Corona Theorems for the Drury-Arveson space and other Besov-Sobolev spaces of holomorphic functions on the unit ball in $\mathbb{C}^n$

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June 18, 2012

Abstract

We discuss the ideas behind the absence of a corona in the multiplier algebra of the Drury-Arveson space. Included are

- the Toeplitz corona theorem,
- the Koszul complex,
- Charpentier's solution operators to d bar equations, and
- the interplay with complex tangential vector fields.

These ideas extend to other Besov-Sobolev spaces of holomorphic functions on the ball having varying degrees of smoothness, as well as to vector-valued settings.

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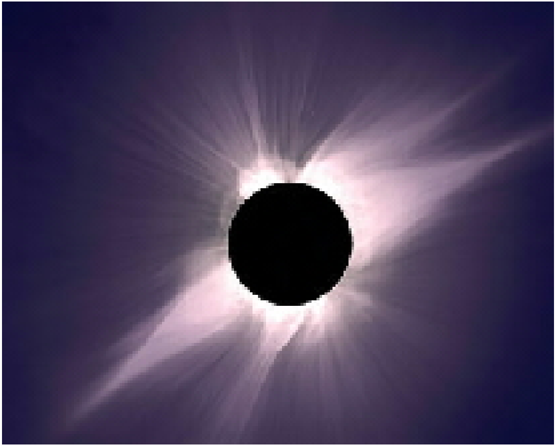
    4. Schur estimates,
    5. operator estimates on iterates of Charpentier's solution kernels, the *rogue* terms.

The Classical Corona Theorem

We discuss Carleson's Corona Theorem, Sibony's counterexample in higher dimensions, and our corona theorem.

The classical corona theorem

- In 1941, Kakutani asked if there was a corona in the maximal ideal space $\triangle$ of $H^\infty(\mathbb{D})$, i.e whether or not the disk $\mathbb{D}$ was dense in $\triangle$.



Carleson's Corona Theorem

- In 1962 Lennart Carleson

demonstrated in [9] the absence of a corona by showing that if $\{g_j\}_{j=1}^N$ is a finite set of functions in $H^\infty(\mathbb{D})$ satisfying

$$\sum_{j=1}^N |g_j(z)| \geq c > 0, \qquad z \in \mathbb{D}, \tag{1}$$

then there are functions $\{f_j\}_{j=1}^N$ in $H^\infty(\mathbb{D})$ with

$$\sum_{j=1}^N f_j(z)\, g_j(z) = 1, \qquad z \in \mathbb{D}. \tag{2}$$

- Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of points in the unit disk $\mathbb{D}$ such each point $e^{i\theta} \in \mathbb{T} = \partial\mathbb{D}$ is a nontangential limit point of $\{a_n\}_{n=1}^{\infty}$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers tending so quickly to 0 that:

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$$V(z) \equiv \prod_{n=1}^{\infty} \left| \frac{z - a_n}{2} \right|^{\lambda_n}$$

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2. the function V is continuous in $\mathbb{D}$, and
3. $V(z) = 0$ if and only if $z = a_n$ for some $n \geq 1$.

A domain of revolution

- Consider the open set of revolution

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$$g(z, w) = \sum_{k=0}^{\infty} h_k(z) w^k, \qquad (z, w) \in U. \tag{3}$$

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- Cauchy's estimates yield

$$|h_k(z)| \leq \left(\frac{1}{e^{-V(z)}} \right)^k \sup_{|w|<e^{-V(z)}} g(z, w) \leq e^{kV(z)} \leq e^k, \qquad z \in \mathbb{D}.$$

Extension of bounded holomorphic functions

Thus for each $k \geq 1$ the function $h_k \in H^\infty(\mathbb{D})$ with $\|h_k\|_\infty \leq e^k$. But since

$$|h_k(a_n)| \leq e^{kV(a_n)} = 1,$$

Fatou's theorem implies that for almost every $e^{i\theta} \in \mathbb{T}$,

$$\left|h_k^*\left(e^{i\theta}\right)\right| = \lim_{a_n \to e^{i\theta} \text{ nontangentially}} |h_k(a_n)| \leq 1.$$

Thus we have the stronger bound $\|h_k\|_\infty \leq 1$, and it follows that the series in (3) converges for all (z, w) in the open unit polydisk $\mathbb{D}^2$. Thus U is a domain of holomorphy that satisfies:

Lemma

Every $g \in H^\infty(U)$ extends uniquely to a function $\widetilde{g} \in H^\infty\left(\mathbb{D}^2\right)$ with the same norm.

Failure of the two-generator Bezout equation

- For any point $(\alpha, \beta) \in \mathbb{D}^2 \setminus \overline{U}$, there is $\delta > 0$ such that

$$|z-\alpha|^2 + |w-\beta|^2 \geq \delta^2 > 0, \qquad (z,w) \in U.$$

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- Yet there cannot exist $g_1, g_2 \in H^\infty(U)$ satisfying

$$(z - \alpha) g_1(z, w) + (w - \beta) g_2(z, w) = 1, \qquad (z, w) \in U,$$

since by unique continuation we would then have

$$(z - \alpha) \widetilde{g_1}(z, w) + (w - \beta) \widetilde{g_2}(z, w) = 1, \qquad (z, w) \in \mathbb{D}^2,$$

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- Sibony then embeds this domain in the boundary of a *smooth* domain $\Omega \subset \mathbb{C}^3$ which is *strongly* pseudoconvex at all boundary points but one, and fails the three-generator Bezout equation.

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- More generally, we obtain the corona theorem for the multiplier algebras $M_{B_2^\sigma(\mathbb{B}_n)}$ of the Besov-Sobolev spaces $B_2^\sigma(\mathbb{B}_n)$, $0 \leq \sigma \leq \frac{1}{2}$, on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$. The space H_n^2 occurs as $B_2^{\frac{1}{2}}(\mathbb{B}_n)$, while $B_2^0(\mathbb{B}_n)$ is the Dirichlet space.

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- The space $B_2^\sigma(\mathbb{B}_n)$ consists of those holomorphic functions f whose derivatives of order $\frac{n}{2} - \sigma$ lie in the classical Hardy space $H^2(\mathbb{B}_n) = B_2^{\frac{n}{2}}(\mathbb{B}_n)$, equivalently whose antiderivatives of order σ lie in the Dirichlet space:

$$\left\{ \sum_{k=0}^{m-1} \left| f^{(k)}(0) \right|^2 + \int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^2 d\lambda_n(z) \right\}^{\frac{1}{2}} < \infty,$$

where $d\lambda_n(z) = \left(1 - |z|^2\right)^{-n-1} dV(z)$ is invariant measure.

The Drury-Arveson Hardy Space H_n^2 on the unit ball in C^n

We discuss von Neumann's classical inequality and Drury's extension to multivariate operators. This introduces the Drury-Arveson Hardy Space H_n^2 and indicates its central role in operator theory.

The DA Hardy space

The first hint that the classical Hardy space $H^2(\mathbb{B}_n)$ (consisting of holomorphic functions on the ball with $L^2(\sigma)$ boundary values) may *not* be the correct generalization of the classical Hardy space on the disk came with the failure of von Neumann's inequality in higher dimensions. Recall the classical inequality:

Theorem

(von Neumann 1951 [17]) Let H be a Hilbert space and let f be a complex-valued polynomial. Then for any contraction T on H,

$$\|f(T)\|_{H\to H} \leq \|f(S^*)\|_{H^2\to H^2} = \|f\|_{H^\infty(\mathbb{D})},$$

where S^* is the backward shift operator on $H^2 = H^2(\mathbb{D})$.

Drury found the correct generalization to the multivariable setting.

Drury's generalization

- Let $A = (A_1, ..., A_n)$ be an n-contraction on a complex Hilbert space $\mathcal{H}$, i.e. an n-tuple of linear operators on $\mathcal{H}$ satisfying

$$A_j A_k = A_k A_j \text{ for all } 1 \leq j, k \leq n, \text{ and } \sum_{j=1}^{n} \|A_j h\|^2 \leq \|h\|^2 \text{ for all } h \in \mathcal{H}$$

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- Drury showed in 1978 [13] that if f is a complex polynomial on $\mathbb{C}^n$, then

$$\|f(A)\| \leq \|f\|_{M_{\mathcal{K}(\mathbb{B}_n)}}, \tag{4}$$

for all n-contractions A on $\mathcal{H}$ where $\|f(A)\|$ is the operator norm of $f(A)$ on $\mathcal{H}$, and $\|f\|_{M_{\mathcal{K}(\mathbb{B}_n)}}$ denotes the multiplier norm of the polynomial f on Drury's Hardy space of holomorphic functions

$$\mathcal{K}(\mathbb{B}_n) = \left\{ \sum_k a_k z^k,\ z \in \mathbb{B}_n : \sum_k |a_k|^2 \frac{k!}{|k|!} < \infty \right\},$$

denoted by H_n^2 in Arveson 1998 [1] (who also proves (4)).

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- Moreover, equality holds in (4) when A is the n-tuple $(S_1^*, ..., S_n^*)$

Chen's identification of the DA space

- In 2003 Chen [11] has identified the Drury-Arveson Hardy space $\mathcal{K}(\mathbb{B}_n) = H_n^2$ as the Besov-Sobolev space $B_2^{\frac{1}{2}}(\mathbb{B}_n)$ consisting of those holomorphic functions $\sum_k a_k z^k$ in the ball with coefficients a_k satisfying

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- We note in passing that a number of operator-theoretic properties of the Hilbert space H_n^2 are developed by Arveson in [1], including some model theory, that point to its central position in multivariable operator theory.

The corona problem for multiplier spaces in C^n

We introduce the baby corona problem, the Toeplitz Corona Theorem, and our baby corona theorem.

