Toeplitz Operators and Hankel Forms on Model Spaces

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- $$k_\zeta(z) = \frac{1 - \overline{\Theta(\zeta)}\Theta(z)}{1 - \bar{\zeta} z}.$$
- $H^1_\Theta =$ the H^1 closure of H^2_Θ =the H^1 closure of K.

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- One can then study the relationship between the operator and the symbol.

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  - What is the condition?

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- We will see echoes of these statements later.

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  - $\Theta = \Theta_{2\alpha}$ the singular inner function generated by a point mass $2\alpha\delta_1$. The RKHS K is equivalent to the RKHS PW_α, the Paley-Wiener space, the subspace of $L^2(\mathbb{R})$ consisting of functions with Fourier transform supported on $[-\alpha, \alpha]$.

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- $\Theta = \Theta_{2\alpha}$: When transported to PW the conjugation becomes $e^{az} \to e^{-\overline{az}}$; i.e. $\widehat{\mathcal{C}f}(\xi) = \overline{\hat{f}(-\xi)}$.

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- For these inner functions the Carleson measure theory for K_Θ is relatively well understood.

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- Special cases of TTO's were studied in the '80's by Bercovici, Foias, Tannenbaum and by RR. However the systematic study of this class began with a 2007 paper of Sarason.

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- If $\Theta = \Theta_{2\alpha}$ and A_ϕ is carried to the Paley Wiener space we obtain (on the Fourier transform side) a Wiener-Hopf convolution operator with symbol $\hat{\phi}$, truncated to an interval; roughly

$$\widehat{Tf}(s) = \int \chi_{[-\alpha,\alpha]}(s) \hat{f}(t) \hat{\phi}(s-t)\, dt.$$

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- This sets up an antilinear isometric bijection between the set of TTOs and the set of THFs. That is, if A_ϕ is given then the equation defines, $B_{\mathcal{C}\phi}$; similarly in the other direction.

Equivalence of the Two Classes

- The relation between the two classes is that

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- This sets up an antilinear isometric bijection between the set of TTOs and the set of THFs. That is, if A_ϕ is given then the equation defines, $B_{\mathcal{C}\phi}$; similarly in the other direction.

- A consequence of this bijection is that many questions and answers can be easily carried back and forth. For instance, questions about finite rank operators/forms, trace class, bounded symbols, etc.

THFs, Examples

- For a kernel function k_ζ, the forms

$$\begin{aligned} B_{k_\zeta}(f,g) &= f(\zeta)g(\zeta) = \langle f, k_\zeta \rangle \langle g, k_\zeta \rangle, \\ B_{\mathcal{C}_{\Theta^2} k_\zeta}(f,g) &= \overline{\mathcal{C}f(\zeta)\mathcal{C}g(\zeta)} = \langle f, \mathcal{C}k_\zeta \rangle \langle g, \mathcal{C}k_\zeta \rangle \end{aligned}$$

  are rank one THFs and are essentially the only ones. This can be shown directly or seen as a consequence of the bijection and the result for TTOs. The first type are analogous to classical Hankel forms; the second class has not classical analog.

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- If $\Theta = z^{n+1}$ the matrix of a THF with respect to the monomial basis is the upper left square section of the Hankel matrix with symbol ϕ.
- If $\Theta = \Theta_{2\alpha}$ then B_ϕ is carried to a bilinear form on the Paley Wiener space of "truncated Hankel" type

$$B(f,g) = \int\int \hat{f}(s)\hat{g}(t)\hat{\phi}(s+t)dsdt.$$

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- The case of finite dimensional K is trivial — but only if one doesn't ask for estimates.
- Symbols of classical Toeplitz's and Hankel's restrict to TTOs and THFs with norms that are no larger.
- In particular, if the TTO or THF has a bounded symbol it is bounded.

