Numerically Effective Corona Problems

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- The Problem: Effective Inversions and Solution of Bezout Equations

(constructive, algorithmic, norm controlled)

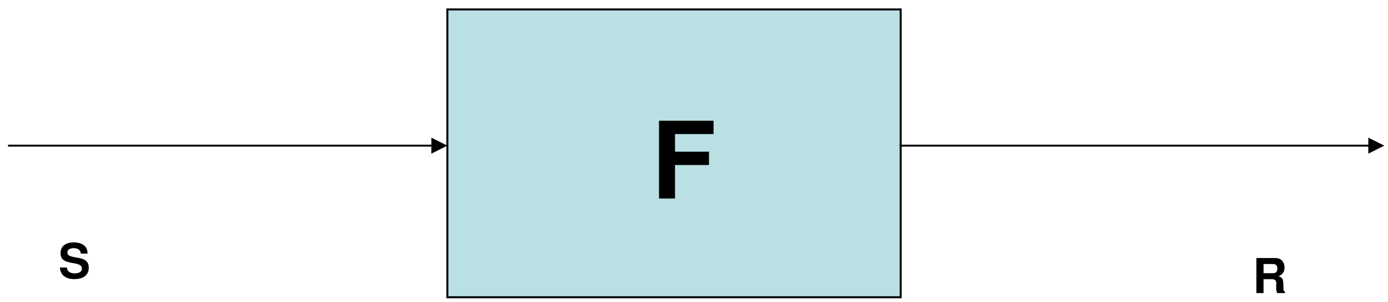
Motivated by

- Inverse Problems of Signal Processing
- Control Theory
- Tx= y, Matrix Numerical Analysis
- Functional Calculi in Operator Theory and Harmonic Analysis
- Others

Stationary frequency filters:

S= signal

R= response



STATIONARY FILTERS

Translation invariant

$$\tau_u F = F \tau_u$$

where $\tau_u f(t) = f(t-u)$. It follows that there exists T (a function, a measure, or a distribution) such that

$$F(S) = S * T,$$

a convolution.

DISCRETE TIME SIGNALS

A signal: $S : \mathbb{z} \longrightarrow \mathbb{c}$, a sequence $S = (S_n)_{n\in\mathbb{z}}$

Translation $\tau_k S = (S_{n-k})_{n\in\mathbb{z}}, \; k \in \mathbb{z}$

Convolution $S * T = (\sum_{k\in\mathbb{z}} S_{n-k}T_k)_n$

Bounded amplitude signals $l^\infty(\mathbb{z}) = \{S : \|S\|_\infty = sup_n|S_n| < \infty\}$

A finite power filter: $Fl^\infty(\mathbb{z}) \subset l^\infty(\mathbb{z}) \Leftrightarrow F(S) = S * T, \; \|T\|_1 = \sum_n |T_n| < \infty$

The Wiener convolution algebra:

$$W(\mathbb{z}) = l^1(\mathbb{z}) = \{T = (T_n)_n : \|T\|_1 = \sum_n |T_n| < \infty\}.$$

REQUENCY CHARACTERISTIC (TRANSFER) FUNCTION

Frequency characteristic function of a filter $F(S) = S * T, T \in W(\mathbb{Z})$

$$f = \mathcal{F}T = \sum_{k \in \mathbb{Z}} T_k e^{-ikx}, \; x \in \mathbb{R}$$

Amplitude and phase factors on an elementary harmonic inputs:

$$S = (e^{ikx})_{k \in \mathbb{Z}} \longrightarrow F(S) = f(x)(e^{ikx})_{k \in \mathbb{Z}}$$

The Wiener algebra of transfer functions

$$A(\mathbb{T}) = \mathcal{F}l^1(\mathbb{Z}) = \{f : \mathbb{T} \longrightarrow \mathbb{C} : \; f = \sum_{k \in \mathbb{Z}} \hat{f}(k)\zeta^k, \; \sum_{k \in \mathbb{Z}} |\hat{f}(k)| < \infty\}$$

FREQUENTLY ASKED QUESTIONS:

▸ **An Identification Problem:** knowing a response $R = S * T$, can one recognize S:

$$R_1 = R_2 \Rightarrow S_1 = S_2\,?$$

FACT: *The IP has a positive solution if and only if $f(\zeta) = \mathcal{F}T(\zeta) \neq 0$ for every $\zeta \in \mathbb{T}$.*

▸ **Well-Posed (norm controlled) Identification Problem:** can one control the amplitude of S in function of the amplitude of R:

$$\exists C > 0 \text{ such that } \|S\|_\infty \leq C\|R\|_\infty\,?$$

Wiener's $1/f$ Theorem (1932): *IP is equivalent to W-P IP, i.e.*

$$f \in A(\mathbb{T}) \text{ and } f(\zeta) \neq 0 \ (\zeta \in \mathbb{T}) \Rightarrow 1/f \in A(\mathbb{T}),$$

and hence $\|S\|_\infty \leq \|1/f\|_1 \|R\|_\infty$ *for every* $R \in l^\infty(\mathbb{Z})$.