The baby corona problem

- Let X be a Hilbert function space and let M_X be its multiplier algebra. The so-called *baby corona problem* for X is this: given $g_1, ...g_N \in M_X$ satisfying
$$\left|g_1(z)\right|^2 + ... + \left|g_N(z)\right|^2 \geq c > 0, \qquad z \in \Omega, \tag{5}$$
is there a constant $\delta > 0$ such that for each $h \in X$ there are $f_1, ...f_N \in X$ satisfying
$$\begin{aligned} \|f_1\|_X^2 + ... + \|f_N\|_X^2 &\leq \frac{1}{\delta}\|h\|_X^2, \\ f_1(z)\,g_1(z) + ... + f_N(z)\,g_N(z) &= h(z), \qquad z \in \Omega? \end{aligned} \tag{6}$$

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- More succinctly, (6) is equivalent to the operator lower bound

$$\mathcal{M}_g\mathcal{M}_g^* - \delta I_X \geq 0, \tag{7}$$

where $g \equiv (g_1, ...g_N)$, $\mathcal{M}_g : \oplus^N X \to X$ by $\mathcal{M}_g f = \sum_{\alpha=1}^N g_\alpha f_\alpha$, and $\mathcal{M}_g^* h = \left(\mathcal{M}_{g_\alpha}^* f\right)_{\alpha=1}^N$.

Toeplitz Corona Theorem

For $f = (f_\alpha)_{\alpha=1}^N \in \oplus^N X$ and $h \in X$, define $\mathbb{M}_f h = (f_\alpha h)_{\alpha=1}^N$ and

$$\|f\|_{Mult(X,\oplus^N X)} = \|\mathbb{M}_f\|_{X \to \oplus^N X} = \sup_{\|h\|_X \leq 1} \|\mathbb{M}_f h\|_{\oplus^N X}.$$

Note that $\max_{1\leq\alpha\leq N} \|\mathcal{M}_{f_\alpha}\|_{M_X} \leq \|f\|_{Mult(X,\oplus^N X)} \leq \sqrt{\sum_{\alpha=1}^N \|\mathcal{M}_{f_\alpha}\|_{M_X}^2}$.

Theorem

(Toeplitz Corona Theorem) *Let X be a Hilbert function space in an open set Ω in $\mathbb{C}^n$ with an irreducible complete Nevanlinna-Pick kernel. Let $\delta > 0$ and $N \in \mathbb{N}$. Then $g_1, \ldots g_N \in M_X$ satisfy the operator lower bound (7) with $\delta > 0$ if and only if there are $f_1, \ldots f_N \in M_X$ such that*

$$\begin{aligned} \|f\|_{Mult(X,\oplus^N X)} &\leq 1, \\ f_1(z)\, g_1(z) + \ldots + f_N(z)\, g_N(z) &= \sqrt{\delta}, \qquad z \in \Omega. \end{aligned} \tag{8}$$

The baby corona problem for two generators

- In 2000 J. M. Ortega and J. Fabrega [18] obtain partial results with $N = 2$ generators in (6) for the Banach spaces $B_p^\sigma(\mathbb{B}_n)$ with $\sigma \in \left[0, \frac{1}{p}\right) \cup \left(\frac{n}{p}, \infty\right)$ and $1 < p < \infty$; and also for the case $N = 2$ with $\sigma = \frac{n}{p}$ when $1 < p \leq 2$. In [19] 2006, they prove the analogous results for the Hardy-Sobolev scale of spaces.

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- To handle $N = 2$ generators they exploit the fact that a 2×2 antisymmetric matrix consists of just one entry up to sign:

$$\begin{aligned} \overline{\partial}\left(\frac{\overline{g}}{|g|^2}\right) &= \overline{\partial}\left(\begin{array}{c} \frac{\overline{g_1}}{|g|^2} \\ \frac{\overline{g_2}}{|g|^2} \end{array}\right) = \left[\begin{array}{cc} 0 & \omega \\ -\omega & 0 \end{array}\right]\left(\begin{array}{c} g_1 \\ g_2 \end{array}\right), \\ \omega &= \frac{\overline{g_2 \partial g_1 - g_1 \partial g_2}}{|g|^4} \text{ is a closed } (0,1)\text{-form.} \end{aligned}$$

Baby Corona Theorem ([12] 2008)

Theorem

Let $0 \leq \sigma < \infty$ *and* $1 < p < \infty$. *Given* $g_1, \ldots, g_N \in M_{B_p^\sigma(\mathbb{B}_n)}$ *satisfying*

$$\sum_{j=1}^{N} |g_j(z)|^2 \geq 1, \qquad z \in \mathbb{B}_n,$$

there is a constant $C_{n,\sigma,N,p}$ *such that for each* $h \in B_p^\sigma(\mathbb{B}_n)$ *there are* $f_1, \ldots, f_N \in B_p^\sigma(\mathbb{B}_n)$ *satisfying*

$$\begin{aligned} \sum_{j=1}^{N} \|f_j\|_{B_p^\sigma(\mathbb{B}_n)}^p &\leq C_{n,\sigma,N,p}(g) \|h\|_{B_p^\sigma(\mathbb{B}_n)}^p, \qquad (9) \\ \sum_{j=1}^{N} f_j(z) g_j(z) &= h(z), \qquad z \in \mathbb{B}_n. \end{aligned}$$

Consequences

Corollary

([12] 2008) Let $0 \leq \sigma \leq \frac{1}{2}$. Then the Banach algebra $M_{B_2^\sigma(\mathbb{B}_n)}$ has no corona, i.e. the linear span of point evaluations $e_z(f) = f(z)$, $f \in M_{B_2^\sigma(\mathbb{B}_n)}$ and $z \in \mathbb{B}_n$, is dense in the maximal ideal space of $M_{B_2^\sigma(\mathbb{B}_n)}$. In particular the multiplier algebra of the Drury-Arveson space H_n^2 has no corona.

- **Proof**: The corollary follows using functional analysis from Theorems 4 and 3 since the spaces $B_2^\sigma(\mathbb{B}_n)$ have an irreducible complete Nevanlinna-Pick kernel when $0 \leq \sigma \leq \frac{1}{2}$.

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- **Proof**: The corollary follows using functional analysis from Theorems 4 and 3 since the spaces $B_2^\sigma(\mathbb{B}_n)$ have an irreducible complete Nevanlinna-Pick kernel when $0 \leq \sigma \leq \frac{1}{2}$.
- The baby corona theorem holds for infinitely many generators $N = \infty$ with appropriately defined norms.

Novelties

In order to treat $N > 2$ generators in (6) with estimates independent of N, we need: to use the Koszul complex, its factorization in the exterior algebra, to invert higher order forms in the $\overline{\partial}$ equation, and in order to obtain results for $\sigma \geq \frac{1}{2}$, to devise new estimates for the Charpentier solution operators for these equations.
In particular the novel estimates include

1. the use of sharp estimates on Euclidean expressions $\left|\left(\overline{w-z}\right)\frac{\partial}{\partial \overline{w}} f\right|$ in terms of the invariant length $\left|1-\overline{w}z\right|\left|\varphi_z\left(w\right)\right|$ multiplied by the invariant derivative $\left|\widetilde{\nabla} f\right|$,
2. the use of the exterior calculus together with the explicit form of Charpentier's solution kernels to handle *rogue* Euclidean factors $\overline{w_j - z_j}$), and
3. the application of generalized operator estimates of Schur type to obtain appropriate boundedness of solution operators.

The Koszul complex

We introduce complex derivatives and differentials and describe how the Koszul complex reduces the corona problem to estimates.

- Define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{i\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{i\partial y}\right)$$

where $z = x + iy$ and $dz = dx + idy$, $d\overline{z} = dx - idy$. Let

$$\overline{\partial} f = \sum_{k=1}^{n} \frac{\partial f}{\partial \overline{z_k}} d\overline{z_k}.$$

The d-bar equation

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$$\overline{\partial} f = \sum_{k=1}^{n} \frac{\partial f}{\partial \overline{z_k}} d\overline{z_k}.$$

- Given a $(0,1)$-form $\eta(z) = \eta_1(z)\, d\overline{z_1} + \dots + \eta_n(z)\, d\overline{z_n}$ in the ball $\mathbb{B}_n$, the $\overline{\partial}$-equation for η is

$$\overline{\partial} f = \eta \quad \text{in the ball } \mathbb{B}_n. \tag{10}$$

- Define

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\frac{\partial}{i\partial y}\right),\qquad \frac{\partial}{\partial \overline{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\frac{\partial}{i\partial y}\right)$$

where $z=x+iy$ and $dz=dx+idy$, $d\overline{z}=dx-idy$. Let

$$\overline{\partial}f=\sum_{k=1}^{n}\frac{\partial f}{\partial \overline{z_k}}d\overline{z_k}.$$

- Given a $(0,1)$-form $\eta\left(z\right)=\eta_1\left(z\right)d\overline{z_1}+...+\eta_n\left(z\right)d\overline{z_n}$ in the ball $\mathbb{B}_n$, the $\overline{\partial}$-equation for η is

$$\overline{\partial}f=\eta \quad \text{in the ball } \mathbb{B}_n. \tag{10}$$

- More generally, we can let $\eta=\sum_{|I|=p,|J|=q+1}\eta_{I,J}\left(z\right)dz^I\wedge d\overline{z}^J$ be a $(p,q+1)$-form in the ball and ask for a (p,q)-form f to satisfy (10).

The Koszul complex

- If $g = (g_j)_{j=1}^N$ satisfies $|g|^2 = \sum_{j=1}^N |g_j|^2 \geq 1$, let

$$\Omega_0^1 = \frac{\overline{g}}{|g|^2} = \left(\frac{\overline{g_j}}{|g|^2} \right)_{j=1}^N = \left(\Omega_0^1 (j) \right)_{j=1}^N ,$$

which we view as a 1-tensor (in $\mathbb{C}^N$) of $(0,0)$-forms with components $\Omega_0^1 (j) = \frac{\overline{g_j}}{|g|^2}$.

