# Some Remarks on Corona Problems

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Workshop on the Corona Problem
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- a Hilbert space approach– the Dichotomy
- multiplier algebras and 1-positive square,$\mathbb{NP}$-kernels
- some successes for $\mathbb{NP}$-kernels
- corona problem for $H^\infty(\mathbb{D}^d)$ and $H^\infty(B^d)$

$F = (f_1, f_2, \dots),\ f_j \in H^\infty(\mathbb{D})$

First, recall $\boxed{\text{Corona Theorem for } H^\infty(\mathbb{D})}$

(Carleson, Wolff, Rosenblum, Tolokonnikov, Uchiyama)

$0 < \epsilon^2 \leq F(z)F(z)^* \leq 1,\ z \in \mathbb{D},$ then

$\exists\, \{u_j\}_{j=1}^\infty \subseteq H^\infty(\mathbb{D})$ such that

$$\text{(a)} \sum_{j=1}^\infty f_j u_j \equiv 1$$

$$\text{(b)} \sup_{z \in D} \sum_{j=1}^\infty |u_j(z)|^2 < \infty \qquad \textit{with bound}, C(\epsilon)$$

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$H^\infty(\mathbb{D})$ is the multiplier algebra for $H^2(T)$.

(A) $\boxed{H^2(T) \text{ Corona Theorem}}$

$$0 < \epsilon^2 \leq F(z)F(z)^* \leq 1, \; z \in \mathbb{D},$$

then

$$\exists\, \delta > 0 \text{ such that } I \geq T_F T_F^* \geq \delta^2 I \qquad i.e.\ T_F \text{ is } onto.$$

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(B) $\boxed{\text{Operator Corona Theorem for } H^2(T)}$

$$I \geq T_F T_F^* \geq \delta^2 I, \text{ then } \exists\, \{u_j\}_{j=1}^{\infty} \subseteq H^\infty(D)$$

such that

$$U^T = (u_1, u_2, \dots), \text{ then (a) } T_F T_U = I$$

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(Arveson, Douglas, Nagy-Foias, Rosenblum, Schubert, ...)

$\mathcal{H}$ is a reproducing kernel Hilbert space on $\Omega$ if $\forall w \in \Omega$ there exists $k_w \in \mathcal{H}$ with $f(z) = \langle f,\ k_w \rangle_{\mathcal{H}}$ for all $f \in \mathcal{H}$.

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Multipliers on $\mathcal{H}$,
$\mathcal{M}(\mathcal{H}) = \{\phi \in \mathcal{H} : \phi h \in \mathcal{H},\ \forall h \in \mathcal{H}\} \approx \{T_\phi \in B(\mathcal{H})\}$

$T_\phi^* k_w = \overline{\phi(w)} k_w$

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Now
$\mathcal{M}(H^2(T))) = H^\infty(\mathbb{D})$, but also $\mathcal{M}(A^2(\mathbb{D}))) = H^\infty(\mathbb{D})$.

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$\exists\, \delta > 0$ such that $I \geq T_F T_F^* \geq \delta^2 I$ *i.e.* $T_F$ *is* a multiplier on $A^2(\mathbb{D})$.

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$$U^T = (u_1, u_2, \dots), \text{ then } \begin{array}{l} \text{(a) } T_F T_U = I \\ \text{(b) } T_U \text{ bounded} \end{array}$$

(B) CLT [reproducing kernel has 1-positive square, complete $\mathbb{NP}$]

$$1 - \frac{1}{k_w(z)} = \sum_{n=1}^{\infty} a_n(z)\overline{a_n(w)} \quad a_n \in \mathcal{H}$$

[Ball-T.- Vinnikov][Agler,McCullough,Quiggen]

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Ex: $\dfrac{1}{1 - z\overline{w}}$ in $H^2(T)$ Non-Ex: $\dfrac{1}{(1 - z\overline{w})^2}$ in $A^2(\mathbb{D})$