SIMILARLY FOR MULTI-CHANNEL FILTERING:

► **Identification Problem:**

- knowing responses $R_k = S * T_k$, $k = 1, ..., n$, can one recognize S, or
- to have a linear (time invariant) desintegration formula

$$\sum_{k=1}^{n} U_k * R_k = \sum_{k=1}^{n} U_k * T_k * S = S, \text{ i.e., } \sum_{k=1}^{n} U_k * T_k = \delta_0 \; ?$$

FACT: *The IP for multi-channel filtering has a positive solution if and only if* $\delta_f =: min_{\zeta \in \tau} |f(\zeta)|^2 = min_{\zeta \in \tau} \sum_{k=1}^{n} |\mathcal{F}T_k(\zeta)|^2 > 0.$

► **Well-Posed (norm controlled) Identification Problem:** to control

$$\exists C = C(\delta_f) > 0 \text{ such that } \|U_k\| \le C(\delta_f) \; ?$$

LOOKING FOR A CONSTRUCTIVE PROOF TO $1/f$ THEOREM:

- an algorithm giving T^{-1} such that $T * T^{-1} = id$

- an estimate of the amplitude amplifier factor $\|1/f\|_{A(\mathbb{T})}$ in terms of the a.a.f. of elementary harmonic signals $1/\delta_f$:

$$\frac{1}{\delta_f} = sup_x \|T^{-1} * (e^{ikx})_{k \in \mathbb{Z}}\|_\infty = \frac{1}{inf_x |f(x)|}.$$

FACT: Wiener's, and then Gelfand's proofs are highly non-constructive

Attempts on a constructive proof
(a brief history)

PRO	CONTRA
A.Calderón, 1950	
P.Cohen, 1961 (Fields Medal 1966)	H.Helson, J.P.Kahane, Y.Katznelson, W.Rudin, 1959
D.Newman, 1965	J.Staffney, 1967
E.Bishop, 1970	G.Björk, 1972 H.Shapiro, 1975
	N.N., 1995

GENERAL SETTING

• Let A be a commutative Banach algebra, $X \subset \mathfrak{m}(A)$ a *"visible part"* of the maximal ideal space.

• For $0 < \delta \leq 1$, the *best upper bound for inverses*

$$c_1(\delta) = c_1(\delta, A, X) =:$$

$$= sup\{\|1/f\|_A : \ f \in A, \ \delta \leq |f(x)| \leq \|f\|_A \leq 1, \ \forall x \in X\}.$$

• The first *critical constant* $\delta_1 = \delta_1(A, X)$ is defined by:

$$\delta_1 < \delta \leq 1 \ \Rightarrow \ c_1(\delta) < \infty,$$

$$0 < \delta < \delta_1 \Rightarrow c_1(\delta) = \infty.$$

GENERAL SETTING (CONTINUED)

• For $0 < \delta \leq 1$ and $n = 1, 2, \ldots$, the *best bound for solutions of n-order Bezout equations*

$$c_n(\delta) = c_n(\delta, A, X) =:$$

$$= \ sup\{\|f^{(-1)}\|_{A^n} : \ f \in A^n, \ \delta \leq |f(x)| \leq \|f\|_{A^n} \leq 1, \ \forall x \in X\},$$

where $f = (f_k)_1^n$, $|f(x)|^2 = \sum_{k=1}^{n} |f_k(x)|^2 \ (x \in X)$, $\|f\|_{A^n}^2 = \sum_{k=1}^{n} \|f_k\|_A^2$,

$$\|f^{(-1)}\|_{A^n} =: \ inf\{\|g\|_{A^n} : \ \sum_{k=1}^{n} g_k f_k = \ id_A\}.$$

• The *n-th critical constant* $\delta_n = \delta_n(A, X)$ is defined by:

$$\delta_n < \delta \leq 1 \ \Rightarrow \ c_n(\delta) < \infty,$$
$$0 < \delta < \delta_n \Rightarrow c_n(\delta) = \infty.$$

THREE EXAMPLES

I. Convolution measure algebra $\mathcal{M}(G)$, where G is a locally compact abelian group.

Visible spectrum is given by the Fourier transform $y \longmapsto \mathcal{F}\mu(y)$, $y \in \hat{G}$ (on the dual group of characters).

II. H^∞ trace algebras $H^\infty|\sigma$, where σ is a Blaschke set.

Visible spectrum is given by the restriction map $z \longmapsto f(z)$, $z \in \sigma$.