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- Then $f = \Omega_0^1 h$ satisfies $f \cdot g = h$, but in general fails to be holomorphic.
- The Koszul complex provides a scheme when g, h are holomorphic for solving a sequence of $\overline{\partial}$ equations that result in a correction term $\Lambda_g \Gamma_0^2$ that when subtracted from f above yields a *holomorphic* solution to $f \cdot g = h$.

Lifting of forms, or division of vectors

- The 1-tensor of $(0,1)$-forms $\overline{\partial}\Omega_0 = \left(\overline{\partial}\frac{\overline{g_j}}{|g|^2}\right)_{j=1}^{N} = \left(\overline{\partial}\Omega_0^1(j)\right)_{j=1}^{N}$ is given by

$$\overline{\partial}\Omega_0^1(j) = \overline{\partial}\frac{\overline{g_j}}{|g|^2} = \frac{1}{|g|^4}\sum_{k=1}^{N} g_k \overline{\{g_k \partial g_j - \partial g_k g_j\}}.$$

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- A key fact is that this 1-tensor of $(0,1)$-forms can be written as

$$\overline{\partial}\Omega_0^1 = \Lambda_g \Omega_1^2 \equiv \left[\sum_{k=1}^N \Omega_1^2(j,k)\, g_k\right]_{j=1}^N,$$

where the 2-tensor Ω_1^2 of $(0,1)$-forms is given by

$$\Omega_1^2 = \left[\Omega_1^2(j,k)\right]_{j,k=1}^N = \left[\frac{\overline{\{g_k \partial g_j - \partial g_k g_j\}}}{|g|^4}\right]_{j,k=1}^N.$$

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- Thus the form $\overline{\partial}\Omega_0^1$ has been factored as (or lifted to) $\Lambda_g \Omega_1^2$ where Ω_1^2 is *alternating*.

Repeating the division

- We can repeat this division process to obtain

$$\overline{\partial}\Omega_1^2 = \Lambda_g \Omega_2^3 \equiv \left[\sum_{\ell=1}^{N} \Omega_2^3 \left(j,k,\ell\right) g_\ell \right]_{j,k=1}^{N},$$

where the alternating 3-tensor Ω_2^3 of $(0,2)$-forms is given by

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where the alternating 3-tensor Ω_2^3 of $(0, 2)$-forms is given by

- $$\begin{aligned} \Omega_2^3 &= \left[\Omega_2^3 (j, k, \ell) \right]_{j,k,\ell=1}^{N} \\ &= \left[2 \frac{\overline{\{ g_\ell (\partial g_k \wedge \partial g_j) + g_k (\partial g_j \wedge \partial g_\ell) - g_j (\partial g_k \wedge \partial g_\ell) \}}}{|g|^6} \right]_{j,k,\ell=1}^{N}. \end{aligned}$$

The division calculation

- Here is the calculation: $\overline{\partial}\Omega_1^2(j,k) =$

$$
\begin{aligned}
& \overline{\partial}\,|g|^{-4}\,\overline{\{g_k\partial g_j - g_j\partial g_k\}} \\
= \quad & \frac{|g|^4\,\overline{2\partial g_k \wedge \partial g_j} - 2\,|g|^2\,\overline{\partial}\,|g|^2 \wedge \overline{\{g_k\partial g_j - g_j\partial g_k\}}}{|g|^8} \\
= \quad & 2\frac{\left(\sum_{\ell=1}^N g_\ell\overline{g_\ell}\right)\overline{\partial g_k \wedge \partial g_j} - 2\left(\sum_{\ell=1}^N g_\ell\overline{\partial g_\ell}\right) \wedge \overline{\{g_k\partial g_j - \partial g_k g_j\}}}{|g|^6} \\
= \quad & \frac{2}{|g|^6}\sum_{\ell=1}^N g_\ell\overline{\{g_\ell\left(\partial g_k \wedge \partial g_j\right) - g_k\left(\partial g_\ell \wedge \partial g_j\right) + g_j\left(\partial g_\ell \wedge \partial g_k\right)\}}.
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\end{aligned}$$

- Now recall that h is holomorphic. Thus $\Omega_2^3 h$ is $\overline{\partial}$-closed since every $(0,2)$-form is $\overline{\partial}$-closed, and this will allow us to solve the $\overline{\partial}$ equations

$$\begin{aligned}
\overline{\partial}\Gamma_1^3 &= \Omega_2^3 h; \\
\overline{\partial}\Gamma_0^2 &= \Omega_1^2 h - \Lambda_g\Gamma_1^3.
\end{aligned}$$

Solving the complex in two dimensions

- Since $\Omega_2^3 h$ is $\overline{\partial}$-closed and alternating, there is an alternating 3-tensor Γ_1^3 of $(0,1)$-forms satisfying

$$\overline{\partial}\Gamma_1^3 = \Omega_2^3 h.$$

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- Now note that the 2-tensor $\Omega_1^2 h - \Lambda_g \Gamma_1^3$ of $(0,1)$-forms is $\overline{\partial}$-closed since both h and g are holomorphic:

$$\overline{\partial}\left(\Omega_1^2 h - \Lambda_g \Gamma_1^3\right) = \overline{\partial}\Omega_1^2 h - \overline{\partial}\Lambda_g \Gamma_1^3 = \Lambda_g \Omega_2^3 h - \Lambda_g \Omega_2^3 h = 0.$$

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- Thus there is an alternating 2-tensor Γ_0^2 of $(0,0)$-forms, i.e. functions, satisfying

$$\overline{\partial}\Gamma_0^2 = \Omega_1^2 h - \Lambda_g \Gamma_1^3.$$

The Bezout equation

- Now
$$f \equiv \Omega_0^1 h - \Lambda_g \Gamma_0^2$$
is holomorphic since Γ_0^2 is alternating:
$$\begin{aligned}\bar{\partial}\left(\Omega_0^1 h - \Lambda_g \Gamma_0^2\right) &= \left(\bar{\partial}\Omega_0^1\right) h - \Lambda_g \bar{\partial}\Gamma_0^2 \\ &= \left(\Lambda_g \Omega_1^2\right) h - \Lambda_g\left(\Omega_1^2 h - \Lambda_g \Gamma_1^3\right) \\ &= \Lambda_g\left(\Lambda_g \Gamma_1^3\right) \equiv \Gamma_0^2(g, g) = 0.\end{aligned}$$

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- Thus we compute that
$$f \cdot g = \Omega_0^1 h \cdot g - \Lambda_g \Gamma_0^2 \cdot g = h - 0 = h.$$

Factorization of the Koszul complex

- With the standard basis $\{\mathbf{e}_i\}_{i=1}^N$ for the exterior algebra on $\mathbb{C}^N$, Andersson and Carlsson obtained the following factorization of the Koszul complex:

$$\begin{aligned}
\Omega_1^2 &= \left[\frac{\overline{\{g_k\partial g_j - \partial g_k g_j\}}}{|g|^4}\right]_{j,k=1}^N \\
&= 2\sum_{1\le j<k\le N} \frac{\overline{\{g_k\partial g_j - \partial g_k g_j\}}}{|g|^4}\mathbf{e}_j \wedge \mathbf{e}_k \\
&= -2\left(\sum_{j=1}^N \frac{\overline{g_j}}{|g|^4}\mathbf{e}_j\right) \wedge \left(\sum_{k=1}^N \frac{\overline{\partial g_k}}{|g|^4}\mathbf{e}_k\right) = \Omega_0^1 \wedge \widetilde{\Omega_0^1}; \\
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\end{aligned}
$$

- The Hilbert-Schmidt norm is multiplicative on tensors, hence quasi-multiplicative on wedge products with constant depending only on the number of factors and not on N.

Charpentier's solution kernels

We describe Charpentier's solutions of the $\overline{\partial}$-equation in the ball, which are superbly adapted to solving the corona problem. But first it is useful to compare the $\overline{\partial}$-equation above with the more familiar gradient ∇ equation in real Euclidean space, and then to introduce the Cauchy-Leray form.

The Cauchy kernel

- In one complex dimension, the equation $\bar{\partial} f = \eta$ is easily solved using the Cauchy kernel,

$$\begin{aligned}\mathcal{C}\eta(z) &= \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{\eta(w)}{w-z} dw \wedge d\overline{w} \\ &= \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{\overline{(w-z)}}{|w-z|^2}\eta(w)\, dw \wedge d\overline{w},\end{aligned}$$

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- together with the distributional equation

$$\frac{\partial}{\partial \overline{z}} \frac{1}{z} = \frac{1}{\pi}\delta_0,$$

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- together with the distributional equation

$$\frac{\partial}{\partial \overline{z}} \frac{1}{z} = \frac{1}{\pi} \delta_0,$$

- to obtain

$$\overline{\partial} \mathcal{C}\eta(z) = \eta(z).$$

Charpentier's solution kernels

We begin with some notation. Denote by $\triangle : \mathbb{C}^n \times \mathbb{C}^n \to [0,\infty)$ the map:

$$\begin{aligned} \triangle(w,z) &= \left|1 - w\overline{z}\right|^2 - \left(1 - |w|^2\right)\left(1 - |z|^2\right) \\ &= \left(1 - |z|^2\right)|w-z|^2 + \left|\overline{z}(w-z)\right|^2 \\ &= \left(1 - |w|^2\right)|w-z|^2 + \left|\overline{w}(w-z)\right|^2 \\ &= \left|1 - w\overline{z}\right|^2 \left|\varphi_w(z)\right|^2 \\ &= \left|1 - w\overline{z}\right|^2 \left|\varphi_z(w)\right|^2 \\ &= \left|P_w(z-w) + \sqrt{1-|w|^2}\,Q_w(z-w)\right|^2 \\ &= \left|P_z(z-w) + \sqrt{1-|z|^2}\,Q_z(z-w)\right|^2 . \end{aligned} \tag{11}$$

The Cauchy-Leray form

$$\mu(\xi, w, z) \equiv \frac{1}{(\xi(w-z))^n} \sum_{i=1}^{n} (-1)^{i-1} \xi_i \left[\wedge_{j \neq i} d\xi_j\right] \wedge_{i=1}^{n} d(w_i - z_i),$$

is a closed form on $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n$. One then lifts the form μ via a section $s : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ to give a closed form on $\mathbb{C}^n \times \mathbb{C}^n$:

$$s^* \mu\,(w, z) \equiv \frac{1}{\left(s\,(w, z)\,(w-z)\right)^n} \sum_{i=1}^{n} (-1)^{i-1} s_i\,(w, z) \left[\wedge_{j \neq i} ds_j\right] \wedge_{i=1}^{n} d\,(w_i -$$

Now fix s to be the following section used by Charpentier:

$$s(w, z) \equiv \overline{w}(1 - w\overline{z}) - \overline{z}(1 - |w|^2). \tag{12}$$

We compute that $s(w, z)(w - z) = \triangle\,(w, z)$ by (11).

