

Corona and completion problems

for A_+

Amol Sasane

Mathematics Department

London School of Economics

(1) Linear control systems ; \mathcal{A}_+

(2) Stabilization problem ; corona theorem

(3) Corona theorem for \mathcal{A}_+

(4) Completion problem for \mathcal{A}_+ .

Linear Control Systems

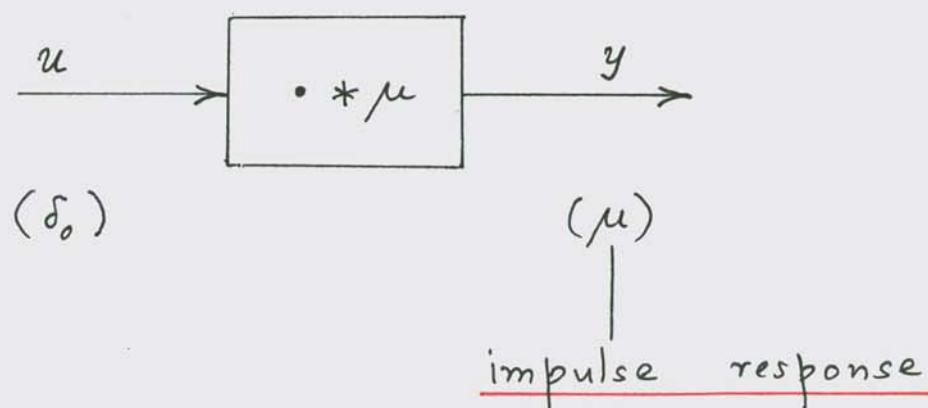
$$u \mapsto y$$

$$y = \mu * u$$

$$u, y : [0, \infty) \longrightarrow \mathbb{C}$$

u : input function

y : output function



$$\left. \begin{array}{l} \frac{d}{dt} x(t) = A x(t) + B u(t) \\ y(t) = C x(t) + D u(t) \end{array} \right\} \quad t \geq 0$$

$$A : D(A) (\subset X) \rightarrow X$$

$$x(t) \in X$$

$$B : \mathbb{C} \rightarrow X$$

$$u(t) \in \mathbb{C}$$

$$C : X \rightarrow \mathbb{C}$$

$$y(t) \in \mathbb{C}$$

$$D : \mathbb{C} \rightarrow \mathbb{C}$$

$$x(0) = 0 \Rightarrow y = \mu * u, \text{ where}$$

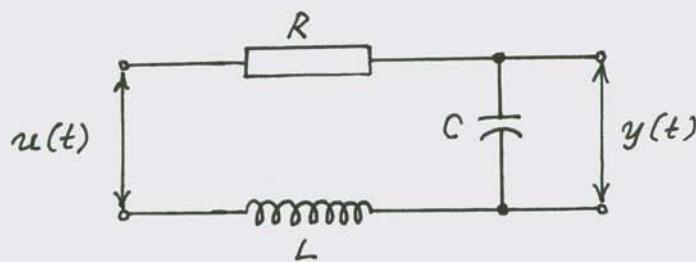
$$\mu = C e^{tA} B \mathbb{1} + D \delta_0$$

$$\hat{y}(s) = \underbrace{\hat{\mu}(s)}_{|} \cdot \hat{u}(s)$$

transfer function

Examples

(1)



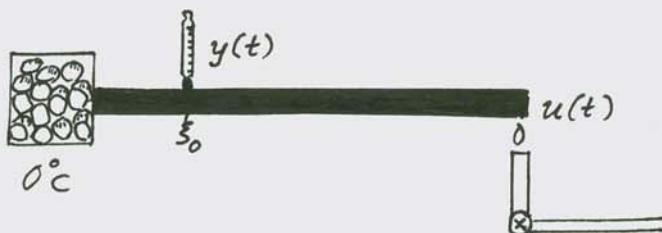
$$\hat{\mu}(s) = \frac{1}{s(LCs + RC)}$$

(2)

$$\begin{cases} \frac{dx}{dt} = x(t-1) + u(t) \\ y(t) = x(t) \end{cases}$$

$$\hat{\mu}(s) = \frac{1}{s - e^{-s}}$$

(3)



$$\hat{\mu}(s) = \frac{\cosh(\sqrt{s}\xi_0)}{\sqrt{s} \sinh \sqrt{s}}$$

Stable control system

Nice inputs $u \mapsto$ nice outputs y

Has transfer function $\hat{\mu} \in \mathcal{A}_+$.