The Question of Bounded Symbols

Theorem (Saragon 1967)

Given K, Θ, α *with* $|\alpha| < 1$, *and* $\phi \in \mathrm{Hol}(\mathbb{D})$.

1. *The TTO* A_ϕ *is bounded iff* $\exists\ \psi \in H^\infty$ *such that*

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- Proof: (1) Commutant lifting theorem; (2) unitary equivalence.
- For each α the set $\left\{A_{\phi(1+\alpha\bar{\Theta})}, \phi \in H^\infty\right\}$ is a commutative subalgebra of $\{\mathrm{TTO}\}$. There is no classical analog of this phenomenon.

Bounded Symbols, A Negative Result

Theorem (Baranov, Chalendar, Frican, Mashreghi, Timotin, 2009)

Suppose Θ *is given and the point evaluation at some* $\zeta \in \mathbb{T}$ *is bounded on* K_Θ*. If, for some* $p > 2$, $k_\zeta \notin L^p$ *then the rank one TTO* $k_\zeta \otimes k_\zeta$ *does not have a bounded symbol.*

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- It is automatic that $k_\zeta \in L^2$.
- Given $p > 2$, classical results give straightforward recipes for building examples for which $k_\zeta \notin L^p$.

Bounded Symbols, CLS Inner Functions

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Theorem (Aleksandrov, 1999)

If Θ is a CLS inner function then $E_1(\Theta^2) = E_2(\Theta^2)$.

- Some have speculated/conjectured the converse of that theorem holds; i.e., if every bounded TTO has a bounded symbol then Θ is CLS.

More Background

- If A is a TTO then

$$A = A_\alpha + A_\beta^*, \quad \alpha, \beta \in K \qquad \text{(split)}$$

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- There is a general theory of Schatten ideals $\mathcal{S}_p$, $0 < p < \infty$.
- It is a classical result (Peller, Rochberg, Semmes; 1980's) that a Hankel operator on the Hardy space is in $\mathcal{S}_p$ if and only if the holomorphic symbol is in the Besov smoothness class $\mathcal{B}_p$.

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- The part I can't prove is, in 2., that A_α and $A_{\bar{\beta}}$ are individually in $\mathcal{S}_p$
- The BCMFT example shows that this implication fails without some hypothesis on Θ.
- The "Theorem" is correct for the Paley-Wiener space (RR '87) and that proof can be extended a bit using ideas in BCMFT.

My Current Thoughts

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- However even that case comes down to a Helson-Szego style question about the angle between past and future, but for the Sobolev space of order $1/2$.

- We would like to know if there is an $\varepsilon > 0$ so that, given $\alpha, \beta \in K$. $\beta(0) = 0$

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- If, for example $\Theta = z^n$ then the left hand side is zero.

Tools for Proofs of Results in This Area

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- *Recent progress on truncated Toeplitz operators,* Garcia and Ross, arXiv:1108.1858 is a nice survey.

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  2. Are there analogs of the AAK results? For instance, is the best finite rank approximation to a TTO itself a TTO? If so, or if not, is there a good intrinsic description of the approximant. These questions are essentially equivalent to the analogous questions for THFs. A positive answer would resolve the open step in the "Theorem".

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- These questions are related to other basic problems in harmonic analysis, for instance characterizing the weights for which there is a two-weight weighted norm inequality for the Hilbert transform.

Broader Questions

- Several interrelated broader questions.
  - Describe the elements of $E_2(\Theta)$, the Carleson measures for K_Θ.
  - Which TTO or, equivalently, THF have bounded symbols? What happens in the other case?
  - Describe $H^2_\Theta \odot H^2_\Theta$ and $\left(H^2_\Theta \odot H^2_\Theta\right)^*$.
- These questions are related to other basic problems in harmonic analysis, for instance characterizing the weights for which there is a two-weight weighted norm inequality for the Hilbert transform.
- Research experience of recent years by a number of people suggests some of these and related questions are quite difficult.

Thank You !