Charpentier's forms

- Define the Cauchy Kernel on $\mathbb{B}_n \times \mathbb{B}_n$ by $\mathcal{C}_n(w,z) \equiv s^* \mu(w,z)$ where s is Charpentier's section.

Definition

For $0 \leq p \leq n$ and $0 \leq q \leq n-1$ we let $\mathcal{C}_n^{p,q}$ be the component of $\mathcal{C}_n(w,z)$ that has bidegree (p,q) in z and bidegree $(n-p, n-q-1)$ in w.

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- Thus if η is a $(p, q+1)$-form in w, then $\mathcal{C}_n^{p,q} \wedge \eta$ is a (p,q)-form in z and a multiple of the volume form in w.

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- We next give explicit formulas for Charpentier's solution kernels $\mathcal{C}_n^{0,q}(w,z)$.

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- We next give explicit formulas for Charpentier's solution kernels $\mathcal{C}_n^{0,q}(w,z)$.
- Let $\omega_n(z) = \bigwedge_{j=1}^n dz_j$. For n a positive integer and $0 \leq q \leq n-1$ let P_n^q denote the collection of all permutations ν on $\{1,\ldots,n\}$ that map to $\{i_\nu, J_\nu, L_\nu\}$ where J_ν is an increasing multi-index with $card(J_\nu) = n-q-1$ and $card(L_\nu) = q$. Let $\epsilon_\nu \equiv sgn(\nu) \in \{-1,1\}$ denote the signature of the permutation ν.

Explicit formulas for Charpentier kernels

Theorem

Let n be a positive integer and suppose that $0 \leq q \leq n-1$. Then

$$\mathcal{C}_n^{0,q}(w,z) = \sum_{\nu \in P_n^q} (-1)^q \, \Phi_n^q(w,z) \, sgn(\nu) \left(\overline{w_{i_\nu}} - \overline{z_{i_\nu}} \right) \times \bigwedge_{j \in J_\nu} d\overline{w_j} \bigwedge_{l \in L_\nu} d\overline{z_l} \bigwedge \omega_n(w).$$

where $\Phi_n^q(w,z) \equiv \frac{(1-w\overline{z})^{n-1-q}\left(1-|w|^2\right)^q}{\triangle(w,z)^n}$ *for* $0 \leq q \leq n-1$.

We can rewrite the formula for $\mathcal{C}_n^{0,q}(w,z)$ as

$$\mathcal{C}_n^{0,q}(w,z) = \Phi_n^q(w,z) \sum_{|J|=q} \sum_{k \notin J} (-1)^{\mu(k,J)} \left(\overline{z_k} - \overline{w_k} \right) \times d\overline{z}^J \wedge d\overline{w}^{(J \cup \{k\})^c} \wedge \omega_n(w).$$

Explicit formulas for kernels in two dimensions

We have the formulas

$$\begin{aligned} & \mathcal{C}_2^{0,0}(w,z) \\ = \; & \frac{(1-w\overline{z})}{\triangle(w,z)^2}\left[(\overline{z}_2-\overline{w}_2)d\overline{w}_1\wedge dw_1\wedge dw_2-(\overline{z}_1-\overline{w}_1)d\overline{w}_2\wedge dw_1\wedge dw_2\right] \end{aligned}$$

and

$$\begin{aligned} & \mathcal{C}_2^{0,1}(w,z) \\ = \; & \frac{(1-|w|^2)}{\triangle(w,z)^2}\left[(\overline{w}_2-\overline{z}_2)d\overline{z}_1\wedge dw_1\wedge dw_2-(\overline{w}_1-\overline{z}_1)d\overline{z}_2\wedge dw_1\wedge dw_2\right]. \end{aligned}$$

An explicit formula for a kernel in three dimensions

- In $n = 3$ dimensions, the simplest kernel is given by

$$\begin{aligned} & \mathcal{C}_3^{0,q}(w,z) \\ = & \sum_{\sigma\in\mathcal{S}_3} sgn(\sigma)\frac{(1-w\overline{z})^{2-q}\left(1-|w|^2\right)^q\left(\overline{z_{\sigma(1)}}-\overline{w_{\sigma(1)}}\right)}{\triangle(w,z)^3} \\ & \times d\overline{\zeta_{\sigma(2)}}\wedge d\overline{\zeta_{\sigma(3)}}\wedge\omega_3(w), \end{aligned}$$

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- where $\mathcal{S}_3$ denotes the group of permutations on $\{1,2,3\}$ and q determines the number of $d\overline{z_i}$ in the form $d\overline{\zeta_{\sigma(2)}} \wedge d\overline{\zeta_{\sigma(3)}}$:

$$d\overline{\zeta_{\sigma(2)}} \wedge d\overline{\zeta_{\sigma(3)}} = \begin{cases} d\overline{w_{\sigma(2)}} \wedge d\overline{w_{\sigma(3)}} & \text{if} \quad q = 0 \\ d\overline{z_{\sigma(2)}} \wedge d\overline{w_{\sigma(3)}} & \text{if} \quad q = 1 \\ d\overline{z_{\sigma(2)}} \wedge d\overline{z_{\sigma(3)}} & \text{if} \quad q = 2 \end{cases}.$$

Real variable spaces

We introduce real variable analogues of the holomorphic Besov-Sobolev spaces.

Invariant derivatives

- Define

$$\nabla_z = \left(\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n} \right) \text{ and } \overline{\nabla_z} = \left(\frac{\partial}{\partial \overline{z_1}}, ..., \frac{\partial}{\partial \overline{z_n}} \right)$$

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- Fix $\alpha \in \mathcal{T}_n$ the Bergman tree and let $a = c_\alpha$. Recall that the gradient with invariant length given by

$$\begin{aligned} \widetilde{\nabla} f(a) &= (f \circ \varphi_a)'(0) = f'(a)\, \varphi_a'(0) \\ &= -f'(a) \left\{ \left(1 - |a|^2\right) P_a + \left(1 - |a|^2\right)^{\frac{1}{2}} Q_a \right\} \end{aligned}$$

fails to be holomorphic in a.

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fails to be holomorphic in a.
- To rectify this, we define for $z \in \mathbb{B}_n$,
$$\begin{aligned}D_a f(z) &= f'(z)\,\varphi_a'(0) \\ &= -f'(z)\left\{\left(1-|a|^2\right)P_a + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a\right\}.\end{aligned} \tag{13}$$

The two-dimensional derivative

- When $n = 2$ we can calculate D_a neatly using the basis

$$\left\{ a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, a^{\perp} = \begin{pmatrix} -\overline{a_2} \\ \overline{a_1} \end{pmatrix} \right\}.$$

The two-dimensional derivative

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- In the basis $\{a, a^\perp\}$, we compute that$-D_a$ is

$$\left(\left(1 - |a|^2\right) \left(a_1 \frac{\partial}{\partial z_1} + a_2 \frac{\partial}{\partial z_2} \right), \left(1 - |a|^2\right)^{\frac{1}{2}} \left(-\overline{a_2} \frac{\partial}{\partial z_1} + \overline{a_1} \frac{\partial}{\partial z_2} \right) \right),$$

where at the point a, $\left(a_1 \frac{\partial}{\partial z_1} + a_2 \frac{\partial}{\partial z_2} \right) f(a) = f'(a)\, a$ is the complex radial derivative of f at a, and $\left(-\overline{a_2} \frac{\partial}{\partial z_1} + \overline{a_1} \frac{\partial}{\partial z_2} \right) f(a) = f'(a)\, a^\perp$ is the complex tangential derivative of f at a.

A tree seminorm

Lemma

Let $a, b \in \mathbb{B}_n$ *satisfy* $\beta(a,b) \leq C$. *There is a positive constant* C_m *depending only on* C *and* m *such that*

$$C_m^{-1} \left| D_b^m f(z) \right| \leq \left| D_a^m f(z) \right| \leq C_m \left| D_b^m f(z) \right|,$$

for all $f \in H(\mathbb{B}_n)$.

