The Toeplitz Corona Problem and a Distance Formula

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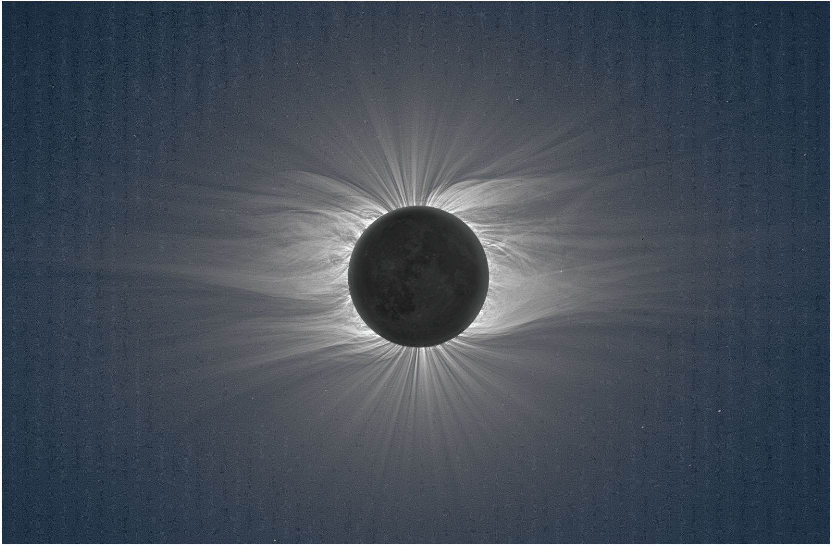

Lemma (Douglas)

For bounded operators A and B, the following are equivalent:

1. $AA^* \geq BB^*$;
2. *There is a contraction C so that $AC = B$.*

Remark: Such a C can be found in $W^*(A, B)$.

The factorization problem for operator algebras

Given an operator algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ and A, B in $\mathcal{A}$, when does the hypothesis $AA^* \geq BB^*$ imply the existence of a contractive C **in** $\mathcal{A}$ so that $AC = B$?
We call this the factorization problem.

Example

- von Neumann algebras have the FP.
- $C([0,1])$ does not.
- Certain classes of C^* algebras do (Fialkow and Salas).
- Nest algebras generally do not (Arveson).
- The analytic Toeplitz algebra $\mathcal{T}(H^\infty)$ has the FP (Nevanlinna-Pick)

The factorization problem for operator algebras

Suppose $\mathcal{H}$ is a reproducing kernel Hilbert space and $\mathcal{M}(\mathcal{H})$ its multiplier algebra.

Theorem (McCullough-Trent 2011)

Let $\mathcal{L}$ be a separable Hilbert space. The algebra $\mathcal{M}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{L})$ has the FP if and only if $\mathcal{H}$ is a complete Nevanlinna-Pick space.

Corollary

The multiplier algebras of Bergman spaces and the Hardy spaces $H^2(\mathbb{D}^d)$ and $H^2(\mathbb{B}_d)$ (tensored with $\mathcal{B}(\mathcal{H})$) do not have the factorization property.

This suggests they probably do not satisfy the Toeplitz corona theorem either.

Theorem (Arveson-1975; Schubert-1978)

Suppose $f_1, \ldots, f_n \in H^\infty$ satisfy

$$\sum_{i=1}^{n} T_{f_i} T_{f_i}^* \geq c^2 I.$$

Then there are functions $g_1, \ldots, g_n \in H^\infty$ so that

$$\sum_{i=1}^{n} f_i g_i = 1, \text{ and } \|[T_{g_i}, \ldots, T_{g_n}]^T\| \leq c^{-1}.$$

Suppose $\{P_m\}_{m\geq 0}$ is an increasing sequence of projections tending strongly to I_H and let $\mathcal{A} := \mathrm{Alg}\{P_m\}_{m\geq 0}$.

Theorem (Arveson-1975)

Suppose $A_1, \ldots, A_n \in \mathcal{A}$ satisfy

$$\sum_{k=1}^{n} A_k P_m A_k^* \geq c^2 P_m \text{ for every } m \geq 0.$$

Then there are $B_1, \ldots, B_n \in \mathcal{A}$ such that

$$\sum_{k=1}^{n} A_k B_k = I_H$$

If $\mathcal{B}$ is an algebra of operators and h a vector, let $\mathcal{B}[h] := \overline{\text{span}\{Bh : B \in \mathcal{B}\}}$.

