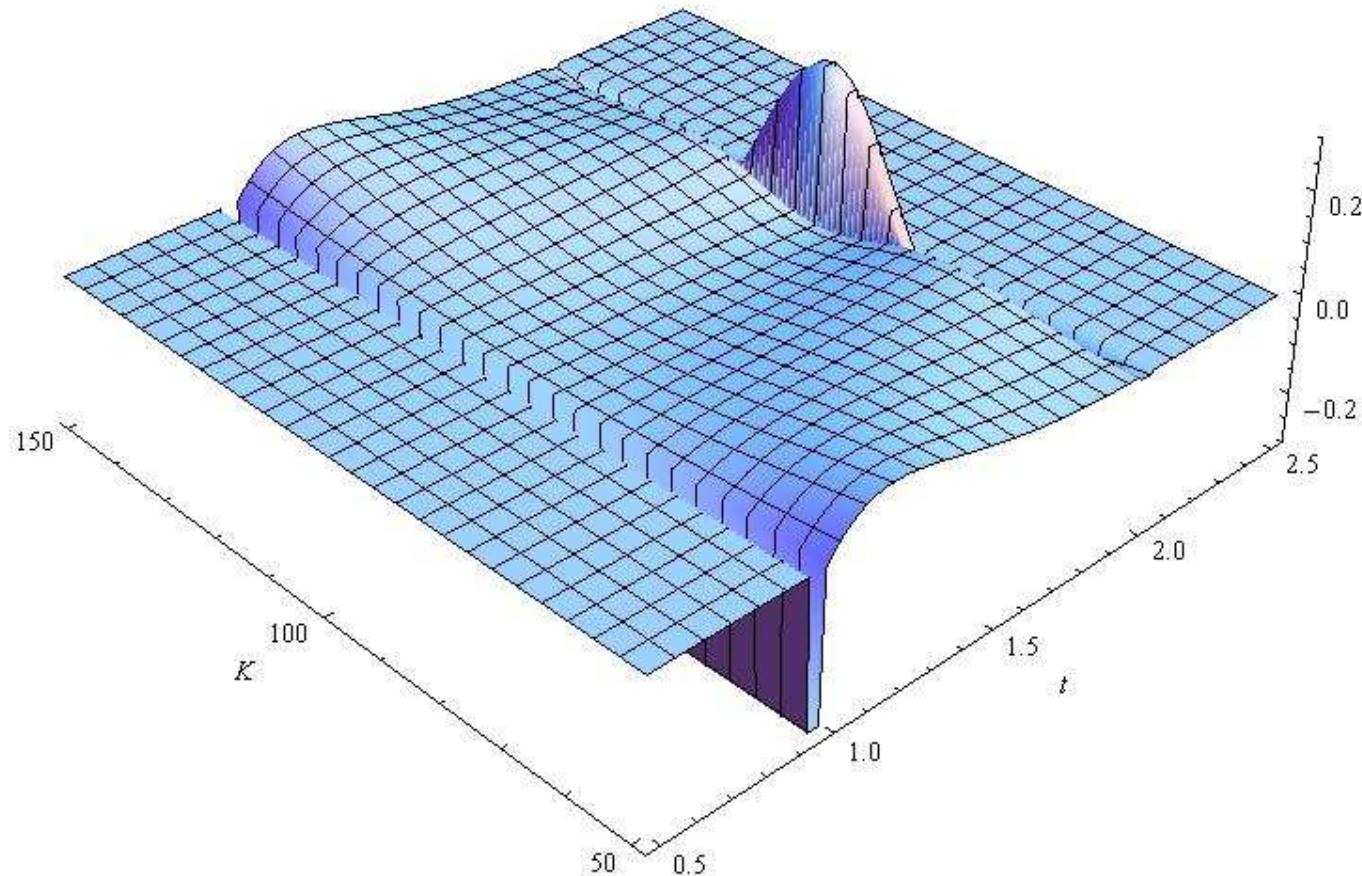
Forward Start - Γ

Payoff $g(X_T) = (S_{T_2} - S_{T_1})^+$

$T_1 = 1, T_2 = 2, S_0 = 100, \sigma = 10$



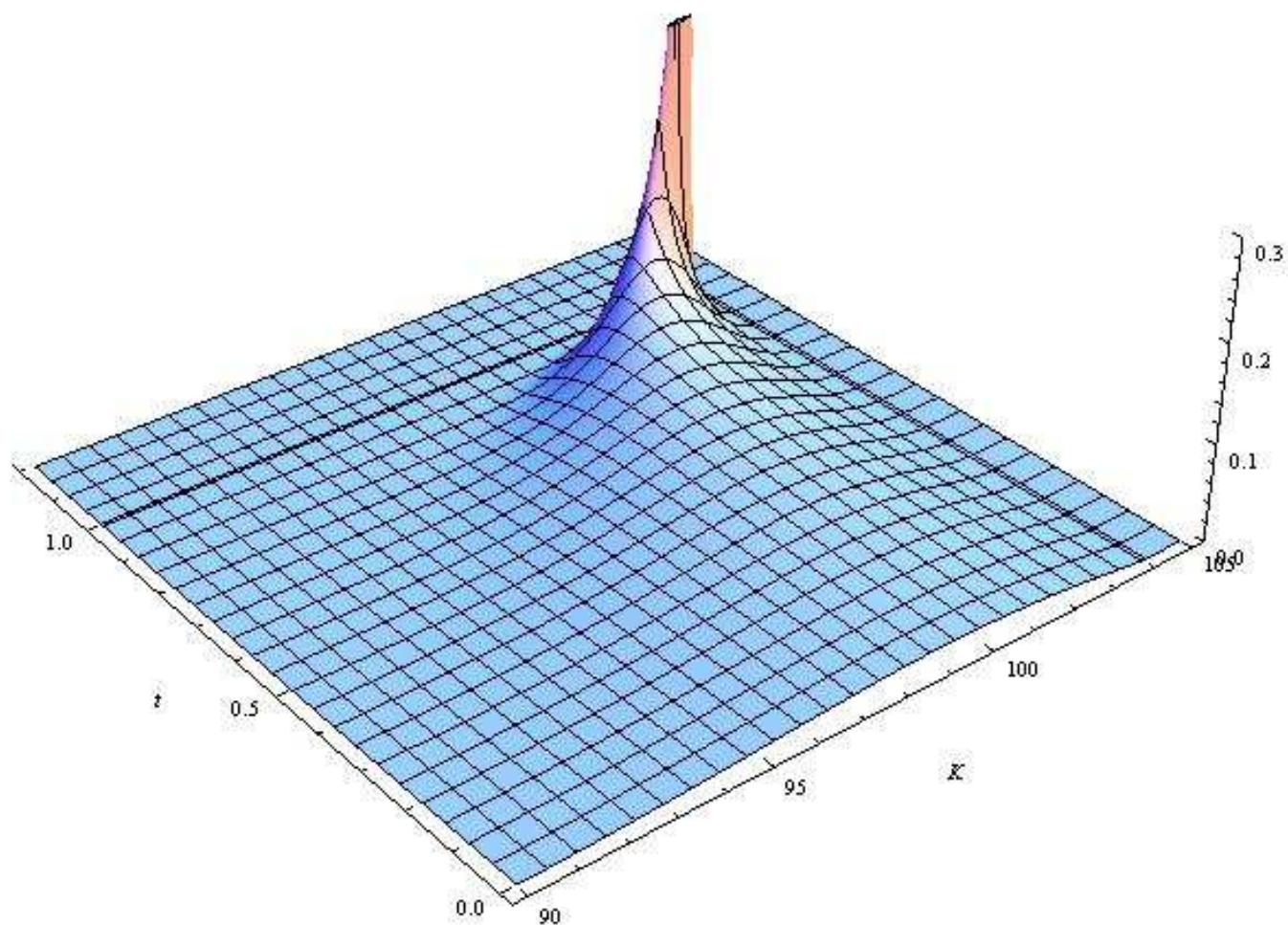
Forward Start Hedge



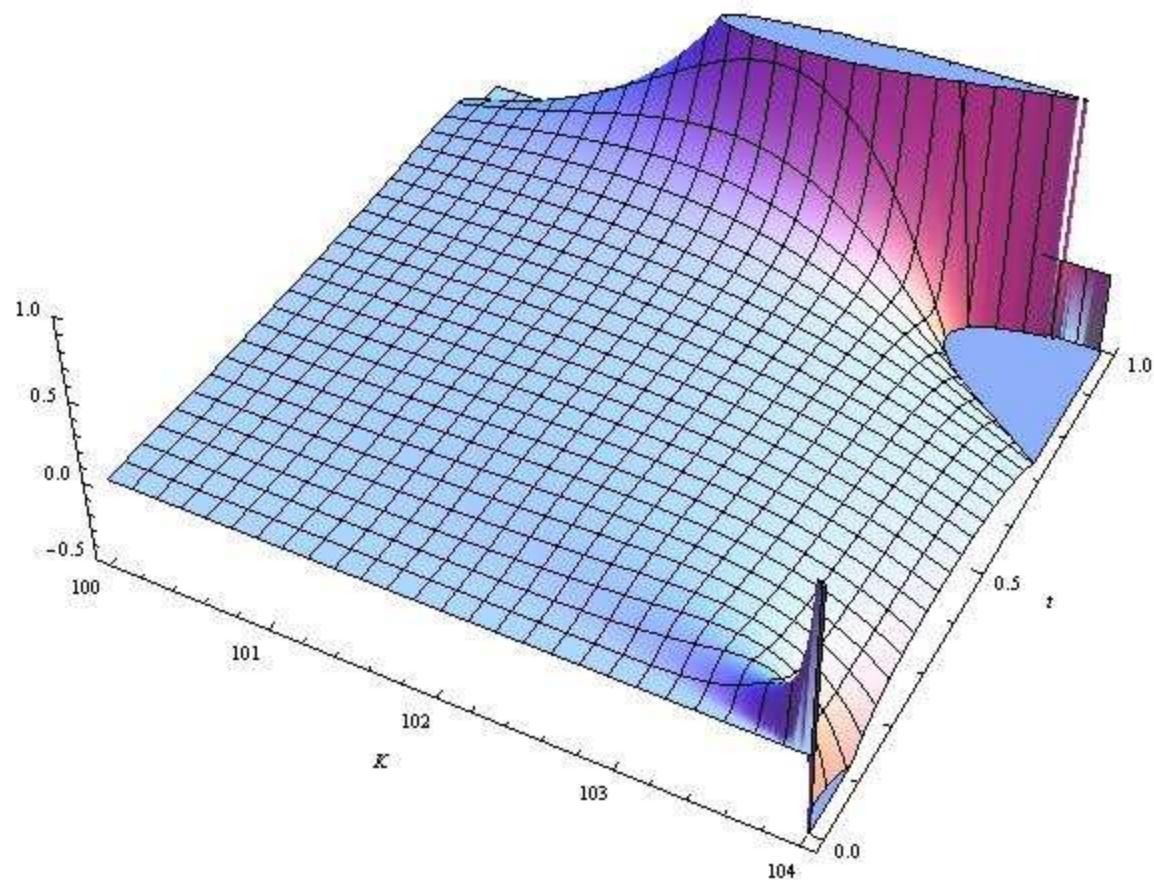
One Touch - Γ

Payoff $g(X_T) = 1_{\{\max S_T > H = 104\}}$

$T_1 = 1, H = 104, S_0 = 100, \sigma = 5$



One Touch Hedge



Link $\Gamma/Vega$

Cancel		In insensitive to
Γ	$\Gamma = 0$	temporary shocks in σ
$Vega$	$\mathbb{E} \left[\int_0^T S_t^2 \Gamma_t dt \right] = 0$	persistent shocks in σ
Superbucket	$\forall S, t \mathbb{E} [\Gamma_t S_t = S] = 0$	<ul style="list-style-type: none"> • any temporary shocks in σ in the future • any small shocks in implied volatility today

Robustness

- Superbucket hedge protects today from first order moves in implied volatilities; what about robustness in the future?
- After delta hedge, the tracking error is
- In the case of discrete hedging in the model or fast mean reversion of vol, variance of TE is proportional to