Definition

Suppose $\sigma \geq 0$, $1 < p < \infty$ and $m \geq 1$. We define a "tree semi-norm" $\|\cdot\|^*_{B^\sigma_{p,m}(\mathbb{B}_n)}$ by

$$\|f\|^*_{B^\sigma_{p,m}(\mathbb{B}_n)} = \left(\sum_{\alpha \in \mathcal{T}_n} \int_{B_d(c_\alpha, C_2)} \left| \left(1 - |z|^2\right)^\sigma D^m_{c_\alpha} f(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}}. \quad (14)$$

Pointwise multipliers

Lemma

We have

$$\|f\|^*_{B^\sigma_{p,m}(\mathbb{B}_n)} + \sum_{j=0}^{m-1} \left|\nabla^j f(0)\right| \approx \|f\|_{B^\sigma_{p,m}(\mathbb{B}_n)}.$$

Let $\varphi \in H^\infty(\mathbb{B}_n) \cap B^\sigma_p(\mathbb{B}_n)$. If $m > \frac{n}{p} - \sigma$ and $0 \le \sigma < \infty$, then φ is a pointwise multiplier on $B^\sigma_p(\mathbb{B}_n)$ if and only if

$$\left|\left(1-|z|^2\right)^{m+\sigma} \nabla^m \varphi(z)\right|^p d\lambda_n(z) \tag{15}$$

is a $B^\sigma_p(\mathbb{B}_n)$-Carleson measure on $\mathbb{B}_n$. If $m > 2\left(\frac{n}{p} - \sigma\right)$ and $0 \le \sigma < \frac{n}{p} + 1$, then (15) can be replaced by

$$\left|\left(1-|z|^2\right)^{\sigma} D^m \varphi(z)\right|^p d\lambda_n(z).$$

The real variable Besov space

Definition

We denote by $\mathcal{X}^m$ the vector of all differential operators of the form $X_1 X_2 ... X_m$ where each X_i is either the identity operator I, the operator $\overline{D}$, or the operator $\left(1-|z|^2\right) R$. We calculate the products $X_1 X_2 ... X_m$ by composing $\overline{D_a}$ and $\left(1-|a|^2\right) R$ and then setting $a=z$ at the end. Note that $\overline{D_a}$ and $\left(1-|a|^2\right) R$ commute since the first is an antiholomorphic derivative and the coefficient z in $R = z \cdot \nabla$ is holomorphic.

Definition

We define the *norm* $\|\cdot\|_{\Lambda^{\sigma}_{p,m}(\mathbb{B}_n)}$ for f smooth on the ball $\mathbb{B}_n$ by

$$\|f\|_{\Lambda^{\sigma}_{p,m}(\mathbb{B}_n)} \equiv \left(\int_{\mathbb{B}_n} \left| \left(1-|z|^2\right)^{\sigma} \mathcal{X}^m f(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}}. \qquad (16)$$

Lemma

$$\left|\left(\overline{z-w}\right)^{\alpha}\frac{\partial^m}{\partial\overline{w}^{\alpha}}F(w)\right|\leq C\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^m\left|\overline{D}^m F(w)\right|,\qquad m=|\alpha|. \tag{17}$$

$$\begin{aligned} |D_z\triangle(w,z)| &\leq C\left\{\left(1-|z|^2\right)\triangle(w,z)^{\frac{1}{2}}+\triangle(w,z)\right\}, \\ \left|\left(1-|z|^2\right)R\triangle(w,z)\right| &\leq C\left(1-|z|^2\right)\sqrt{\triangle(w,z)}, \end{aligned} \tag{18}$$

$$\begin{aligned} \left|D_z^m\left\{\left(1-\overline{w}z\right)^k\right\}\right| &\leq C\left|1-\overline{w}z\right|^k\left(\frac{1-|z|^2}{|1-\overline{w}z|}\right)^{\frac{m}{2}}, \\ \left|\left(1-|z|^2\right)^m R^m\left\{\left(1-\overline{w}z\right)^k\right\}\right| &\leq C\left|1-\overline{w}z\right|^k\left(\frac{1-|z|^2}{|1-\overline{w}z|}\right)^m. \end{aligned} \tag{19}$$

Integration by parts

We generalize the integration by parts formulas of Ortega and Fabrega. These formulas serve to relax the singularities of the kernel on the diagonal and boundary at the expense of differentiating the form.

A reproducing vector field

- Let $\overline{\mathcal{Z}} = \overline{\mathcal{Z}_{z,w}}$ be the vector field acting in the variable $w = (w_1, w_2)$ and parameterized by $z = (z_1, z_2)$ given by

$$\overline{\mathcal{Z}} = \overline{\mathcal{Z}_{z,w}} = (\overline{w_1} - \overline{z_1}) \frac{\partial}{\partial \overline{w_1}} + (\overline{w_2} - \overline{z_2}) \frac{\partial}{\partial \overline{w_2}}. \tag{20}$$

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- The $(0,1)$-form $\overline{\mathcal{Z}}^m \eta$ is obtained from η by componentwise differentiation holding monomials in $\overline{w} - \overline{z}$ fixed, and acts on the vector field $\overline{\mathcal{Z}}$ to obtain

$$\begin{aligned} \left(\overline{\mathcal{Z}}^m \eta\right)\left[\overline{\mathcal{Z}}\right] &= \left(\sum_{k=1}^{2} \overline{\mathcal{Z}}^j \eta_k(w)\, d\overline{w_k}\right)\left(\sum_{j=1}^{2} (\overline{w_j} - \overline{z_j}) \frac{\partial}{\partial \overline{w_j}}\right) \\ &= \sum_{k=1}^{n} (\overline{w_1} - \overline{z_1}) \overline{\mathcal{Z}}^m \eta_1(w) + (\overline{w_2} - \overline{z_2}) \overline{\mathcal{Z}}^m \eta_2(w). \end{aligned}$$

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- A key property is $\overline{\mathcal{Z}} \triangle(w,z) = \triangle(w,z)$.

The covariant derivative

- The Charpentier kernel $\mathcal{C}_n^{0,q}(w,z)$ takes $(0,q+1)$-forms in w to $(0,q)$-forms in z. In order to express the solution operator $\mathcal{C}_n^{0,q}$ in terms of a volume integral, our definition of $\overline{\mathcal{D}}^m\eta$, must include an appropriate exchange of w-differentials for z-differentials.

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Lemma

Let $m \geq 0$. For a $(0,q+1)$-form $\eta = \sum_{|I|=q+1} \eta_I d\overline{w}^I$ in the variable w, define the $(0,q)$-form $\overline{\mathcal{D}}^m\eta$ in the variable z by

$$\overline{\mathcal{D}}^m \eta(w) = \sum_{|J|=q} \overline{\mathcal{Z}}^m \left(\eta \lrcorner d\overline{w}^J\right) \left[\overline{\mathcal{Z}}\right](w)\, d\overline{z}^J.$$

Explanation of the derivative

The effect of $\overline{\mathcal{D}}^m$ on a basis element $\eta_I d\overline{w}^I$ is to replace a differential $d\overline{w_k}$ from $d\overline{w}^I$ ($I = J \cup \{k\}$) with the factor $(-1)^{\mu(k,J)} (\overline{w_k} - \overline{z_k})$ (and this is accomplished by acting a $(0,1)$-form on $\overline{\mathcal{Z}}$), replace the remaining differential $d\overline{w}^J$ with $d\overline{z}^J$, and then to apply the differential operator $\overline{\mathcal{Z}}^m$ to the coefficient η_I. We will refer to the factor $(\overline{w_k} - \overline{z_k})$ introduced above as a *rogue* factor since it is not associated with a derivative $\frac{\partial}{\partial \overline{w_k}}$ in the way that $\left(\overline{w-z}\right)^\alpha$ is associated with $\frac{\partial^m}{\partial \overline{w}^\alpha}$. The point of this distinction will be explained later when dealing with estimates for solution operators.

The following lemma expresses $\mathcal{C}_n^{0,q}\eta(z)$ in terms of integrals involving $\overline{\mathcal{D}}^j\eta$ for $0 \leq j \leq m$. Note that the overall effect is to reduce the singularity of the kernel on the diagonal by m factors of $\sqrt{\triangle(w,z)}$, at the cost of increasing by m the number of derivatives hitting the form η. Let

$$\Phi_n^\ell(w,z) \equiv \frac{(1-w\overline{z})^{n-1-\ell}\left(1-|w|^2\right)^\ell}{\triangle(w,z)^n}.$$

Lemma

Let $q \geq 0$. For all $m \geq 0$ we have the formula,

$$\mathcal{C}_n^{0,q}\eta(z) = \sum_{k=0}^{m-1} c_k \mathcal{S}_n\left(\overline{\mathcal{D}}^k\eta\right)(z) + \sum_{\ell=1}^{q} c_\ell \Phi_n^\ell\left(\overline{\mathcal{D}}^m\eta\right)(z). \qquad (21)$$

Relaxing boundary singularities with radial derivatives

- Recall $R = \sum_{j=1}^{n} w_j \frac{\partial}{\partial w_j}$ and $R_b = \frac{n+b+1}{b+1} I + \frac{1}{b+1} R$.

Lemma

Let $b > -1$. For $\Psi \in C\left(\overline{\mathbb{B}_n}\right) \cap C^{\infty}\left(\mathbb{B}_n\right)$ we have

$$\int_{\mathbb{B}_n} \left(1 - |w|^2\right)^b \Psi(w)\, dV(w) = \int_{\mathbb{B}_n} \left(1 - |w|^2\right)^{b+1} R_b \Psi(w)\, dV(w).$$

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- The important point is that combinations of radial derivatives R and the identity I are played off against powers of $1 - |w|^2$, thus relaxing the singularity of the kernel on the boundary at the expense of differentiating the function.

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Lemma

Let $b>-1$. For $\Psi\in C\left(\overline{\mathbb{B}_n}\right)\cap C^{\infty}\left(\mathbb{B}_n\right)$ we have

$$\int_{\mathbb{B}_n}\left(1-|w|^2\right)^b\Psi(w)\,dV(w)=\int_{\mathbb{B}_n}\left(1-|w|^2\right)^{b+1}R_b\Psi(w)\,dV(w).$$

- The important point is that combinations of radial derivatives R and the identity I are played off against powers of $1-|w|^2$, thus relaxing the singularity of the kernel on the boundary at the expense of differentiating the function.
- Typically this lemma is applied with $\Psi(w)=\frac{1}{(1-\overline{w}z)^s}\psi(w,z)$, where z is a parameter in the ball $\mathbb{B}_n$ and then $R\Psi(w)=\frac{1}{(1-\overline{w}z)^s}R\psi(w,z)$ since $\frac{1}{(1-\overline{w}z)^s}$ is antiholomorphic in w.