Proposition

If $\hat{\mu} \in \mathcal{A}_+$, $1 \leq p \leq +\infty$, $u \in L^p[0, \infty)$,

then $y \in L^p[0, \infty)$.

Moreover, $\sup_{0 \neq u \in L^p} \frac{\|y\|_p}{\|u\|_p} \leq \|\hat{\mu}\|_{\mathcal{A}_+}$.

What is \mathcal{A}_+ ?

$$\mathcal{A}_+ = \left\{ \widehat{\mu} : \mathbb{C}_{\geq 0} \rightarrow \mathbb{C} \mid \begin{array}{l} \mu \text{ is a complex Borel measure on } \mathbb{R} \\ \text{whose support is in } [0, \infty), \\ \text{without a singular nonatomic part} \end{array} \right\}$$

$$= \left\{ s \mapsto \widehat{f_a}(s) + \sum_{k \geq 0} f_k e^{-st_k} \quad \begin{array}{l} f_a \in L^1[0, \infty) \\ (f_k)_{k \geq 0} \in \ell^1 \\ t_0 = 0 < t_1, t_2, t_3, \dots \end{array} \right\}$$

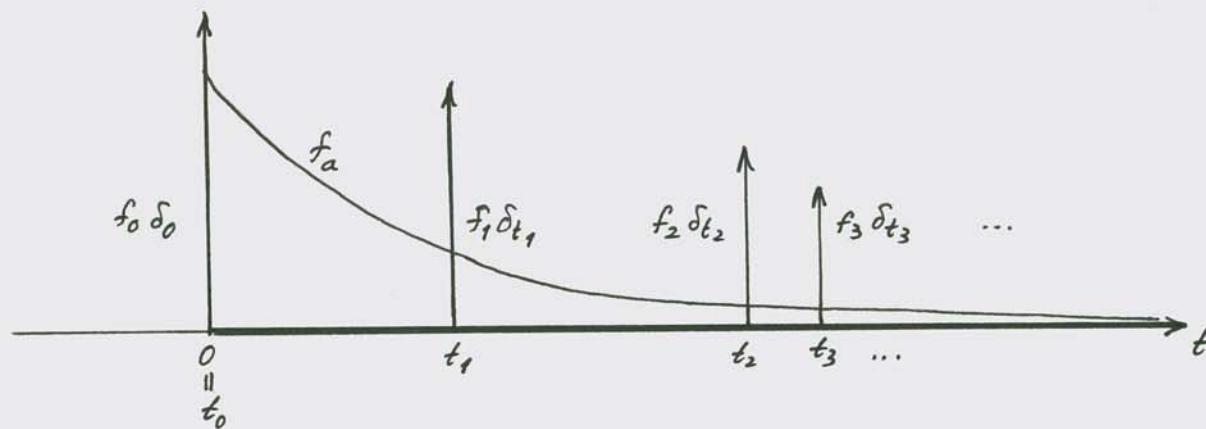
Proposition

\mathcal{A}_+ is a Banach algebra with pointwise operations and $\|\cdot\|_{\mathcal{A}_+}$ given by

$$\begin{aligned} \|\widehat{\mu}\|_{\mathcal{A}_+} &:= |\mu|([0, \infty)) \\ &= \|f_a\|_{L^1} + \|(f_k)_{k \geq 0}\|_{\ell^1}. \end{aligned}$$

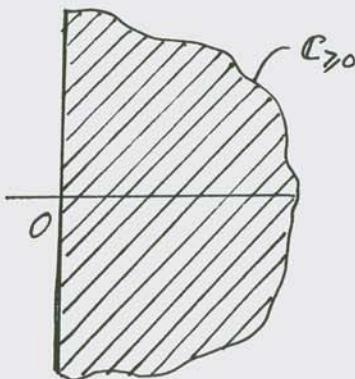
Time domain

$$\mu = f_a + \sum_{k>0} f_k \delta_{t_k}$$



$$\hat{\mu}(s) = \int_0^\infty e^{-st} d\mu(t)$$

Frequency domain



$$\begin{aligned} \hat{\mu} &= \hat{f}_a + F_{AP} \\ &\text{/} \\ &\text{bounded, continuous on } C_{>0}, \\ &\text{holomorphic in } C_{>0}. \end{aligned}$$