Theorem (Raghupathi-Wick 2010)

Suppose $\mathcal{A}$ is a unital, weak-closed subalgebra of H^∞ and $f_1, \ldots, f_n \in \mathcal{A}$ satisfy*

$$\sum_{i=1}^{n} T_{f_i} P_L T_{f_i}^* \geq c^2 P_L$$

for every L of the form $\mathcal{A}[h]$ where h is an outer function. Then there are $g_1, \ldots, g_n \in \mathcal{A}$ so that

$$\sum_{i=1}^{n} f_i g_i = 1 \text{ and } \|[T_{g_1}, \ldots, T_{g_n}]^T\| \leq c^{-1}$$

Theorem (Amar 2003; Trent-Wick 2008)

Suppose $f_1, \ldots, f_n \in H^\infty(\mathbb{D}^d)$ *(resp.* $H^\infty(\mathbb{B}_d)$*) satisfy*

$$\sum_{i=1}^{n} T_{f_i}^{\nu}(T_{f_i}^{\nu})^* \geq c^2 I_\nu$$

for measure of the form $\nu = |f|^2\mu$ *where* $f \in H^2(\mathbb{D}^d)$ *(reps.* $H^2(\mathbb{B}_d)$*). Then there are functions* $g_1, \ldots, g_n \in H^\infty(\mathbb{D}^2)$ *(resp.* $H^\infty(\mathbb{B}_d)$*) so that*

$$\sum_{i=1}^{n} f_i g_i = 1, \text{ and } \|[T_{g_i}, \ldots, T_{g_n}]^T\| \leq c^{-1}.$$

Definition

Suppose $\mathcal{L}$ is any Hilbert space. A *vector-valued reproducing kernel Hilbert space* is a Hilbert space $\mathcal{H}$ of $\mathcal{L}$-valued functions on some domain X such that point evaluation is norm continuous for $\mathcal{H}$.

Every RKHS admits a positive semidefinite kernel $K : X \times X \to \mathcal{B}(\mathcal{L})$.
When $\mathcal{L} = \mathbb{C}$, we use the notation $k(x, y) = \langle k_y, k_x \rangle$ for the kernel function for $\mathcal{H}$.

Definition

Given two RKHS $\mathcal{H}_1(K_1, \mathcal{L}_1, X_1)$ and $\mathcal{H}_2(K_2, \mathcal{L}_2, X_2)$, a *multiplier* between $\mathcal{H}_1$ and $\mathcal{H}_2$ is a function

$$F : X \to \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$$

such that $Ff \in \mathcal{H}_2$ for every $f \in \mathcal{H}_1$.

Every multiplier F determines a bounded operator $M_F \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ determined by $M_F f(x) := F(x)f(x)$.
Let $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ denote the operator space of all multiplication operators.

Example

- For $\Omega \in \{\mathbb{B}_d, \mathbb{D}^d\}$, the Hardy space $H^2(\Omega)$.
  $\mathcal{M}(H^2(\Omega)) = H^\infty(\Omega)$.
- For $\Omega \subset \mathbb{C}^d$ bounded and open, the Bergman spaces $L^2_a(\Omega)$.
  $\mathcal{M}(L^2_a(\Omega)) = H^\infty(\Omega)$.
- The DA spaces H^2_d. $\mathcal{M}(H^2_d) \subsetneq H^\infty(\mathbb{B}_d)$.
  $k^{H^2(d)}(z, w) = (1 - \langle z, w\rangle)^{-1}$.
- A kernel k is complete Nevanlinna-Pick iff $1 - 1/k$ is a positive semidefinite function on $X \times X$. All such spaces admit natural embeddings into H^2_∞.

The factorization problem for multipliers

Suppose $M_F \in \mathcal{M}(\mathcal{H}_1, \mathcal{H})$ and $M_G \in \mathcal{M}(\mathcal{H}_2, \mathcal{H})$ satisfy

$$M_F M_F^* \geq M_G M_G^*.$$

When can we find a multiplier $M_H \in \mathcal{M}(\mathcal{H}_2, \mathcal{H}_1)$ such that $FH = G$?

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When can we find a multiplier $M_H \in \mathcal{M}(\mathcal{H}_2, \mathcal{H}_1)$ such that $FH = G$?

Facts:

- Multiplier spaces are weak-$*$ closed.
- Point evaluation is weak-$*$ continuous for multipliers.

A local approach to the factorization problem

Suppose for the moment we could ‘solve’ the factorization problem on finite subsets of X. That is, for every $E = \{x_1, \ldots, x_n\} \subset X$, we can find a contractive multiplier H_E so that

$$F(x)H_E(x) = G(x) \, x \in E.$$

Then any weak cluster point of $\{H_E\}_{E \subset X}$ will solve the factorization problem!