Exchanging boundary singularities for tamer ones

Lemma

Suppose that $s > n$ *and* $0 \leq q \leq n-1$. *For all* $m \geq 0$ *and smooth* $(0, q+1)$*-forms* η *in* $\overline{\mathbb{B}_n}$ *we have the formula,*

$$\mathcal{C}_{n,s}^{0,q}\eta\left(z\right) = \sum_{k=0}^{m-1} c'_{k,n,s}\mathcal{S}_{n,s}\left(\overline{\mathcal{D}}^{k}\eta\right)\left[\overline{\mathcal{Z}}\right]\left(z\right) + \sum_{\ell=0}^{q} c_{\ell,n,s}\Phi_{n,s}^{\ell}\left(\overline{\mathcal{D}}^{m}\eta\right)\left(z\right),$$

where the ameliorated operators $\mathcal{S}_{n,s}$ *and* $\Phi_{n,s}^{\ell}$ *have kernels given by,*

$$\begin{aligned}
\mathcal{S}_{n,s}\left(w,z\right) &= c_{n,s}\left(\frac{1-\left|w\right|^{2}}{1-\overline{w}z}\right)^{s-n-1}\frac{1}{\left(1-\overline{w}z\right)^{n+1}},\\
\Phi_{n,s}^{\ell}(w,z) &= \Phi_{n}^{\ell}\left(w,z\right)\left(\frac{1-\left|w\right|^{2}}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-\ell-1} c_{j,n,s}\left(\frac{\left(1-\left|w\right|^{2}\right)\left(1-\left|z\right|^{2}\right.}{\left|1-w\overline{z}\right|^{2}}\right.
\end{aligned}$$

The point is to exchange boundary singularities with tamer ones.

Schur's Test

Scur's test is surprisingly effective in dealing with boundedness of the ameliorated Charpentier solution operators on the real-variable Besov spaces.

Schur's Test

Here we characterize boundedness of the positive operators that arise as majorants of the solution operators below. The case $c = 0$ of the following lemma is Theorem 2.10 in Zhu [27]. Note the balance of powers in numerator and denominator.

Lemma

Let $a, b, c, t \in \mathbb{R}$. *Then the operator*

$$T_{a,b,c}f(z) = \int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^a \left(1-|w|^2\right)^b \left(\sqrt{\triangle(w,z)}\right)^c}{|1-w\overline{z}|^{n+1+a+b+c}} f(w)\, dV(w)$$

is bounded on $L^p\left(\mathbb{B}_n; \left(1-|w|^2\right)^t dV(w)\right)$ *if and only if* $c > -2n$ *and*

$$-pa < t+1 < p(b+1). \tag{22}$$

Operator estimates on iterates of Charpentier's solution kernels

We describe how to use integration by parts, the crucial inequalities, and Schur estimates to prove the baby corona theorem. The *rogue* terms are discussed.

The main estimate in two dimensions

From the Koszul complex we must show that $f = \Omega_0^1 h - \Lambda_g \Gamma_0^2 \in B_p^\sigma(\mathbb{B}_n)$ where

$$
\begin{aligned}
f &= \Omega_0^1 h - \Lambda_g \Gamma_0^2 \\
&= \Omega_0^1 h - \Lambda_g \mathcal{C}_{2,s_1}^{0,0}\left(\Omega_1^2 h - \Lambda_g \Gamma_1^3\right) \\
&= \Omega_0^1 h - \Lambda_g \mathcal{C}_{2,s_1}^{0,0} \Omega_1^2 h + \Lambda_g \mathcal{C}_{2,s_1}^{0,0} \Lambda_g \mathcal{C}_{2,s_2}^{0,1} \Omega_2^3 h \\
&\equiv \mathcal{F}^0 - \mathcal{F}^1 + \mathcal{F}^2.
\end{aligned}
$$

Estimates

The goal is to establish

$$\left\| f \right\|_{B_p^\sigma(\mathbb{B}_2)} \leq C\left(g\right) \left\| h \right\|_{\Lambda_p^\sigma(\mathbb{B}_2)},$$

which we accomplish by showing that

$$\left\| \mathcal{F}^\mu \right\|_{B_{p,m_1}^\sigma(\mathbb{B}_2)} \leq C\left(g\right) \left\| h \right\|_{\Lambda_{p,m_\mu}^\sigma(\mathbb{B}_2)}, \qquad 0 \leq \mu \leq 2, \tag{23}$$

for a choice of integers m_μ satisfying

$$\frac{2}{p} - \sigma < m_1 < m_2.$$

Recall that we defined both of the norms $\left\| F \right\|_{B_{p,m_\mu}^\sigma(\mathbb{B}_2)}$ and $\left\| F \right\|_{\Lambda_{p,m_\mu}^\sigma(\mathbb{B}_2)}$ for smooth functions F in the ball $\mathbb{B}_2$.

The setup

- The norms $\|\cdot\|_{\Lambda^{\sigma}_{p,m}(\mathbb{B}_2)}$ will now be used to estimate the composition of Charpentier solution operators in each function

$$\begin{aligned}\mathcal{F}^2 &= \Lambda_g \mathcal{C}^{0,0}_{2,s_1} \Lambda_g \mathcal{C}^{0,1}_{2,s_2} \Omega^3_2 h,\\ \mathcal{F}^1 &= \Lambda_g \mathcal{C}^{0,0}_{2,s_1} \Omega^2_1 h,\\ \mathcal{F}^0 &= \Omega^1_0 h,\end{aligned}$$

as follows.

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\mathcal{F}^1 &= \Lambda_g \mathcal{C}^{0,0}_{2,s_1} \Omega^2_1 h, \\
\mathcal{F}^0 &= \Omega^1_0 h,
\end{aligned}
$$

as follows.

- We will use the facts that

$$
\begin{aligned}
\|h\|^p_{B^{\sigma}_p(\mathbb{B}_2)} &\approx \int_{\mathbb{B}_2} \left| \left(1-|z|^2\right)^{\sigma} \mathcal{X}^m h(z) \right|^p d\lambda_2(z), \qquad (\\
\|g\|^p_{M_{B^{\sigma}_p(\mathbb{B}_2)}} &\approx \|g\|^p_{\infty} + \left\| \left| \left(1-|z|^2\right)^{\sigma} \mathcal{X}^m g(z) \right|^p d\lambda_n(z) \right\|_{B^{\sigma}_p(\mathbb{B}_2)-Carle}
\end{aligned}
$$

for $0 \le \sigma < \frac{2}{p} + 1$ and $m > 2\left(\frac{2}{p} - \sigma\right)$.

- Fix attention for the moment on the function $\mathcal{F}^2 = \mathcal{F}_0^2$ and write

$$\begin{aligned}
\mathcal{F}_0^2 &= \Lambda_g \mathcal{C}_{2,s_1}^{0,0} \left\{ \Lambda_g \mathcal{C}_{2,s_2}^{0,1} \Omega_2^3 h \right\} = \Lambda_g \mathcal{C}_{2,s_1}^{0,0} \left\{ \mathcal{F}_1^2 \right\}, \\
\mathcal{F}_1^2 &= \Lambda_g \mathcal{C}_{2,s_2}^{0,1} \left\{ \Omega_2^3 h \right\} = \Lambda_g \mathcal{C}_{2,s_2}^{0,1} \left\{ \mathcal{F}_2^\mu \right\}, \\
\mathcal{F}_2^2 &= \Omega_2^3 h,
\end{aligned}$$

so that $\mathcal{F}_q^2 = \Lambda_g \mathcal{C}_{2,s_{q+1}}^{0,q} \left\{ \mathcal{F}_{q+1}^2 \right\}$ where $\mathcal{F}_q^2$ is a $(0, q)$-form.

The iterations

- Fix attention for the moment on the function $\mathcal{F}^2 = \mathcal{F}_0^2$ and write

$$\begin{aligned}
\mathcal{F}_0^2 &= \Lambda_g \mathcal{C}_{2,s_1}^{0,0} \left\{ \Lambda_g \mathcal{C}_{2,s_2}^{0,1} \Omega_2^3 h \right\} = \Lambda_g \mathcal{C}_{2,s_1}^{0,0} \left\{ \mathcal{F}_1^2 \right\}, \\
\mathcal{F}_1^2 &= \Lambda_g \mathcal{C}_{2,s_2}^{0,1} \left\{ \Omega_2^3 h \right\} = \Lambda_g \mathcal{C}_{2,s_2}^{0,1} \left\{ \mathcal{F}_2^\mu \right\}, \\
\mathcal{F}_2^2 &= \Omega_2^3 h,
\end{aligned}$$

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- The same can be done for $\mathcal{F}^1 = \mathcal{F}_0^1$:

$$\begin{aligned}
\mathcal{F}_0^1 &= \Lambda_g \mathcal{C}_{2,s_1}^{0,0} \left\{ \Omega_1^2 h \right\} = \Lambda_g \mathcal{C}_{2,s_1}^{0,0} \left\{ \mathcal{F}_1^1 \right\}, \\
\mathcal{F}_1^1 &= = \Omega_2^3 h.
\end{aligned}$$

Integration by parts in each factor

We perform integration by parts in $\mathcal{F}_q^{\mu} = \Lambda_g \mathcal{C}_{n,s_{q+1}}^{0,q} \left\{ \mathcal{F}_{q+1}^{\mu} \right\}$ to obtain

$$\begin{aligned} \mathcal{F}_q^{\mu} &= \Lambda_g \mathcal{C}_{n,s_{q+1}}^{0,q} \mathcal{F}_{q+1}^{\mu} \qquad (25) \\ &= \sum_{j=0}^{m'_{q+1}-1} c'_{j,n,s_{q+1}} \Lambda_g \mathcal{S}_{n,s_{q+1}} \left(\overline{\mathcal{D}}^{j} \mathcal{F}_{q+1}^{\mu} \right)(z) \\ &\quad + \sum_{\ell=0}^{\mu} c_{\ell,n,s_{q+1}} \Lambda_g \Phi_{n,s_{q+1}}^{\ell} \left(\overline{\mathcal{D}}^{m'_{q+1}} \mathcal{F}_{q+1}^{\mu} \right)(z) . \end{aligned}$$