Stable control systems

$$\hat{\mu} \in \mathcal{A}_+$$

Unstable control systems

$$\hat{\mu} \in \mathbb{F}(\mathcal{A}_+) = \left\{ \frac{n}{d} : n, d \in \mathcal{A}_+, d \neq 0 \right\}$$

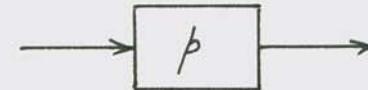
Example : $\frac{1}{s-1} \notin \mathcal{A}_+$

$$\frac{1}{s-1} = \frac{\frac{1}{s+1}}{\frac{s-1}{s+1}} \in \mathbb{F}(\mathcal{A}_+).$$

Stabilization problem

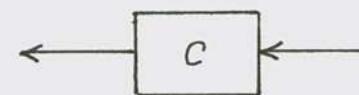
Given

$$\beta \in \mathbb{F}(\mathcal{A}_+)$$



Find

$$c \in \mathbb{F}(\mathcal{A}_+)$$

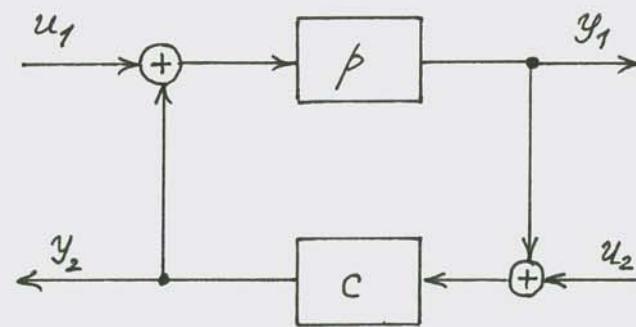


such that

$$\begin{bmatrix} \frac{\beta}{1-\beta c} & \frac{\beta c}{1-\beta c} \\ \frac{\beta c}{1-\beta c} & \frac{c}{1-\beta c} \end{bmatrix} \in \mathcal{A}_+^{2 \times 2}$$

transfer function of

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$



Solution to the stabilization problem

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$p \in \mathbb{F}(A_+)$ is stabilizable



p has a coprime factorization

$$p = \frac{n}{d}, \quad n, d \in A_+ \quad \text{s.t.} \quad \exists \quad x, y \in A_+ \quad \text{with} \quad \underline{\underline{nx + dy = 1}}$$

($c := -\frac{x}{y}$ stabilizes p ;

$$\frac{p}{1-pc} = \frac{\frac{n}{d}}{1 - \frac{n}{d} \cdot \frac{-x}{y}} = \frac{n}{d} \cdot \frac{dy}{\cancel{nx + dy}} = n \cdot y \in A_+$$

Relevance of the corona theorem

Given $b = \frac{n}{d}$, $n, d \in \mathbb{A}_+$, do there exist $x, y \in \mathbb{A}_+$ s.t. $nx + dy = 1$?

Theorem (Callier and Desoer, 1978)

Let $n, d \in \mathbb{A}_+$. The following are equivalent:

$$(1) \quad \exists x, y \in \mathbb{A}_+ \text{ s.t. } nx + dy = 1.$$

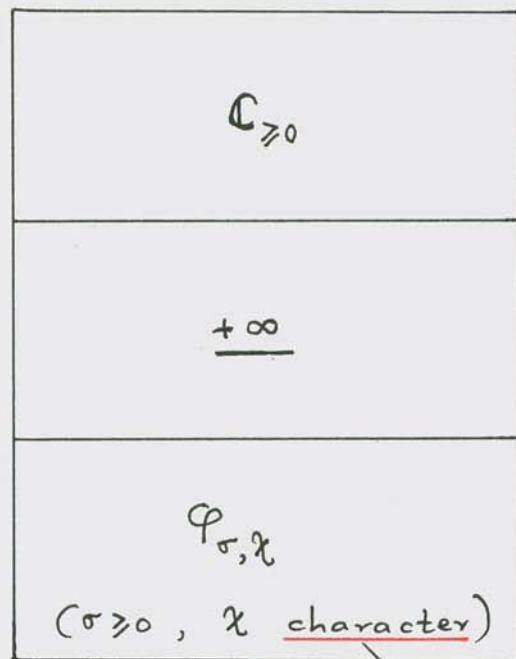
$$(2) \quad \exists \delta > 0 \text{ s.t. } |n(s)| + |d(s)| \geq \delta \quad (s \in \mathbb{C}_{\geq 0}).$$

Proof . relies on the description of $M(\mathbb{A}_+)$ given in Hille and Phillips, Functional Analysis and Semigroups, 1957.