$\dfrac{1}{z\overline{w}} \ln \dfrac{1}{1 - z\overline{w}}$ in $\mathcal{D}^2(\mathbb{D})$ $\displaystyle\prod_{j=1}^{d} \frac{1}{(1 - z_j\overline{w}_j)}$ in $H^2(T^d)$ $d > 1$

$\dfrac{1}{1 - \sum_{i=1}^{d} z_i\overline{w}_i}$ in $\mathcal{F}^d$ $\dfrac{1}{(1 - \sum_{i=1}^{d} z_i\overline{w}_i)^d}$ in $H^2(B^d)$ $d > 1$

($\mathcal{F}^d$ Drury-Arveson or symmetric Fock space)

## Aside: $\mathcal{A} = \{$analytic Toeplitz operators on $H^2(T)\}$

$A \in \mathcal{A}$ and $A A^* \geq \epsilon^2 I$

$\boxed{\text{Operator Corona Theorem}} \Rightarrow \exists\ B \in \mathcal{A}$ s.t. $A B = I$ and $\|B\| \leq \frac{1}{\epsilon}$

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Of course, for any subalgebra, $\mathcal{A}$, of $B(\mathcal{H})$

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Of course, for any subalgebra, $\mathcal{A}$, of $B(\mathcal{H})$

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$$\Rightarrow \quad (\textit{Douglas})\ \exists\, X \in B(\mathcal{H}) \text{ s.t. } AX = I \text{ and } \|X\| \leq \frac{1}{\epsilon}$$

but $X$ might not belong to $\mathcal{A}$

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McCullough-T. For $\mathcal{A} = \mathcal{M}(\mathcal{H})$ and nice, TFAE.

Whenever $B, A \in \mathcal{A} \otimes B(\mathcal{H})$ with $A A^* \geq B B^*$

$$\exists\, C \in \mathcal{A} \otimes B(\mathcal{H})$$

with $\|C\| \leq 1$ and $B = A C$.

$\mathcal{H}$ has an $\mathbb{NP}$-kernel.

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For $\mathcal{A}$ equal to the multiplier algebras on

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$\boxed{\text{Question 0}}$ Can such an example be found with $B = I$ ?

i.e. does there exist $A \in \mathcal{A} \otimes B(\mathcal{H})$ right invertible in $B(\mathcal{H}) \otimes B(\mathcal{H})$, but not right invertible in $\mathcal{A} \otimes B(\mathcal{H})$?

(B) $\boxed{\text{Operator Corona Theorem for } H^2(T)}$

$F = (f_1, \dots)$ with $f_j \in H^\infty(\mathbb{D})$.

$$I \geq T_F T_F^* \geq \delta^2 I, \ \text{then } \exists \{u_j\}_{j=1}^\infty \subseteq H^\infty(D)$$

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$\boxed{\text{Nagy-Foias}}$ Let $\begin{array}{cc} & \begin{array}{cc}\mathcal{H} & K_1\end{array} \\ \begin{array}{c}\mathcal{H} \\ K_2\end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix}\end{array} \in B(\mathcal{H} \oplus K_1, \mathcal{H} \oplus K_2)$

and $\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \| \leq 1.$

Then for $z \in \mathbb{D}$ and $\varphi(z) = D + z\, C(I - z\, A)^{-1} B,$

$$\|\varphi(z)\|_{B(K_1, K_2)} \leq 1.$$

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<u>Proof</u>: $$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} z(I - z\, A)^{-1} Bh \\ h \end{pmatrix} = \begin{pmatrix} (I - z\, A)^{-1} B\, h \\ \varphi(z)\, h \end{pmatrix}.$$

$$\|\varphi(z)\, h\|^2 \leq \|h\|^2 - (1 - |z|^2)\, \|(I - z\, A)^{-1} B\, h\|^2. \qquad \square$$

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$$\|\varphi(z)\, h\|^2 \leq \|h\|^2 - (1 - |z|^2)\, \|(I - z\, A)^{-1} B\, h\|^2. \quad \square$$