The main terms

Now we compose these formulas for $\mathcal{F}_k^\mu$ to obtain an expression for $\mathcal{F}^\mu$ that is a complicated sum of compositions of the individual operators in (25) above. For now we will concentrate on the main terms $\Lambda_g \Phi^\mu_{n,s_{k+1}} \left(\overline{\mathcal{D}}^{m'_{k+1}} \mathcal{F}^\mu_{k+1} \right)$ that arise in the second sum above when $\ell = \mu$. The composition of these main terms is

$$\begin{aligned} & \left(\Lambda_g \Phi^\mu_{n,s_1} \overline{\mathcal{D}}^{m'_1} \right) \mathcal{F}^\mu_1 \qquad (26) \\ = & \left(\Lambda_g \Phi^\mu_{n,s_1} \overline{\mathcal{D}}^{m'_1} \right) \left(\Lambda_g \Phi^\mu_{n,s_2} \overline{\mathcal{D}}^{m'_2} \right) \mathcal{F}^\mu_2 \\ = & \left(\Lambda_g \Phi^\mu_{n,s_1} \overline{\mathcal{D}}^{m'_1} \right) \left(\Lambda_g \Phi^\mu_{n,s_2} \overline{\mathcal{D}}^{m'_2} \right) \ldots \left(\Lambda_g \Phi^\mu_{n,s_\mu} \overline{\mathcal{D}}^{m'_\mu} \right) \Omega^{\mu+1}_\mu h. \end{aligned}$$

The rogue factors

- At this point we would like to take absolute values inside all of these integrals and use the crucial inequalities to obtain a composition of positive operators of the type considered in Lemma 18. However, there is a difficulty in using inequality (17) to estimate the derivative $\overline{\mathcal{D}}^m$ on $(0, q+1)$-forms η given by

$$\overline{\mathcal{D}}^m \eta(z) = \sum_{|J|=q} \sum_{k \notin J} \sum_{|\alpha|=m} (-1)^{\mu(k,J)} \overline{(w_k - z_k)} \overline{(w-z)^\alpha} \frac{\partial^m}{\partial \overline{w}^\alpha} \eta_{J \cup \{k\}}(w)$$

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- The problem is that the factor $\overline{(w_k - z_k)}$ has no derivative $\frac{\partial}{\partial w_k}$ naturally associated with it, as do the other factors in $\overline{(w-z)^{\alpha}}$. We refer to the factor $\overline{(w_k - z_k)}$ as a *rogue* factor, as it requires special treatment in order to apply (17). Note that we cannot simply estimate $\overline{(w_k - z_k)}$ by $|w-z|$ because this is much larger in general than the estimate $\sqrt{\triangle (w,z)}$ obtained in (17).

When $n=2$ and $\mu=2$, we show that

$$\int_{\mathbb{B}_2}\left|\left(1-|z|^2\right)^{m_1+\sigma}R^{m_1}\Lambda_g\mathcal{C}^{0,0}_{n,s_1}\Lambda_g\mathcal{C}^{0,1}_{n,s_2}\Omega^3_2 h\right|^p d\lambda_2(z) \tag{27}$$
$$\leq C\int_{\mathbb{B}_2}\left|\left(1-|z|^2\right)^{\sigma}\left(1-|z|^2\right)^{m_3''}R^{m_3''}\overline{D}^{m_3'+2}\Omega^3_2 h(z)\right|^p d\lambda_2(z).$$

We will have to deal with a *rogue* term $\left(\overline{z_2}-\overline{\xi_2}\right)$ where there is no derivative $\frac{\partial}{\partial\overline{\xi_2}}$ to associate to the factor $\left(\overline{z_2}-\overline{\xi_2}\right)$. We perform integration by parts m_2' times in the first iterated integral in $\mathcal{C}^{0,0}_{n,s_1}\Lambda_g\mathcal{C}^{0,1}_{n,s_2}\Omega^3_2$, and m_3' times in the second iterated integral. We also perform integration by parts in the *radial* derivative m_2'' times in the *first* iterated integral, so that the additional factor $\left(1-|\xi|^2\right)^{m_2''}$ can be used crucially in the second iterated integral, and also m_3'' times in the *second* iterated integral for use in acting on Ω^3_2.

A typical part of the resulting kernel of the operator $\mathcal{C}_{2,s_1}^{0,0}\mathcal{C}_{2,s_2}^{0,1}\Omega_2^3(z)$ is

$$\begin{aligned}
&\int_{\xi\in\mathbb{B}_2}\frac{(1-\xi\overline{z})}{\triangle(\xi,z)^n}\left(\frac{1-|\xi|^2}{1-\xi\overline{z}}\right)^{s_1-n}\left(\overline{z_2}-\overline{\xi}_2\right) \qquad (28)\\
&\times\left(1-|\xi|^2\right)^{m_2'}R^{m_2'}\overline{\mathcal{D}}^{m_2''}\int_{w\in\mathbb{B}_2}\frac{(1-|w|^2)}{\triangle(w,\xi)^n}\left(\frac{1-|w|^2}{1-w\overline{\xi}}\right)^{s_2-n}\\
&\times\left(\overline{w_1}-\overline{\xi}_1\right)\left(1-|w|^2\right)^{m_3'}R^{m_3'}\overline{\mathcal{D}}^{m_3''}\Omega_2^3h(w)\,dV(w)\,dV(\xi),
\end{aligned}$$

where we have chosen $\left(\overline{z_2}-\overline{\xi}_2\right)$ and $\left(\overline{w_1}-\overline{\xi}_1\right)$ as the *rogue* factors. Now we recall that $\Omega_2^3h(w)$ must have both derivatives $\frac{\partial g}{\partial w_1}$ and $\frac{\partial g}{\partial w_2}$ occurring in it along with other harmless powers of g that we ignore. We set $h\equiv 1$ for simplicity.

So with

$$\overline{z_2} - \overline{\xi_2} = \left(\overline{z_2} - \overline{w_2}\right) - \left(\overline{\xi_2} - \overline{w_2}\right)$$

we write the above iterated integral as

$$\int_{\xi \in \mathbb{B}_2} \frac{\left(1 - \xi \overline{z}\right)}{\triangle\left(\xi, z\right)^2} \left(\frac{1 - \left|\xi\right|^2}{1 - \xi \overline{z}}\right)^{s_1 - 2}$$

$$\times \int_{w \in \mathbb{B}_2} \left(1 - \left|\xi\right|^2\right)^{m_2''} R^{m_2''} \overline{\mathcal{D}}^{m_2'} \left\{ \frac{\left(1 - \left|w\right|^2\right)}{\triangle\left(w, \xi\right)^2} \left(\frac{1 - \left|w\right|^2}{1 - w \overline{\xi}}\right)^{s_2 - 2} \right\}$$

$$\times \left[\left(1 - \left|w\right|^2\right)^{m_3''} R^{m_3''} \left(\overline{\xi_2} - \overline{w_2}\right) \frac{\partial}{\partial \overline{w_2}} \overline{\mathcal{D}}^{m_3' - \ell} g \right]$$

$$\times \left[\left(1 - \left|w\right|^2\right)^{m_3''} R^{m_3''} \left(\overline{\xi_1} - \overline{w_1}\right) \frac{\partial}{\partial \overline{w_1}} \overline{\mathcal{D}}^{\ell} g \right] dV\left(w\right) dV\left(\xi\right)$$

minus

$$\int_{\xi\in\mathbb{B}_2}\frac{(1-\xi\overline{z})}{\triangle(\xi,z)^n}\left(\frac{1-|\xi|^2}{1-\xi\overline{z}}\right)^{s_1-n}$$

$$\times\int_{w\in\mathbb{B}_2}\left(1-|\xi|^2\right)^{m_2''}R^{m_2''}\overline{\mathcal{D}}^{m_2'}\left\{\frac{\left(1-|w|^2\right)}{\triangle(w,\xi)^n}\left(\frac{1-|w|^2}{1-w\overline{\xi}}\right)^{s_2-n}\right\}$$

$$\times\left[\left(1-|w|^2\right)^{m_3''}R^{m_3''}\left(\overline{z_2}-\overline{w_2}\right)\frac{\partial}{\partial\overline{w_2}}\overline{\mathcal{D}}^{m_3'-\ell}g\right]$$

$$\times\left[\left(1-|w|^2\right)^{m_3''}R^{m_3''}\left(\overline{\xi_1}-\overline{w_1}\right)\frac{\partial}{\partial\overline{w_1}}\overline{\mathcal{D}}^{\ell}g\right]dV(w)\,dV(\xi).$$