III. Fourier-Hadamard multiplier algebra $Mult(L^2(\mathbb{T}, w))$ for L^2 Muckenhoupt weighted space ($w \in (A_2)$).

Visible spectrum is given by eigenvalues $\mu = (\mu_k)_{k\in\mathbb{Z}}$; $z^k \longmapsto \mu_k z^k$ ($k \in \mathbb{Z}$) extends to a linear bounded map $L^2(\mathbb{T}, w) \longrightarrow L^2(\mathbb{T}, w)$.

I. Measure algebra $\mathcal{M}(G)$, G is a LCAG. *Visible spectrum is given by the Fourier transform $y \longmapsto \mathcal{F}\mu(y)$, $y \in \hat{G}$.*

• **N.Wiener and H.R.Pitt phenomenon (1938):** $\exists \mu \in \mathcal{M}(\mathbb{R})$ s.t. $inf_{y\in\mathbb{R}}|\mathcal{F}\mu(y)| > 0$ BUT there exists NO $\nu \in \mathcal{M}(\mathbb{R})$ s.t. $\nu * \mu = \delta_0$, i.e.

$$\delta_1(\mathcal{M}(\mathbb{R})) > 0.$$

• **E.Hewitt:** $\mathcal{M}(G)$ has a Wiener-Pitt phenomenon iff G is NON DISCRETE.

• **H.Helson, J-P.Kahane, Y.Katznelson, and W.Rudin (1959), J.Staffney (1967):** $\delta_1(\mathcal{M}(G)) > 0$ for every infinite LCAG G.

• **Y.Katznelson and H.S.Shapiro's conjecture (1975):** $\delta_1(\mathcal{M}(\mathbb{Z})) = 1/2$.

• **N.N., 1999:** $\delta_1(\mathcal{M}(\mathbb{Z}_+)) = 1/2$ and $1/2 \leq \delta_n(\mathcal{M}(\mathbb{Z})) \leq 1/\sqrt{2}$ for every $n = 1, 2, \ldots$.

• **O.ElFallah, M.Zarrabi, and N.N., 1999:** $1/2 \leq \delta_n(\mathcal{M}(G)) \leq 1/\sqrt{2}$ for every infinite LCAG G, $n = 1, 2, \ldots$.

Measure algebra $\mathcal{M}(G)$: COMMENTS

• **Weighted algebras** $\mathcal{M}(G,w) = \{\mu \in \mathcal{M}(G) : \int_G w d|\mu| < \infty\}$, $w(x+y) \leq w(x)w(y)$, G is **not compact** and $lim_{x \longrightarrow \infty} w(x) = \infty$ (w is "regularly varying" weight).

Then $\delta_n(\mathcal{M}(G,w)) = 0$ for all $n \geq 1$.
The same for weighted convolution algebras $L^p(G,w)$.

• Let G be an infinite **DISCRETE LCAG**. Then $\mathfrak{M}(\mathcal{M}(G)) = \hat{G}$ (**NO corona**), but there is a NUMERICALLY DETECTABLE CORONA:

$$c_n(\delta, \hat{G}) = \infty \text{ for every } \delta,\ 0 < \delta \leq 1/2.$$

For **"flat" data** $\mu \in \mathcal{M}(G)$, $1/\sqrt{2} < \delta \leq |\hat{\mu}(y)| \leq \|\mu\| \leq 1$ ($y \in \hat{G}$) a corona cannot be detected: $c_n(\delta, \hat{G}) < \infty$.

II. Trace algebras $H^\infty|\sigma$ and quotient algebras $H^\infty/\Theta H^\infty$ (Θ inner).

- *Visible spectrum is given by the restriction* $f \longmapsto f|\sigma$, $f \in H^\infty$.
- σ *is a Blaschke set in* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, *then* $H^\infty|\sigma = H^\infty/BH^\infty$, $B = \Pi_\lambda b_\lambda$, $b_\lambda = (z-\lambda)/(1-\overline{\lambda}z)$ *a Blaschke factor.*

- **THEOREM (P.Gorkin, R.Mortini, N.N.; 2008).** *Given a Blaschke set* σ *in* $\mathbb{D}$, *TFAE:*

(1) $\delta_1(H^\infty|\sigma) = 0$.
(2) $H^\infty|\sigma$ *inverse closed:* $f \in H^\infty|\sigma, inf_{\lambda\in\sigma}|f(\lambda)| > 0 \Rightarrow 1/f \in H^\infty|\sigma$.
(3) There is NO corona: σ *is dense in* $\mathfrak{M}(H^\infty|\sigma)$.
(4) $\delta_n(H^\infty|\sigma) = 0$ *for every* $n = 1, 2, ...$
(5) Weak Embedding Property: $\forall\epsilon > 0\ \exists C(\epsilon) > 0$ *s.t.*

$$inf_{\lambda\in\sigma}|b_\lambda(z)| = \epsilon > 0 \Rightarrow \sum_{\lambda\in\sigma} \frac{(1-|z|^2)(1-|\lambda|^2)}{|1-\overline{\lambda}z|^2} \leq C(\epsilon).$$

(6) $\forall\epsilon > 0\ \exists\eta(\epsilon) > 0$ *s.t.* $|B_\sigma(z)| \leq \eta(\epsilon) \Rightarrow inf_{\lambda\in\sigma}|b_\lambda(z)| \leq \epsilon$.