Proposition

$M(A_+)$

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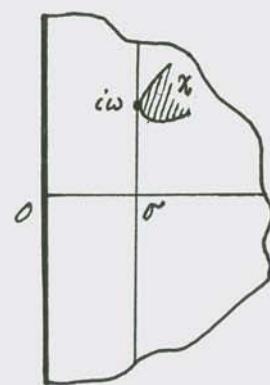
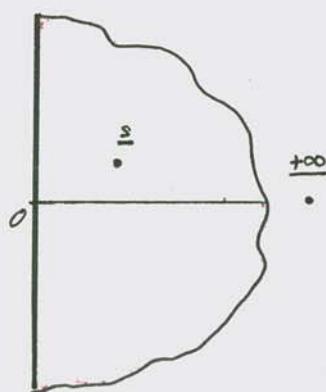
$$\hat{\mu} \xrightarrow{s} \hat{\mu}(s)$$

$$\hat{\mu} = \hat{f}_a + f_0 \cdot 1 + \sum_{k \geq 0} f_k e^{-st_k} \xrightarrow{s \rightarrow +\infty} \lim_{s \rightarrow +\infty} \hat{\mu}(s) = f_0$$

$$\hat{\mu} = \hat{f}_a + \sum_{k \geq 0} f_k e^{-st_k} \xrightarrow{\varphi_{\sigma, \chi}} \sum_{k \geq 0} f_k e^{-\sigma t_k} \chi(t_k)$$

$\chi : [0, \infty) \rightarrow \mathbb{C}$ s.t. $\forall t \geq 0, |\chi(t)| = 1$ and

$$\forall t_1, t_2 \geq 0, \chi(t_1 + t_2) = \chi(t_1) \cdot \chi(t_2).$$



Let $\varphi \in M(\mathcal{A}_+) \setminus (\mathbb{C}_{\geq 0} \cup \{\pm\infty\})$.

(1)

$$\forall f_a \in L^1[0, \infty), \quad \varphi(\widehat{f}_a) = 0.$$

$$\boxed{\widehat{L^1[0, \infty)} + \mathbb{C}} = \left\{ \widehat{f_a + f_0} : \begin{array}{l} f_a \in L^1[0, \infty) \\ f_0 \in \mathbb{C} \end{array} \right\} \subset \mathcal{A}_+ = \left\{ \widehat{f_a + f_0 + \sum_{k>0} f_k e^{-st_k}} : \dots \right\}$$

has maximal ideal space $\mathbb{C}_{\geq 0} \cup \{\pm\infty\}$.

$$\varphi(\widehat{f_{a*}}) \neq 0 \Rightarrow \varphi|_{\widehat{L^1[0, \infty)} + \mathbb{C}} \in M\left(\widehat{L^1[0, \infty)} + \mathbb{C}\right) = \mathbb{C}_{\geq 0} \cup \{\pm\infty\}.$$

$$\varphi|_{\widehat{L^1[0, \infty)} + \mathbb{C}} \neq +\infty \quad \left(\text{otherwise } \varphi(\widehat{f_{a*}}) = \lim_{s \rightarrow +\infty} \widehat{f_{a*}}(s) = 0 \right).$$

$$s_0 \quad \varphi|_{\widehat{L^1[0, \infty)} + \mathbb{C}} = \underline{s_0}, \quad s_0 \in \mathbb{C}_{\geq 0}.$$

$\widehat{\mu} \in \mathcal{A}_+ \Rightarrow \mu * f_{a*} \in L^1[0, \infty) \quad (\text{since weighted superposition of shifted versions of } f_{a*} \in L^1[0, \infty])$

$$\Rightarrow \varphi(\widehat{\mu}) = \frac{\varphi(\widehat{\mu} \cdot \widehat{f_{a*}})}{\varphi(\widehat{f_{a*}})} = \frac{(\widehat{\mu} \cdot \widehat{f_{a*}})(s_0)}{\widehat{f_{a*}}(s_0)} = \frac{\widehat{\mu}(s_0) \cdot \widehat{f_{a*}}(s_0)}{\widehat{f_{a*}}(s_0)} = \widehat{\mu}(s_0) = \underline{s_0}(\widehat{\mu}).$$