Now we apply $\left(1-|z|^2\right)^{\sigma}\left(1-|z|^2\right)^{m_1''}R^{m_1''}D^{m_1'}$ to these operators. Using the crucial inequalities, the result of this application on the above integrals is dominated respectively by these two iterated integrals:

$$
\int_{\xi\in\mathbb{B}_2} \frac{\left(1-|z|^2\right)^{\sigma} |1-\xi\overline{z}|}{\triangle\left(\xi,z\right)^{m_1'+m_1''+2}} \left[\left(1-|z|^2\right)\sqrt{\triangle\left(\xi,z\right)}\right]^{m_1''}
$$

$$
\times \left\{ \left[\left(1-|z|^2\right)\sqrt{\triangle\left(\xi,z\right)}\right]^{m_1'} + \triangle\left(\xi,z\right)^{m_1'} \right\} \left|\frac{1-|\xi|^2}{1-\xi\overline{z}}\right|^{s_1-2}
$$

$$
\times \int_{w\in\mathbb{B}_2} \frac{\left(1-|\xi|^2\right)^{m_2''}\left(1-|w|^2\right)}{\triangle\left(w,\xi\right)^{m_2'+m_2''+2}} \left(\frac{\sqrt{\triangle\left(\xi,z\right)}}{1-|\xi|^2}\right)^{m_2'} \left[\left(1-|\xi|^2\right)\sqrt{\triangle\left(w\right.}\right.
$$

$$
\times \left\{ \left[\left(1-|\xi|^2\right)\sqrt{\triangle\left(w,\xi\right)}\right]^{m_2'} + \triangle\left(w,\xi\right)^{m_2'} \right\} \left|\frac{1-|w|^2}{1-w\overline{\xi}}\right|^{s_2-2}
$$

$$
\times \left(\frac{\sqrt{\triangle\left(w,\xi\right)}}{1-|w|^2}\right)^{m_3'} \left(\frac{\sqrt{\triangle\left(w,\xi\right)}}{1-|w|^2}\right)^{2}
$$

$$
\times \left|\left(1-|w|^2\right)^{m_3''} R^{m_3''}\overline{D^{m_3'+2}}g\left(w\right)\right| dV\left(w\right) dV\left(\xi\right);
$$

$$\int_{\xi\in\mathbb{B}_2}\frac{\left(1-|z|^2\right)^{\sigma}|1-\xi\overline{z}|}{\triangle(\xi,z)^{m_1'+m_1''+2}}\left[\left(1-|z|^2\right)\sqrt{\triangle(\xi,z)}\right]^{m_1''}$$

$$\times\left\{\left[\left(1-|z|^2\right)\sqrt{\triangle(\xi,z)}\right]^{m_1'}+\triangle(\xi,z)^{m_1'}\right\}\left|\frac{1-|\xi|^2}{1-\xi\overline{z}}\right|^{s_1-2}$$

$$\times\int_{w\in\mathbb{B}_2}\frac{\left(1-|\xi|^2\right)^{m_2''}\left(1-|w|^2\right)}{\triangle(w,\xi)^{m_2'+m_2''+2}}\left(\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}\right)^{m_2'}\left[\left(1-|\xi|^2\right)\sqrt{\triangle(w$$

$$\times\left\{\left[\left(1-|\xi|^2\right)\sqrt{\triangle(w,\xi)}\right]^{m}+\triangle(w,\xi)^{m_2'}\right\}\left|\frac{1-|w|^2}{1-w\overline{\xi}}\right|^{s_2-2}$$

$$\times\left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^{m_3'}\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)\left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)$$

$$\times\left|\left(1-|w|^2\right)^{m_3''}R^{m_3''}\overline{D^{m_3'+2}}g(w)\right|dV(w)\,dV(\xi).$$

- The only difference between these two iterated integrals is that one of the factors $\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}$ that occur in the first is replaced by the factor $\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}$ in the second.

- The only difference between these two iterated integrals is that one of the factors $\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}$ that occur in the first is replaced by the factor $\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}$ in the second.
- Now for the iterated integral in (29), we can separate it into the composition of two operators. One factor is the operator

$$\int_{\xi\in\mathbb{B}_2} \frac{\left(1-|z|^2\right)^{\sigma}|1-\xi\overline{z}|}{\triangle(\xi,z)^{m_1'+m_1''+2}}\left[\left(1-|z|^2\right)\sqrt{\triangle(\xi,z)}\right]^{m_1''} \tag{31}$$

$$\times\left\{\left[\left(1-|z|^2\right)\sqrt{\triangle(\xi,z)}\right]^{m_1'}+\triangle(\xi,z)^{m_1'}\right\}$$

$$\times\left(\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}\right)^{m_2'}\left|\frac{1-|\xi|^2}{1-\xi\overline{z}}\right|^{s_1-2}\left(1-|\xi|^2\right)^{-\sigma}F(\xi)\,dV(\xi),$$

- and the other factor is the operator

$$F(\xi) = \int_{w \in \mathbb{B}_2} \frac{\left(1-|\xi|^2\right)^{\sigma}\left(1-|w|^2\right)}{\triangle(w,\xi)^{m_2'+m_2''+2}} \left[\left(1-|\xi|^2\right)\sqrt{\triangle(w,\xi)}\right]^{m_2''} \tag{32}$$
$$\times \left\{\left[\left(1-|\xi|^2\right)\sqrt{\triangle(w,\xi)}\right]^{m_2'} + \triangle(w,\xi)^{m_2'}\right\} \left|\frac{1-|w|^2}{1-w\overline{\xi}}\right|^{s_2-2}$$
$$\times \left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^{m_3'+2} \left(1-|w|^2\right)^{-\sigma} f(w)\, dV(w),$$

where $f(w) = \left(1-|w|^2\right)^{\sigma} \left|\left(1-|w|^2\right)^{m_3''} R^{m_3''} \overline{D^{m_3'+2}} g(w)\right|$.

- and the other factor is the operator

$$F(\xi) = \int_{w\in\mathbb{B}_2} \frac{\left(1-|\xi|^2\right)^{\sigma}\left(1-|w|^2\right)}{\triangle(w,\xi)^{m_2'+m_2''+2}} \left[\left(1-|\xi|^2\right)\sqrt{\triangle(w,\xi)}\right]^{m_2''} \quad (32)$$

$$\times\left\{\left[\left(1-|\xi|^2\right)\sqrt{\triangle(w,\xi)}\right]^{m_2'} + \triangle(w,\xi)^{m_2'}\right\}\left|\frac{1-|w|^2}{1-w\overline{\xi}}\right|^{s_2-2}$$

$$\times\left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^{m_3'+2}\left(1-|w|^2\right)^{-\sigma} f(w)\,dV(w),$$

where $f(w) = \left(1-|w|^2\right)^{\sigma}\left|\left(1-|w|^2\right)^{m_3''} R^{m_3''}\overline{D^{m_3'+2}}g(w)\right|$.

- We can apply Lemma 18 to each of these operators to obtain the appropriate boundedness.

- To handle the integral in (30) we must first deal with the *rogue* factor $\sqrt{\triangle(w,z)}$ whose variable pair (w,z) doesn't match that of either of the denominators $\triangle(\xi,z)$ or $\triangle(w,\xi)$.

- To handle the integral in (30) we must first deal with the *rogue* factor $\sqrt{\triangle(w,z)}$ whose variable pair (w,z) doesn't match that of either of the denominators $\triangle(\xi,z)$ or $\triangle(w,\xi)$.
- For this we use the fact that

$$\sqrt{\triangle(w,z)} = |1 - w\overline{z}| \, |\varphi_z(w)| = \delta(w,z)^2 \rho(w,z),$$

where $\rho(w,z) = |\varphi_z(w)|$ is the invariant pseudohyperbolic metric on the ball and where $\delta(w,z) = |1 - w\overline{z}|^{\frac{1}{2}}$ satisfies the triangle inequality on the ball.

- Thus we have

$$
\begin{aligned}
\rho(w,z) &\leq \rho(\xi,z)+\rho(w,\xi), \\
\delta(w,z) &\leq \delta(\xi,z)+\delta(w,\xi),
\end{aligned}
$$

- Thus we have

$$\begin{aligned}\rho(w,z) &\leq \rho(\xi,z)+\rho(w,\xi),\\ \delta(w,z) &\leq \delta(\xi,z)+\delta(w,\xi),\end{aligned}$$

- and so also

$$\begin{aligned}\sqrt{\triangle(w,z)} &\leq 2\left[\delta(\xi,z)^2+\delta(w,\xi)^2\right]\left(|\varphi_z(\xi)|+\left|\varphi_\xi(w)\right|\right)\\ &= 2\left(1+\frac{|1-w\overline{\xi}|}{|1-\xi\overline{z}|}\right)\sqrt{\triangle(\xi,z)}\\ &\quad +2\left(1+\frac{|1-\xi\overline{z}|}{|1-w\overline{\xi}|}\right)\sqrt{\triangle(w,\xi)}.\end{aligned}$$

- Thus we can write

$$
\begin{aligned}
&\frac{\sqrt{\triangle(w,z)}}{1-|w|^2} \\
\lesssim\ &\frac{1-|\xi|^2}{1-|w|^2}\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}+\frac{|1-w\overline{\xi}|}{1-|w|^2}\frac{1-|\xi|^2}{|1-\xi\overline{z}|}\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2} \\
&+\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}+\frac{|1-\xi\overline{z}|}{1-|\xi|^2}\frac{1-|\xi|^2}{|1-w\overline{\xi}|}\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}.
\end{aligned}
\tag{33}
$$

- Thus we can write

$$
\begin{aligned}
&\frac{\sqrt{\triangle(w,z)}}{1-|w|^2} \\
\lesssim\ & \frac{1-|\xi|^2}{1-|w|^2}\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}+\frac{|1-w\overline{\xi}|}{1-|w|^2}\frac{1-|\xi|^2}{|1-\xi\overline{z}|}\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2} \\
&+\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}+\frac{|1-\xi\overline{z}|}{1-|\xi|^2}\frac{1-|\xi|^2}{|1-w\overline{\xi}|}\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}.
\end{aligned}
\tag{33}
$$

- All of the terms on the right hand side of (33) are of an appropriate form to distribute throughout the iterated integral, and again Lemma 18 applies to obtain the appropriate boundedness, but with $c=\pm 1$ this time as the remaining parameters have already been chosen.

Open Problems

- Does the algebra $H^{\infty}(\mathbb{B}_n)$ of bounded analytic functions on the ball have a corona in its maximal ideal space? The lack of Blaschke products in higher dimensions precludes the nonlinear best approximation used in the original proof of Carleson. The failure of the complete Nevanlinna-Pick property for $H^2(\mathbb{B}_n)$ when $n \geq 2$ precludes the use of the Toeplitz corona theorem.

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- Does the corona theorem for the multiplier algebra of the Drury-Arveson space H_n^2 extend to more general domains in $\mathbb{C}^n$, such as strictly pseudoconvex domains?
- Can we prove a corona theorem for *any* algebra in higher dimensions that is not the multiplier algebra of a Hilbert space with the complete Nevanlinna-Pick property?
- Is there a more robust notion of completing point evaluations by holomorphy extension, that circumvents the example of Sibony, and produces an empty "corona"?

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