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Then any weak cluster point of $\{H_E\}_{E \subset X}$ will solve the factorization problem!

For any reasonable space, we can always find H_E (not necessarily contractive). What we require is a uniform bound on all the H_E.

If $\mathcal{H}$ is a *scalar*-valued RKHS, then $\mathcal{H} \otimes \ell^2$ is a RKHS and

$$\mathcal{M}(\mathcal{H} \otimes \ell^2, \mathcal{H}) = \mathrm{Row}(\mathcal{M}(\mathcal{H}))$$

$$\mathcal{M}(\mathcal{H}, \mathcal{H} \otimes \ell^2) = \mathrm{Col}(\mathcal{M}(\mathcal{H}))$$

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By letting $F = (f_1, f_2, \dots)$ and $G = c$, the factorization problem for

$$M_F M_F^* = \sum M_{f_i} M_{f_i}^* \geq c^2 I$$

is precisely the Toeplitz corona problem.

- Now suppose $\mathcal{A}$ is *any* weakly closed algebra of multipliers on $\mathcal{H}$ and $M_F \in \mathrm{Row}(\mathcal{A})$ satisfies $M_F M_F^* \geq c^2 I$.
- We wish to find a contractive $G_E \in \mathrm{Col}(\mathcal{A})$ so that $F(x_i)G_E(x_i) = \sum f(x_i)g(x_i) = 1$ for every $x_i \in E$.

A return to the Toeplitz corona problem

- Now suppose $\mathcal{A}$ is *any* weakly closed algebra of multipliers on $\mathcal{H}$ and $M_F \in \mathrm{Row}(\mathcal{A})$ satisfies $M_F M_F^* \geq c^2 I$.
- We wish to find a contractive $G_E \in \mathrm{Col}(\mathcal{A})$ so that $F(x_i)G_E(x_i) = \sum f(x_i)g(x_i) = 1$ for every $x_i \in E$.
- More generally, we seek to solve the interpolation problem: Given a finite set E, vectors $v_1, \dots, v_n$ in ℓ^2 and complex numbers $w_1, \dots, w_n$, find a contractive $G_E \in \mathrm{Col}(\mathcal{A})$ so that

$$\langle G_E(x_i), v_i \rangle = \overline{w_i}.$$

Taking $w_i = c$ and $v_i = F(x_i)^*$ yields the result.

If k is any reproducing kernel for which $\mathcal{A}$ is contained in its multiplier algebra, we have

$$M_F^* k_x = F(x_i)^* k_x$$

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$$M_F M_F^* = \sum_{i=1}^{n} M_{f_i} M_{f_i}^* \geq c^2 I_H$$

$$\left\langle \left(M_F M_F^* - c^2 \right) h, h \right\rangle \geq 0, \; h \in H$$

Our approach

If k is any reproducing kernel for which $\mathcal{A}$ is contained in its multiplier algebra, we have

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$$M_F M_F^* = \sum_{i=1}^{n} M_{f_i} M_{f_i}^* \geq c^2 I_H$$

$$\left\langle \left(M_F M_F^* - c^2 \right) h, h \right\rangle \geq 0, \; h \in H$$

Now take $h = \sum a_i k_{x_i}$.

$$\left[\left(\langle F(x_i)^*, F(x_j)^* \rangle - c^2 \right) \langle k_{x_i}, k_{x_j} \rangle \right]_{i,j=1}^{n} \geq 0.$$

We have reduced the problem to a statement about interpolation.

Definition

Suppose $x_1, \ldots, x_k \in X$, $v_1, \ldots, v_k \in \ell^2_n$ and $w_1, \ldots, w_k \in \mathbb{C}$.
A collection of kernels $\{k^\alpha\}$ is said to be a **tangential family** if the following statement holds:

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Definition

Suppose $x_1, \ldots, x_k \in X$, $v_1, \ldots, v_k \in \ell^2_n$ and $w_1, \ldots, w_k \in \mathbb{C}$.
A collection of kernels $\{k^\alpha\}$ is said to be a **tangential family** if the following statement holds:
There is a contractive column multiplier $M_G = [M_{g_1}, \ldots, M_{g_n}]^T$ with $g_i \in \mathcal{A}$ such that $\langle G(x_i), v_i \rangle_{\mathbb{C}^n} = w_i$ for each i if and only if

$$\left[\left(\langle v_i, v_j \rangle - w_i \overline{w_j} \right) \left\langle k^\alpha_{x_i}, k^\alpha_{x_j} \right\rangle \right]_{i,j=1}^{k}, \text{ all } \alpha$$

is positive semidefinite.