Note: Every $\phi \in Ball_1(H^\infty_{l^2}(\mathbb{D}))$ or $\phi \in Ball_1(H^\infty_{l^2}(\mathbb{D}^2))$ has this form! Agler

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$$= \langle P^* k_w, P^* k_z \rangle + \delta^2.$$

$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} \overline{w}\, P^* k_w \\ \overline{f_1(w)} \\ \overline{f_2(w)} \end{pmatrix} = \begin{pmatrix} P^* k_w \\ \\ \delta \end{pmatrix}.$$

Proof of $\boxed{\text{Operator Corona Theorem for } H^2(\mathbb{D})}$ $F = (f_1, f_2)$

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$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} \overline{w}\, P^* k_w \\ \overline{f_1(w)} \\ \overline{f_2(w)} \end{pmatrix} = \begin{pmatrix} P^* k_w \\ \\ \delta \end{pmatrix}.$$

$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}$ is a partial isometry from $\mathcal{H} \oplus \mathbb{C}^2$ to $\mathcal{H} \oplus \mathbb{C}$.

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$$C^* \begin{pmatrix} \overline{f_1(w)} \\ \overline{f_2(w)} \end{pmatrix} = (I - \overline{w}\, A^*)\, P^* k_w$$

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$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} \overline{w}\, P^* k_w \\ \overline{\dfrac{f_1(w)}{f_2(w)}} \end{pmatrix} = \begin{pmatrix} P^* k_w \\ \delta \end{pmatrix}$$

$$C^* \left( \overline{\frac{f_1(w)}{f_2(w)}} \right) = (I - \overline{w}\, A^*)\, P^* k_w$$

$$D^* \left( \overline{\frac{f_1(w)}{f_2(w)}} \right) + \overline{w}\, B^* P^* k_w = \delta$$

$$[D^* + \overline{w}\, B^* (I - \overline{w}\, A^*)^{-1} C^*] \left( \overline{\frac{f_1(w)}{f_2(w)}} \right) = \delta$$

$$(f_1(w), f_2(w))\, \frac{\phi(w)}{\delta} = 1 \ \text{ or } \ T_F\, T_{\frac{\phi}{\delta}} = I.$$

$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} \overline{w}\, P^* k_w \\ \overline{f_1(w)} \\ \overline{f_2(w)} \end{pmatrix} = \begin{pmatrix} P^* k_w \\ \delta \end{pmatrix}$$

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$$(f_1(w), f_2(w)) \frac{\phi(w)}{\delta} = 1 \text{ or } T_F\, T_{\frac{\phi}{\delta}} = I.$$

Note: A refinement of the above shows that if the input depends smoothly on a parameter, so does the corona solution $\frac{\phi}{\delta}$.

Replace $H^2(T)$ by $\mathcal{D}^2_\alpha(\mathbb{D})$

$H^\infty(\mathbb{D})$ by $\mathcal{M}(\mathcal{D}^2_\alpha(\mathbb{D}))$

$\forall z \in \mathbb{D}$, $F(z)F(z)^* \leq 1$ by $\|(M_{f_1}, M_{f_2}, \dots)^T\| < \infty$

$$\sup_{z\in\mathbb{D}} \sum_{j=1}^{\infty} |u_j(z)|^2 \text{ by } \|(M_{u_1}, M_{u_2}, \dots)^T\| < \infty$$

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(Banjade-T.)

$\boxed{\text{Corona theorem for } \mathcal{M}(\mathcal{F}^d)\text{!!}}$

(Costea-Sawyer-Wick)

$\boxed{\text{Matrix version}}$

(Fuhrmann, Vasyunin, T., T.-Zhang)

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$$\text{(a) } \mathcal{M}(\mathcal{H}) = H^\infty(\Omega)$$

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Warning!-Scheinberg

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Amar Extension to $H^\infty(\mathbb{D}^d)$

and $H^\infty(B^d)$.