$$s_0 \quad \varphi = \underline{s_0} \quad \not\vdash \quad \square$$

$$(2) \quad \boxed{\forall \tau \geq 0, \varphi(e^{-s\tau}) \neq 0.}$$

$$\varphi(e^{-sT_*}) = 0 \Rightarrow \varphi(e^{-s\tau}) = 0 \quad \forall \tau \geq T_*$$



$$\varphi(e^{-s\tau}) = \varphi(e^{-sT_*} \cdot e^{-s(\tau - T_*)})$$

$$= \underbrace{\varphi(e^{-sT_*})}_{=0} \cdot \varphi(e^{-s(\tau - T_*)}) = 0.$$

$$(\varphi(e^{-sT_{*}/2}))^2 = \varphi(e^{-sT_*})$$



$$\varphi = \underline{+\infty} \quad \not\Leftarrow \quad \square$$



$$(3) \quad g(\tau) := \log |\varphi(e^{-s\tau})|, \quad \tau \geq 0.$$

$$g(0) = \log |\varphi(1)| = \log 1 = 0$$

$$g(\tau_1 + \tau_2) = \log |\varphi(e^{-s\tau_1} \cdot e^{-s\tau_2})| = \log |\varphi(e^{-s\tau_1}) \varphi(e^{-s\tau_2})| = g(\tau_1) + g(\tau_2).$$

$$|\varphi(e^{-s\tau})| \leq \|\varphi\| \|e^{-s\tau}\|_{\mathcal{L}_+} \leq 1 \cdot 1 = 1.$$

$$\Rightarrow g(\tau) \leq 0 \quad \forall \tau \geq 0.$$

As g is additive and bounded above, it can't be "weird".

$$\Rightarrow \exists \sigma \geq 0 \text{ s.t. } g(\tau) = -\sigma\tau$$

$$\Rightarrow |\varphi(e^{-s\tau})| = e^{-\sigma\tau}, \quad \tau \geq 0.$$

Define χ by

$$\chi(\tau) = \frac{\varphi(e^{-s\tau})}{|\varphi(e^{-s\tau})|}, \quad \tau \geq 0.$$

Then: $|\chi(\tau)| = 1 \quad \forall \tau \geq 0,$

$$\chi(\tau_1 + \tau_2) = \chi(\tau_1) \cdot \chi(\tau_2) \quad \forall \tau_1, \tau_2 \geq 0.$$

So χ is a character.

$$\begin{aligned} \varphi(\hat{\mu}) &= \varphi\left(\hat{f}_a + \sum_{k \geq 0} f_k e^{-st_k}\right) = \underbrace{\varphi(\hat{f}_a)}_{=0} + \sum_{k \geq 0} f_k \varphi(e^{-st_k}) \\ &= \sum_{k \geq 0} f_k \underbrace{\frac{|\varphi(e^{-st_k})|}{e^{-\sigma t_k}}}_{\chi(t_k)} \cdot \underbrace{\frac{\varphi(e^{-st_k})}{|\varphi(e^{-st_k})|}}_{\chi(t_k)} = \sum_{k \geq 0} f_k e^{-\sigma t_k} \cdot \chi(t_k) \\ &= \varphi_{\sigma, \chi}(\hat{\mu}) \quad \forall \hat{\mu} \in \mathcal{A}_+. \end{aligned}$$

We've shown $\varphi \in M(\mathcal{A}_+) \setminus (\mathbb{C}_{\geq 0} \cup \{\infty\}) \Rightarrow \varphi = \varphi_{\sigma, \chi}$ for some $\sigma \geq 0$, χ character. \square

Corona theorem for \mathcal{A}_+

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Suppose CC holds : $\forall s \in \mathbb{C}_{>0}$, $|n(s)| + |d(s)| \geq \delta > 0$.

$\langle n, d \rangle \neq \mathcal{A}_+ \Rightarrow \exists \varphi \in M(\mathcal{A}_+) \text{ s.t. } \langle n, d \rangle \subset \ker \varphi \subset \mathcal{A}_+$
 and so $\varphi(n) = 0, \varphi(d) = 0$.