- Let $\mathcal{J}$ be the submodule of $\mathrm{Col}(\mathcal{A})$ consisting of those H which satisfy $\langle H(x_i), v_i \rangle = 0$.
- If G is *any* column (not necessarily contractive) which satisfies $\langle G(x_i), v_i \rangle = w_i$, then $G + H$ also interpolates the data for any $H \in \mathcal{J}$.
- Thus, the minimal possible norm for a solution is $\mathrm{dist}(G, \mathcal{J})$.

Deriving a tangential family

Suppose $\mathcal{A}$ is a weakly closed algebra of multiplication operators for $\mathcal{H}$ and $\mathcal{L}$ is an invariant subspace for $\mathcal{A}$.

- The reproducing kernel on $\mathcal{L}$ is given by $k_x^{\mathcal{L}} := P_{\mathcal{L}} k_x$
- Any $M_F \in \mathcal{A}$ determines the multiplication operator $M_F^{\mathcal{L}} := M_F|_{\mathcal{L}}$ on $\mathcal{L}$.

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Lemma

Let $\mathcal{L}$ be any cyclic invariant subspace for $\mathcal{A}$ and let $\mathcal{M}_{\mathcal{L}} := \operatorname{span}\{k_{x_1}^{\mathcal{L}} \otimes v_1, \ldots, k_{x_n}^{\mathcal{L}} \otimes v_n\}$. Then $\{k^{\mathcal{L}}\}$ is a tangential family for $\mathcal{A}$ if and only if

$$\operatorname{dist}(G, \mathcal{J}) = \sup_{\mathcal{L}} \|(M_G^{\mathcal{L}})^*|_{\mathcal{M}_{\mathcal{L}}}\|$$

Deriving a tangential family

It is immediate that $\|(M_G^{\mathcal{L}})^*|_{\mathcal{M}_{\mathcal{L}}}\| \leq 1$ if and only if the matrix

$$\left[\left(\langle v_i, v_j \rangle - w_i \overline{w_j} \right) \langle k_{x_i}^{\mathcal{L}}, k_{x_j}^{\mathcal{L}} \rangle \right]_{i,j=1}^{k}$$

is positive semidefinite. To summarize:

It is immediate that $\|(M_G^{\mathcal{L}})^*|_{\mathcal{M}_{\mathcal{L}}}\| \leq 1$ if and only if the matrix

$$\left[\left(\langle v_i, v_j\rangle - w_i\overline{w_j}\right)\langle k_{x_i}^{\mathcal{L}}, k_{x_j}^{\mathcal{L}}\rangle\right]_{i,j=1}^{k}$$

is positive semidefinite. To summarize:

Theorem

Suppose $\{k^{\mathcal{L}}\}_{L\ cyclic}$ is a tangential family for $\mathcal{A}$. Then the following are equivalent

1. $\sum(M_{f_i}^{\mathcal{L}})(M_{f_i}^{\mathcal{L}})^* \geq c^2 I_{\mathcal{L}}$ *for all $\mathcal{L}$ cyclic;*
2. *There is a column $G = (g_1, g_2, \dots)$ in* Col $\mathcal{A}$ *such that*

   $$\sum f_i g_i = 1$$

   and $\|M_G\| \leq c^{-1}$.

Main Result

Definition

A weakly closed subspace $\mathcal{S} \subset \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ of operators is said to have *property* $\mathbb{A}_1(r)$ if for every contractive weak-$*$ continuous functional φ on $\mathcal{S}$, there are vectors $f \in \mathcal{H}$ and $g \in \mathcal{H}$ so that

$$\varphi(A) = \langle Af, g\rangle \text{ and } \|f\|\|g\| < r$$

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Example

- Brown 1978: Subnormal operators have $\mathbb{A}_1(r)$ for some $r \geq 1$ (Bercovici-Conway: $\mathbb{A}_1(1)$).
- Bercovici 1988: Any algebra isometrically isomorphic to H^∞ has $\mathbb{A}_1(1)$.
- BCP operators have $\mathbb{A}_1(1)$.
- Arias-Popescu: $\mathcal{M}(H^2_d)$ has $\mathbb{A}_1(1)$.
- Still open for subnormal tuples.

Main Result

Theorem

Let $\mathcal{A}$ be any weakly closed algebra of multiplication operators on $\mathcal{H}$. Then $\{k^{\mathcal{L}}\}_{\mathcal{L} \text{ cyclic}}$ is a tangential family if $\mathrm{Col}(\mathcal{A})$ has property $\mathbb{A}_1(1)$. More generally, if $\mathrm{Col}(\mathcal{A})$ has property $\mathbb{A}_1(r)$, a solution may be found of norm at most rc^{-1}.