∞ T

Hedgeability

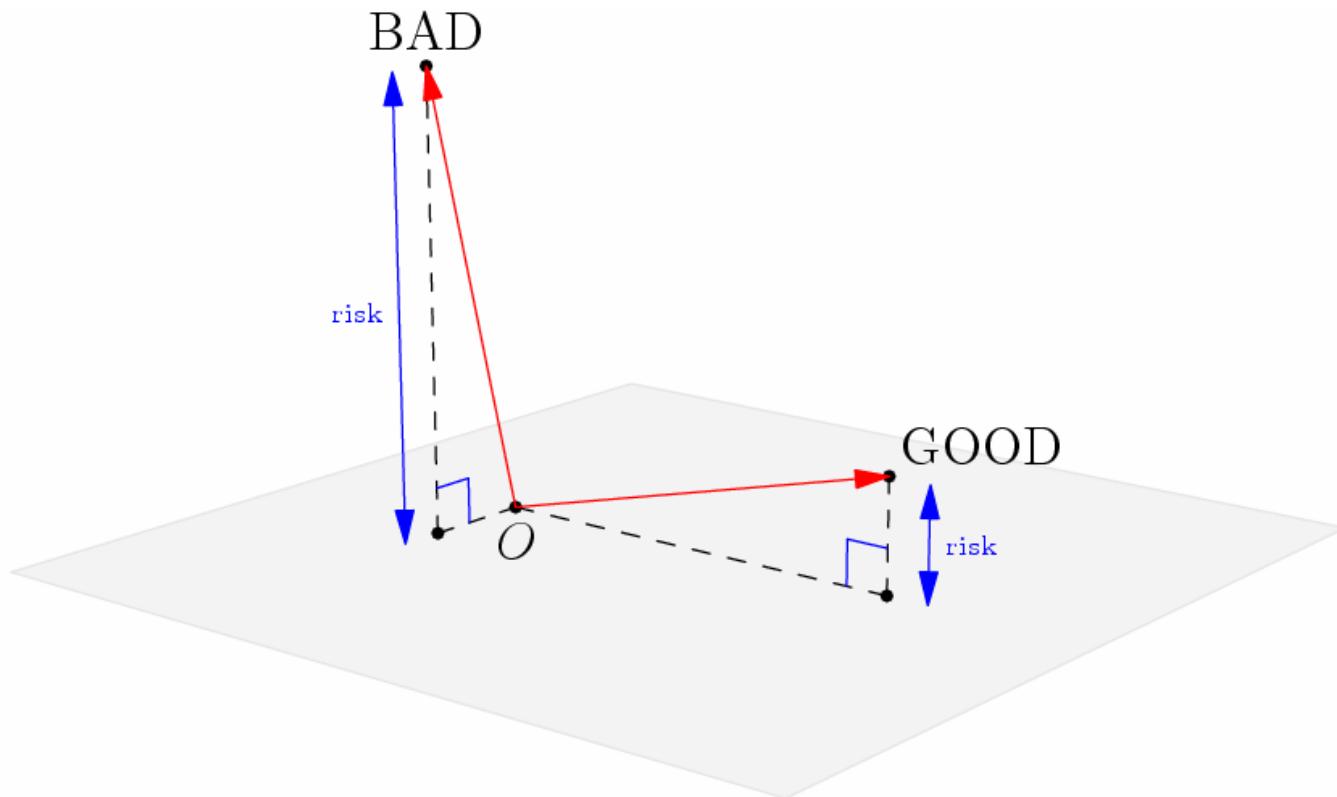
- The optimal vanilla hedge PF minimizes

and thus coincides with the Superbucket Hedge as

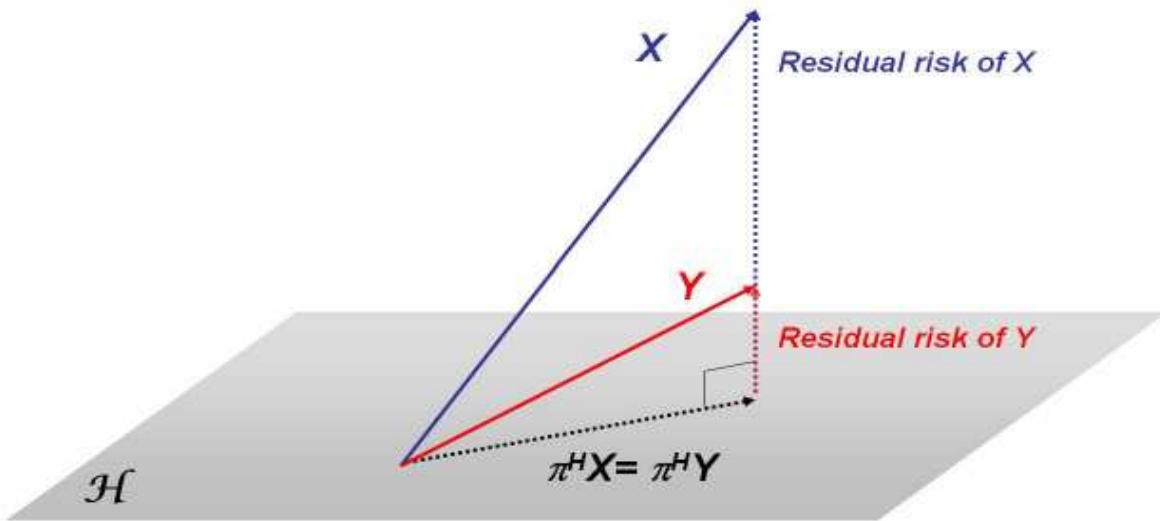
- The residual risk is

- The hedgeability of an option is linked to the dependency of the functional gamma on the path