1° $\varphi \neq \pm\infty$ (otherwise $0 = |n(s)| + |d(s)| \geq \delta > 0 \not\rightarrow$)

2° $\varphi \neq \pm\infty$ (otherwise $0 = \lim_{s \rightarrow +\infty} |n(s)| + |d(s)| \geq \delta > 0 \not\rightarrow$)

3° $\varphi = \varphi_{\sigma, \chi} \quad \sigma \geq 0, \chi \text{ character}$

$$n = \hat{n}_a + n_{AP} = \hat{n}_a + \sum_{k \geq 0} n_k e^{-st_k}$$

$$d = \hat{d}_a + d_{AP} = \hat{d}_a + \sum_{k \geq 0} d_k e^{-s\tau_k} e^{-i\omega t_k}$$

$$\left. \begin{array}{l} \varphi(n) = 0 \\ \varphi(d) = 0 \end{array} \right\} \Rightarrow \begin{aligned} 0 &= \sum_{k \geq 0} n_k e^{-\sigma t_k} \boxed{\chi(t_k)} \\ 0 &= \sum_{k \geq 0} d_k e^{-\sigma \tau_k} \boxed{\chi(\tau_k)} \end{aligned}$$

Kronecker's Approximation Theorem (1884)

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$v \in \mathbb{R}^n$ has rationally independent components

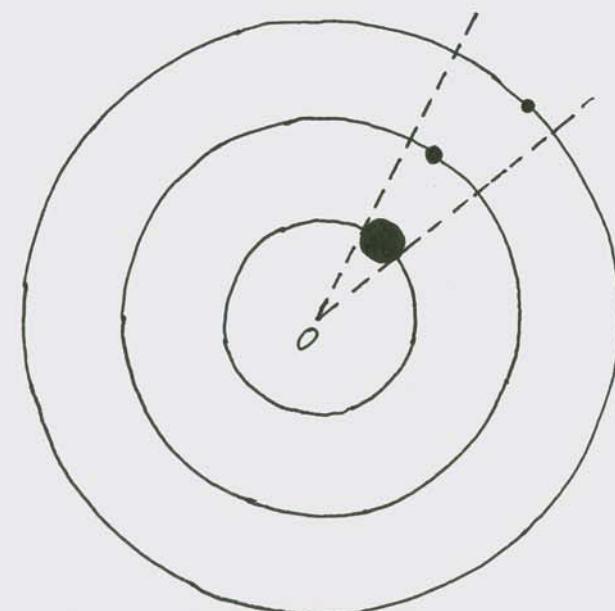


$\mathbb{R}v + \mathbb{Z}^n$ is dense in \mathbb{R}^n

(Chapter 23 of Theory of Numbers, Hardy and Wright, 1938)

Astronomical illustration

If the angular velocities
are rationally independent,
then occultations recur continually
whatever the initial positions.



$\exists \omega_* \in \mathbb{R}$ st.

$$\left| \sum_{k \geq 0} n_k e^{-\sigma t_k} \cdot e^{-i\omega_* t_k} \right| < \frac{\delta}{8}$$

$$\left| \sum_{k \geq 0} d_k e^{-\sigma \tau_k} \cdot e^{-i\omega_* \tau_k} \right| < \frac{\delta}{8}$$

i.e., $|n_{AP}(\sigma + i\omega_*)| + |d_{AP}(\sigma + i\omega_*)| < \frac{\delta}{4}$.

But $n_{AP}(\sigma + i\omega)$, $d_{AP}(\sigma + i\omega)$ are almost periodic,

since these are sums of exponentials $e^{-i\omega t_k}$, $e^{-i\omega \tau_k}$:

$$\omega \mapsto n_{AP}(\sigma + i\omega) = \sum_{k \geq 0} (n_k e^{-\sigma t_k}) \cdot e^{-i\omega t_k}$$