$\boxed{\text{T.-Wick}}$ Suffices to consider $d\mu = |H|^2\, d\sigma$, where

$$\Omega = \mathbb{D}^d \quad d\sigma = d\sigma_1 \dots d\sigma_d$$
$$\Omega = B^d \quad d\sigma = dm_{\partial B^d}$$

and $H \in H^2(d\sigma)$ satisfies

(1) $H$ is nonvanishing in $\Omega$.
(2) $|H(\bullet)| \geq 1$ a.e. on $T^d$, if $\Omega = \mathbb{D}^d$
on $\partial B^d$, if $\Omega = B^d$ .

## Consequences of T.-Wick

Let

$$\mathcal{H}^{(d)} = \{H \in H^2(T^d) : \begin{array}{l} (1)\ H \text{ is nonvanishing on } \mathbb{D}^d \\ (2)\ |H(e^{it_1}, \ldots, e^{it_d})| \geq 1,\ \sigma\text{-a.e. on } T^d \end{array}\}$$

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The corona theorem for $H^\infty(\mathbb{D}^d)$ holds if and only if

$$\forall\, H \in \mathcal{H}^{(d)},\ 1 \in ran\,(T_F^H) \text{ where}$$

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i.e., need only find $\{u_j\}_{j=1}^n \subset H^2(|H|^2\, d\sigma)$ s.t. $\sum f_j u_j \equiv 1$ in $\mathbb{D}^d$.

Note: Let $H \in \mathcal{H}^{(2)}$ $S = M_z, W = M_w$ on $H^2(|H|^2\, d\sigma)$

$$F = (f_1, \ldots, f_n) \quad f_i \in H^\infty(\mathbb{D}^2).$$

Assume

$$T_F T_F^* \geq \delta^2 I.$$

We want to show that there exists $\epsilon > 0$ such that

$$T_F^H T_F^{H^*} \geq \epsilon^2 I,$$

i.e.

$$F(S, W)\, F(S, W)^* \geq \epsilon^2 I.$$

"reverse von-Neumann Inequality"

Let $\varphi \in C^1(\overline{\mathbb{D}}),\ z \in \mathbb{D}.$

$$\varphi(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{\varphi(w)}{w - z}\, dw - \frac{1}{\pi} \int_{\mathbb{D}} \frac{\overline{\partial}\, \varphi(w)}{w - z}\, dA(w).$$

$$\varphi(0) = \int_{-\pi}^{\pi} \varphi(e^{it})\, d\sigma(t) - \frac{1}{4\pi} \int_{\mathbb{D}} \Delta\varphi(w) \ln \frac{1}{|w|^2}\, dA(w).$$

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Def: Cauchy transform $\widehat{\varphi}(z) = -\frac{1}{\pi}\int_{\mathbb{D}} \frac{\varphi(w)}{w-z}\, dA(w)$.

(1) $\overline{\partial}\,\widehat{\varphi} = \varphi$

(2) $P_{H^2}^{\perp}\,\varphi = \widehat{(\overline{\partial}\,\varphi)}$ on $\partial\mathbb{D}$.

Fix $\quad F = (f_1, f_2), \quad F(z)\,F(z)^* \geq \epsilon^2 > 0 \quad z \in \mathbb{D}^2$

$\quad Q = \begin{pmatrix} f_2 \\ -f_1 \end{pmatrix} \quad$ so $FQ = 0$.

Fix $\quad H \in \mathcal{H}^{(2)}$. We want $\underline{u} \in \bigoplus_1^2 H^2(|H|^2\, d\sigma_1 d\sigma_2)$ with $F\,\underline{u} \equiv 1$.