The Geometry of Hedging



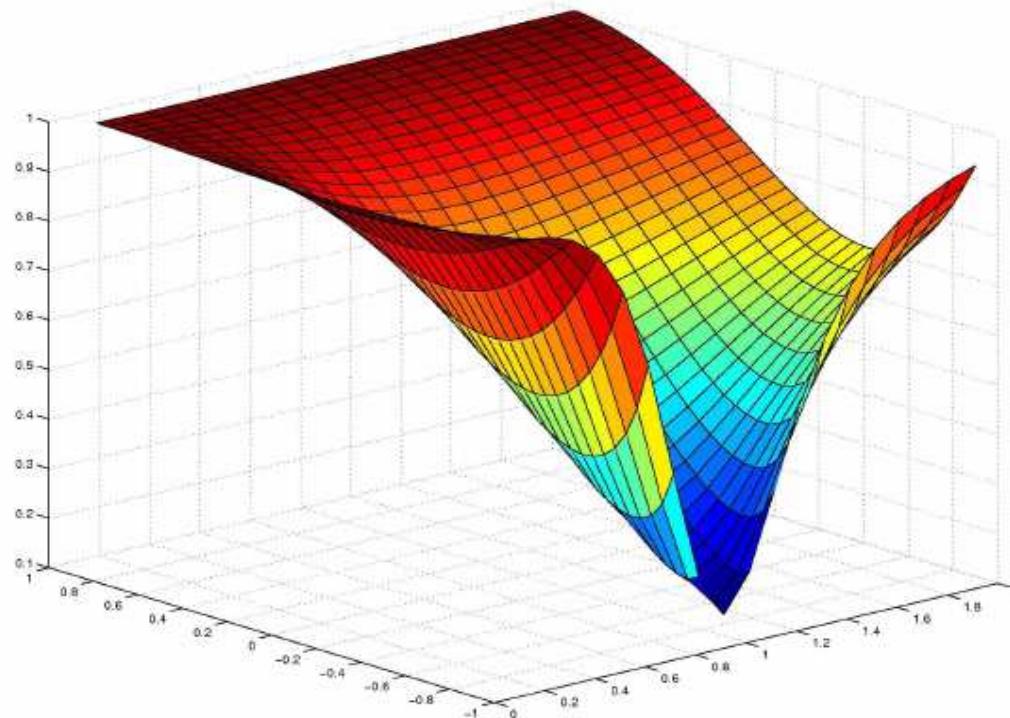
- In practice, determining the best hedge is not sufficient
Need to measure of residual risk



Measure of hedging performance

$$H^2 = 1 - \frac{\text{remaining risk after hedging}}{\text{initial risk before hedging}} = \frac{\|\pi^H X\|^2}{\|X\|^2} = 1 - \frac{\|X - \pi^H X\|^2}{\|X\|^2}$$

$$H^2 \text{ for } 0 < \frac{\sigma_Y}{\sigma_X} < 2 \text{ and } -1 < \rho < 1:$$