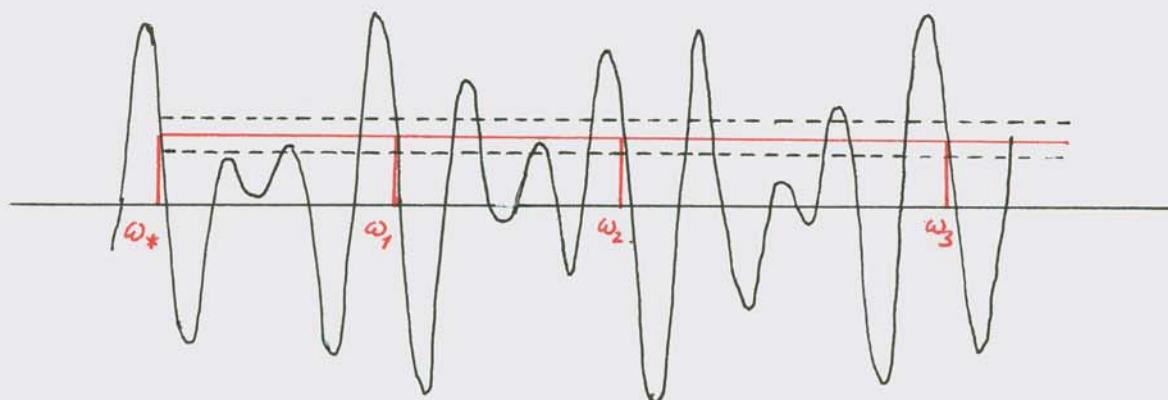
$$\omega \mapsto d_{AP}(\sigma + i\omega) = \sum_{k \geq 0} (d_k e^{-\sigma \tau_k}) \cdot e^{-i\omega \tau_k}$$

For almost periodic functions F ,

" ϵ -translation numbers are relatively dense in \mathbb{R} ", i.e.,

$\forall \epsilon > 0, \exists L > 0$ s.t. every interval of length L has a λ s.t.

$$\forall \omega \in \mathbb{R} \quad |F(\omega) - F(\omega + \lambda)| < \epsilon.$$



So $\exists (\omega_n)_{n \in \mathbb{N}}$ s.t. $\omega_n \xrightarrow{n \rightarrow \infty} +\infty$ and

$$\left| n_{AP}(\sigma + i\omega_*) - n_{AP}(\sigma + i\omega_n) \right| < \frac{\delta}{8}$$

$$\left| d_{AP}(\sigma + i\omega_*) - d_{AP}(\sigma + i\omega_n) \right| < \frac{\delta}{8}.$$

Riemann - Lebesgue Lemma \Rightarrow

$$\widehat{n}_a(\sigma + i\omega_n) \xrightarrow{n \rightarrow \infty} 0$$

$$\widehat{d}_a(\sigma + i\omega_n) \xrightarrow{n \rightarrow \infty} 0.$$

Taking $s = \sigma + i\omega_n$ in CC, we have for large n :

$$\underbrace{|\widehat{n}_a(s)| + |\widehat{d}_a(s)|}_{\text{as small as we please}} + \underbrace{|n_{AP}(s)| + |d_{AP}(s)|}_{< \frac{\delta}{2}} \geq \delta \quad \checkmark$$

So $\langle n, d \rangle = \mathcal{A}_+$, i.e., $\exists x, y \in \mathcal{A}_+$ s.t. $nx + dy = 1$. \square

Remark : Proof is nonconstructive.

Open question

For corona data of the form

$$a_0 + \frac{a_1}{s+1} + \frac{a_2}{(s+1)^2} + \dots + \frac{a_n}{(s+1)^n} + b_0 + b_1 e^{-sT} + b_2 e^{-2sT} + \dots + b_m e^{-msT},$$

is there a constructive method

to find $x, y \in \mathbb{A}_+$ s.t. $nx + dy = 1$?

Completion problem

In matrix versions of the stabilization problem

(i.e., when $u(t) \in \mathbb{C}^m$, $y(t) \in \mathbb{C}^p$)

$$\hat{\mu} \in \mathbb{F}(A_+)^{p \times m}$$

Notions of right / left coprime factorizations:

$$\hat{\mu} = ND^{-1} \quad \text{with} \quad NX + DY = I_m$$

$$\hat{\mu} = \tilde{D}^{-1}\tilde{N} \quad \text{with} \quad \tilde{X}\tilde{N} + \tilde{Y}\tilde{D} = I_p$$

For a general ring, \exists right coprime $\nleftrightarrow \exists$ left coprime factorization

If we have both, then we can solve the stabilization problem.

What happens if the ring is A_+ ?

Completion problem

$\forall n, k \in \mathbb{N}, n > k \quad \forall f \in \mathbb{R}^{n \times k} \text{ s.t. } \exists g \in \mathbb{R}^{k \times n} \text{ s.t. }$

$$\begin{matrix} g \\ \boxed{f} \end{matrix} = \boxed{I_k}$$

$\exists f_c \in \mathbb{R}^{n \times (n-k)}, G \in \mathbb{R}^{n \times n} \text{ s.t. }$

$$\begin{matrix} G \\ \boxed{f} \end{matrix} = \boxed{I_n}$$

Rings with this property are called Hermite rings.