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Now $\quad F\,\underline{x} = 1 \quad$ with $\quad \underline{x} \in \bigoplus_1^2 L^2(|H|^2\, d\sigma_1 d\sigma_2)$

$$\begin{aligned} \underline{x} &= \frac{F^*}{F\,F^*} - Q\underline{y}, \quad \underline{y} \in \bigoplus_1^2 L^2(|H|^2\, d\sigma_1 d\sigma_2) \\ &= \frac{F^*}{F\,F^*} - Q\frac{\underline{z}}{H}, \quad \underline{z} \in \bigoplus_1^2 L^2(d\sigma_1 d\sigma_2) \end{aligned}$$

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We want $\underline{z}$ so that

$$\langle\frac{F^*}{F\,F^*}-Q\frac{\underline{z}}{H},w_\perp\rangle_{\sigma_1\sigma_2}=0 \quad \forall \quad w_\perp\ \perp\ \bigoplus_1^2 H^2(d\sigma_1 d\sigma_2)$$

We want $\quad C_0 < \infty \quad$ such that

$$(*)\ |\langle \frac{F^*}{F\,F^*}, w_\perp \rangle_{\sigma_1\sigma_2}| \overset{?}{\leq} C_0 \| \frac{Q^* w_\perp}{\overline{H}} \|_{2,\sigma_1\sigma_2}\ \forall\, w_\perp\ \perp\ \bigoplus_1^2 H^2(d\sigma_1 d\sigma_2)$$

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Define: $\widetilde{W} = \widehat{W}_1^1 + \widehat{W}_2^2 - \widehat{\widehat{(W_1)_{\overline{w}}}^{1}}^{2}$, where $W_1 = \frac{Q^* F_z^*}{(FF^*)^2}$

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$(*)$ follows if and only if

$$\sup_{0<r<1}\ \inf_{\underline{p}\,\text{poly}} \|H\,\widetilde{W}_r - H\underline{p}\|_{2,\sigma_1\sigma_2} \overset{?}{<} \infty.$$

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But

$$\begin{aligned}(*) &= \sup_{0<r<1} \|P^{\perp}_{H^2}\,(H\,\widetilde{W_r})\|_{2,\sigma_1\sigma_2} \\ &= \sup_{0<r<1} \|\widetilde{H\,W_r}\|_{2,\sigma_1\sigma_2} \\ &\leq C_0 \|H\|_{2,\sigma_1\sigma_2}.\end{aligned}$$

See also Douglas-Sarkar.

$\boxed{\text{Question 1}}$ If $H \in \mathcal{H}^{(2)}$, does there exist $\mathcal{O}$ outer

in $H^2(d\sigma_1 d\sigma_2)$ with $|H| \leq |\mathcal{O}|$ a.e. on $T^2$?

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$H \in H^2(d\sigma_1 d\sigma_2)$ by Rudin.

For $d = 1$ the answer is YES, so the corona theorem follows for $H^\infty(\mathbb{D})$.

$\boxed{\text{Question 2}}$ $F = [f_{ij}],\ f_{ij} \in H^\infty(\mathbb{D})$
Suppose that

$$M_F \in B(\bigoplus_1^\infty A^2(\mathbb{D})) \text{ and } M_F M_F^* \geq \epsilon^2 I,\ \epsilon > 0$$

Is $T_F T_F^*$ invertible?

Here $A^2(\mathbb{D}) = \{f \in Hol(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 dm_2(z) < \infty\}$

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(Note: $\epsilon^2 I \leq F(z)F(z)^* \leq C_0 I \quad \forall\, z \in \mathbb{D}$ Treil counterexample)

$F(z) = [f_{ij}(z)] \quad f_{ij} \in H^\infty(\mathbb{D})$

Assume that

$$\epsilon^2 I \leq F(z)F(z)^* \leq I \quad \forall\, z \in \mathbb{D}$$

(Nagy) Does there exist $G(z) = [g_{ij}(z)] \quad g_{ij} \in H^\infty(\mathbb{D})$

s.t. $F(z)G(z) \equiv I \quad z \in \mathbb{D}$

and $\sup_{z\in\mathbb{D}} \|G(z)\|_{B(l^2)} < \infty$?

Treil -No!

Thus, if the answer to Question 2 is "yes",

then the corona theorem holds for $H^\infty(\mathbb{D}^2)$

Question 2 is "no",

then there is a stronger counterexample than Treil's,

since in Question 2, we assume that $F$ also satisfies

$$M_F \in B\left(\bigoplus_1^\infty A^2(\mathbb{D})\right)$$

and ask the Nagy question.