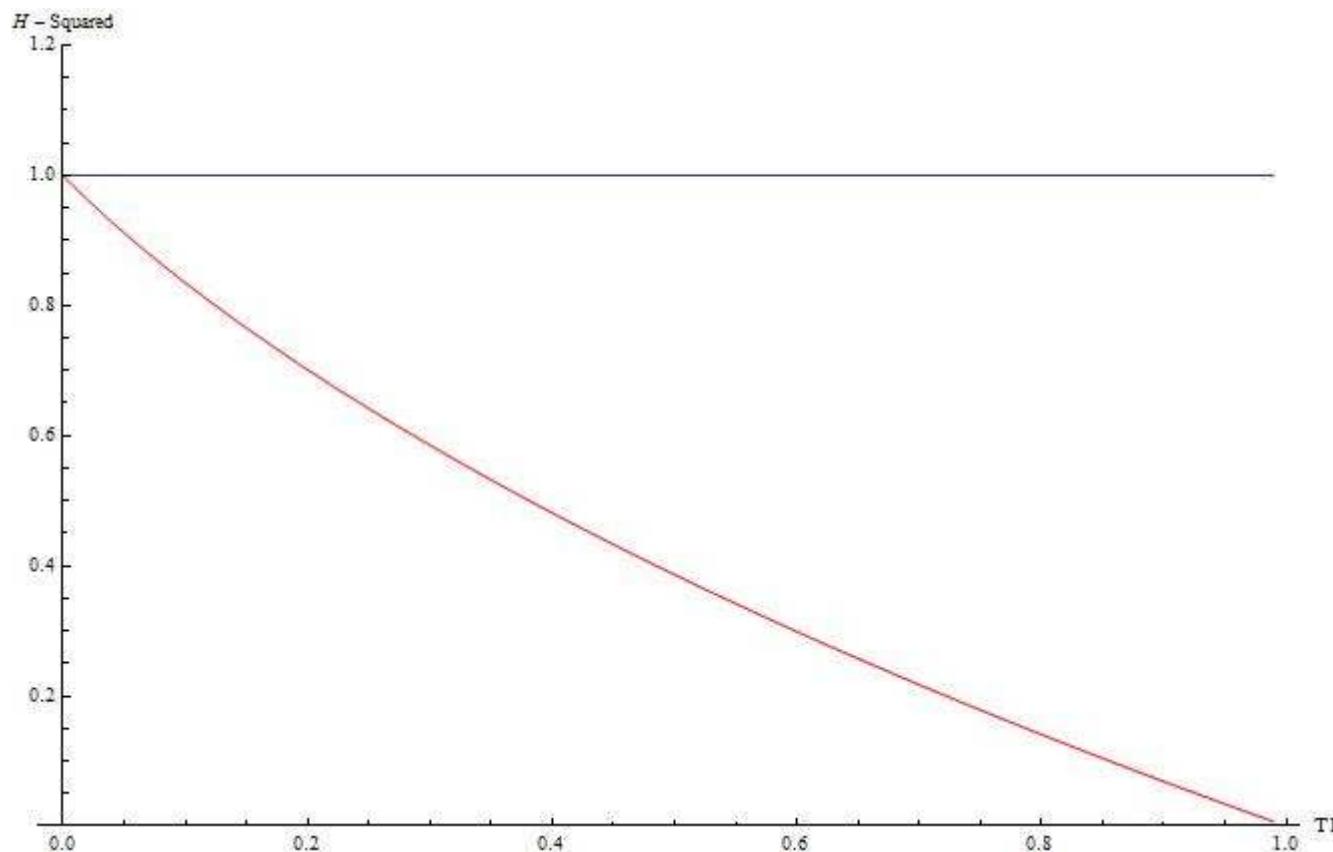


If $\rho \rightarrow 1$ or $\frac{\sigma_Y}{\sigma_X} \rightarrow 0$: $H^2 \rightarrow 1$.

If $\sigma_X = \sigma_Y$ and $\rho \rightarrow -1$: $H^2 \rightarrow 0$.

X & Y highly correlated,
with similar volatility \Rightarrow Basket options hedgeable.
 \Rightarrow Spread options not hedgeable.

H-Squared Parabola vs Cube



Ranking on Risk Scale: Comparing Vanillas, Asian and Barriers

	Price	std Risk (Δ)	$\frac{\text{std Risk } \Delta \text{ (/PriceX)}}{\text{std Risk } \Delta(\text{Call}) \text{ (/PriceCall)}}$
Call	7.98	0.68 (8.5 %)	1
Asian	4.61	0.47 (10.2 %)	0.69 (1.20)
DOC ($H = 80$)	7.81	0.63 (8.1 %)	0.93 (0.95)
UOC ($H = 140$)	6.16	1.66 (27.0 %)	2.44 (3.16)
UOC ($H = 120$)	1.46	1.13 (77.5 %)	1.67 (9.10)
UIC ($H = 120$)	6.52	1.34 (20.6 %)	1.97 (2.41)
OT ($H = 120$)	0.32	0.07 (21.6 %)	0.10 (2.57)

Figure: Summary of residual risk values for ATM exotic options T=1

Rank	Δ (Relative)
1	DOC (8.1%)
2	Call (8.5%)
3	Asian (10.2 %)
4	UIC (20.6 %)
5	One-Touch (21.6 %)
6	UOC ($H = 140$) (27.0 %)
7	UOC ($H = 120$) (77.5 %)

Conclusion

- Itô calculus can be extended to functionals of price paths
- Price difference between 2 models can be computed
- We get a variational calculus on volatility surfaces
- It leads to a strike/maturity decomposition of the volatility risk of the full portfolio