Is \mathbb{C}_+ Hermite?

Theorem (V. Ya. Lin, 1973)

Let R be a complex unital commutative semisimple Banach algebra.

Then $M(R)$ contractible $\Rightarrow R$ is Hermite.

Contractability

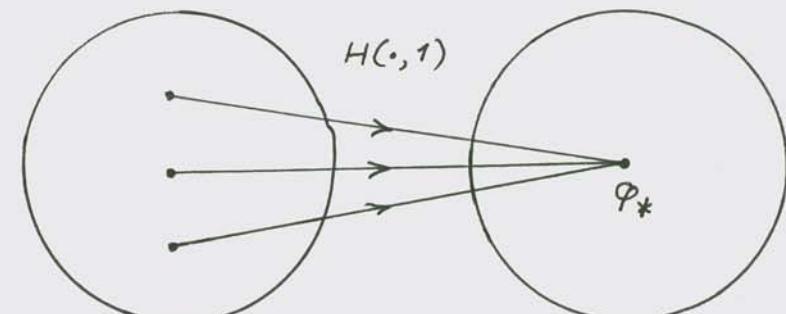
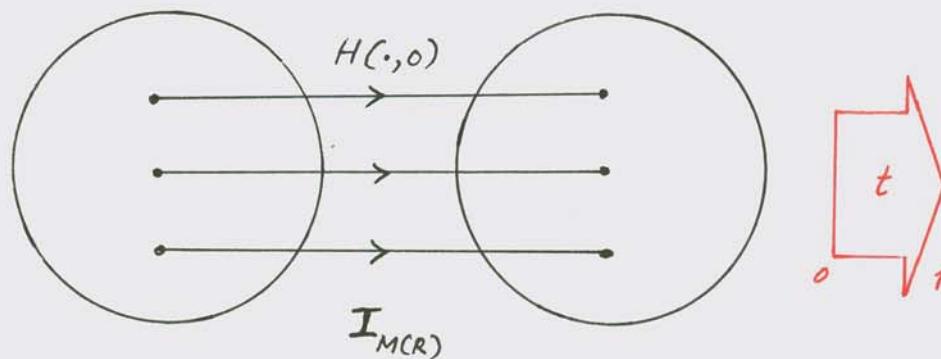
$M(R)$ is contractible if $I_{M(R)} \simeq \varphi_*$, i.e.,

$\exists \varphi_* \in M(R)$ and a continuous $H : M(R) \times [0, 1] \xrightarrow[t]{} M(R)$ s.t.

$$H(\varphi, 0) = \varphi \quad (\varphi \in M(R))$$

and

$$H(\varphi, 1) = \varphi_* \quad (\varphi \in M(R)).$$



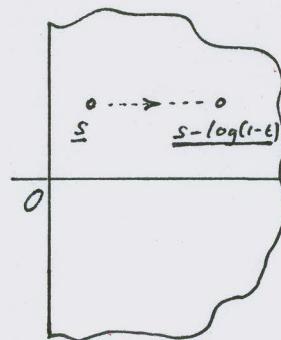
Theorem (A.S., 2010)

(1) $M(\mathcal{A}_+)$ is contractible.

(2) \mathcal{A}_+ is Hermite.

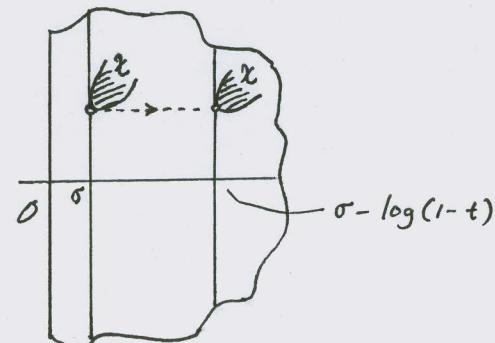
Proof. Define $H: M(\mathcal{A}_+) \times [0,1] \rightarrow M(\mathcal{A}_+)$ by

$$H(\underline{s}, t) = \underline{s - \log(1-t)}$$



$$H(\underline{+\infty}, t) = \underline{+\infty}$$

$$H(\varphi_{\sigma, x}, t) = \varphi_{\sigma - \log(1-t), x}$$



$$\text{Then } I_{M(\mathcal{A}_+)} \simeq \underline{+\infty}.$$

□