Trace algebras $H^\infty|\sigma$ and $\delta_n(H^\infty|\sigma)$ (continued).

• **THEOREM (N.N., V.Vasyunin; 2011).** *Given a number δ, $0<\delta<1$, there exists a Blaschke set σ in $\mathbb{D}$ such that $\delta_1(H^\infty|\sigma)=\delta$. Moreover, there is $f\in H^\infty$ such that $\delta\le |f(z)|\le\|f\|_\infty\le 1$ $(z\in\sigma)$ but $1/f\notin H^\infty|\sigma$.*

A question on the sequence $(\delta_n(H^\infty|\sigma))_{n\ge1}$. Clearly, $\delta_1\le\delta_2\le\delta_3\le ....$ *Is it necessary that $\delta_1=\delta_2$* (as it is if $\delta_1=0$)? Or *δ_2 can be any number in $[\delta_1,1)$?*

• **F.Nazarov (unpublished):** *Given an integer $n\ge1$, there exists a set X and a uniform function algebra A on X such that 1) every Bezout eqution of order $\le n$ is solvable in A; 2) there is a Bezout equation of order $n+1$ having no solution in A.*

III. Fourier-Hadamard multipliers.

• Let X be a space of functions or distributions on the torus $\mathbb{T}$ containing trigo polynomials as a dense subset.

A sequence $\mu = (\mu_k)_{k\in\mathbb{Z}}$ is *a multiplier of* X, $\mu \in Mult(X)$, if the map

$$T_\mu : e^{ikx} \longmapsto \mu_k e^{ikx}, \; k \in \mathbb{Z},$$

extends to a bdd linear operator on X.

• *A multiplier = a convolution operator = a singular integral operator.*
• Popular example: *the Hilbert transform*, $\mu_k = 1$ for $k \geq 0$, $\mu_k = -1$ for $k < 0$.
• Always, $Mult(X) \subset l^\infty(\mathbb{Z})$, and $Mult(X) = l^\infty(\mathbb{Z}) \Leftrightarrow (e^{ikx})$ is an unconditional basis of X.

Fourier-Hadamard multipliers (continued).

• **Problem:** What is the spectrum of a multiplier?

A *visible (point) spectrum of* T_μ is $\{\mu_k : k \in \mathbb{Z}\}$, and hence $clos\{\mu_k : k \in \mathbb{Z}\}$.

• Should it be $\sigma(T_\mu) = clos\{\mu_k : k \in \mathbb{Z}\}$? (the Spectral Localization Property = SLP).

• Is there *a corona* $\mathfrak{M}(Mult(X))\backslash clos\{\varphi_k : k \in \mathbb{Z}\} \neq \emptyset$?
($\varphi_k(T_\mu) = \mu_k$, evaluation functionals).

• **NO SLP** for **BAD spaces** X or/and for **BAD geometry** of eigenfunctions (e^{ikx}).

• **A NICE CASE:** $X = L^2(\mathbb{T}, w)$ a Hilbert space, (e^{ikx}) a Schauder basis, i.e. $w \in (A_2)$ (a *Muckenhoupt weight*).

Fourier-Hadamard multipliers (continued).

Theorem 1 (N.N., 2009). (1) $\forall w \in L^1(\mathbb{T})$ **SLP** $\Leftrightarrow$ **"No corona".**

(2) $\exists w \in (A_2)$ s.t. $Mult(L^2(w))$ has **NO SLP** (there exists a unimodular multiplier $\mu = (\mu_k)$ with $\sigma(T_\mu) = \overline{\mathbb{D}}$).

Theorem 2 (I.Verbitsky + N.N., 2012). Any weight w having a *finite number of "singularities of Schoenberg type"* satisfies the SLP (condition $w \in (A_2)$ is for free).

A I.J.Schoenberg weight: $w^{\pm 1} \in L^1(\mathbb{T})$, $w << 0$ (negatively definite). A "Schoenberg type weight" is equivalent to w or $1/w$, where w is a rotation of a Schoenberg weight.

Example: $w(e^{ix}) = |1 - e^{ix}|^\alpha$, $0 < \alpha < 1$, is equivalent to a Schoenberg weight $w_0 = \Sigma_{k \geq 1} k^{-1-\alpha} Sin^2(kx/2)$.

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THE END

THANK YOU!