Let G be any solution to the tangential interpolation problem.

- A standard duality argument shows that there is a contractive weak-$*$ functional φ on Col($\mathcal{A}$) such that

$$\varphi(G) \approx \mathrm{dist}(G, \mathcal{J}) \text{ and } \varphi|_{\mathcal{J}} = 0.$$

- Find $f \in \mathcal{H}$ and $g \in \mathcal{H} \otimes \ell^2$ so that $\varphi(G) = \langle Gf, g \rangle$.
- Replace M_G with $P_{\mathcal{M}_{\mathcal{L}}} M_G^{\mathcal{L}}$.
- Thus $\mathrm{dist}(G, \mathcal{J}) \leq \|(M_G^{\mathcal{L}})^*|_{\mathcal{M}_{\mathcal{L}}}\| \|f\| \|g\| \leq r \|(M_G^{\mathcal{L}})^*|_{\mathcal{M}_{\mathcal{L}}}\|$.

Proof sketch

- It is always the case that for any $\mathcal{L}$, we have $\text{dist}(G, \mathcal{J}) \geq \|(M_G^{\mathcal{L}})^*|_{\mathcal{M}_{\mathcal{L}}}\|$.
- It follows that we have

$$\sup_{\mathcal{L}} \|M_G^{\mathcal{L}}|_{\mathcal{M}_{\mathcal{L}}}\|\| \leq \text{dist}(G, \mathcal{J}) \leq r \sup_{\mathcal{L}} \|M_G^{\mathcal{L}}|_{\mathcal{M}_{\mathcal{L}}}\|\|$$

- Thus, if the corona hypothesis holds for all members of the tangential family, the right hand side is at most r, and so we obtain a solution of norm at most r.

Suppose $\mathcal{H}$ is any Nevanlinna-Pick space and $\mathcal{A}$ a weakly closed algebra of multipliers on $\mathcal{H}$. We say that a function $h \in \mathcal{H}$ is **outer** if $\mathcal{M}(\mathcal{H})[h] = \mathcal{H}$.

Theorem

The column space Col($\mathcal{A}$) *is elementary and every* $M_F \in (\mathrm{Col}\,\mathcal{A})_*$ *can be factored as*

$$\varphi(M_F) = \langle Fg, h\rangle$$

where h is an outer function.
In other words $\{k^{\mathcal{L}}\}_{\{\mathcal{L}=\mathcal{A}[h]:h \text{ outer}\}}$ *is a tangential family for* $\mathcal{A}$.

Corollary

Let $\mathcal{H}$ be any Nevanlinna-Pick space. Suppose $\mathcal{A}$ is a unital, weak*-closed subalgebra of $\mathcal{M}(\mathcal{H})$ and $f_1, \ldots, f_n \in \mathcal{A}$ satisfy

$$\sum_{i=1}^{n} M_{f_i}^L (M_{f_i}^L)^* \geq c^2 I_L$$

for every L of the form $\mathcal{A}[h]$ where h is an outer function. Then there are $g_1, \ldots, g_n \in \mathcal{A}$ so that

$$\sum_{i=1}^{n} f_i g_i = 1 \text{ and } \|[M_{g_1}, \ldots, M_{g_n}]^T\| \leq c^{-1}$$

When $\mathcal{A} = \mathcal{M}(\mathcal{H})$, this is the Ball-Trent-Vinnikov result. For $\mathcal{H} = H^2$ it is the Raghupathi-Wick result.

For $\Omega \subset \mathbb{C}^d$, let $L^2_a(\Omega)$ denote Bergman space. $(\mathcal{M}(L^2_a(\Omega)) = H^\infty(\Omega).)$

Theorem (Bercovici 1987)

For any weakly closed subalgebra $\mathcal{A} \subset H^\infty(\Omega)$, the finite column space $\mathrm{Col}_n(H^\infty(\Omega))$ has property $\mathbb{A}_1(\sqrt{n})$.

Thus, the hypothesis that $\sum_{i=1}^n (M_{f_i}^{\mathcal{L}})(M_{f_i}^{\mathcal{L}})^* \geq c^2 I_{\mathcal{L}}$ implies that the solution G satisfies

$$\|[g_1, \ldots, g_n]^T\| \leq \sqrt{n} c^{-1}.$$

More generally, if $\mathcal{H}$ is *any* space such that $\mathcal{H} \subset L^2(X, \mu)$ for a suitable measure μ, we have

Theorem (Prunaru 2011)

$\mathrm{Col}(\mathcal{M}(\mathcal{H}))$ *has property* $\mathbb{A}_1(\sqrt{n})$.

In particular, this applies to weighted versions of Bergman spaces.