CORONA PROBLEM: CONNECTIONS WITH OPERATOR THEORY AND COMPLEX GEOMETRY

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I. ORIGINS OF CORONA PROBLEM
II. SHORT EXACT SEQUENCES OF HILBERT MODULES
III. TOEPLITZ-CORONA PROBLEM

I. ORIGINS OF CORONA PROBLEM

(STONE) X COMPACT HAUSDORFF SPACE

$C(X) = \{f : X \to \mathbb{C}, \text{CONTINUOUS}\}$

$x_0 \in X \leftrightarrow$ MAXIMAL IDEAL $\{f \in C(X) : f(x_0) = 0\}$.

$$X \to C(X)$$

I. ORIGINS OF CORONA PROBLEM

$\ell^1(\mathbb{Z})$ CONVOLUTION MULTIPLICATION

$$(f * g)(n) = \sum_{m \in \mathbb{Z}} f(n-m)g(m)$$

(WIENER) $f * \ell^1(\mathbb{Z}) = \ell^1(\mathbb{Z})$ IFF $\hat{f}(e^{it}) \neq 0$, $e^{it} \in \mathbb{T}$.

$$\hat{f}(e^{it}) = \sum_{n \in \mathbb{Z}} f(n)e^{int} \in \mathbb{C}$$

$\mathbb{T} =$ MAXIMAL IDEAL SPACE OF $\quad \ell^1(\mathbb{Z})$

$f \in \ell^1(\mathbb{Z}), \hat{f} \neq 0 \Rightarrow \frac{1}{f} \in \ell^1(\mathbb{Z}$

I. ORIGINS OF CORONA PROBLEM

$$A(\mathbb{D}) = \{\varphi \in C(\mathbb{T}) : \int_0^{2\pi} \varphi(e^{it})e^{ikt}dt = 0 \text{ for } k > 0\}.$$

$A(\mathbb{D}) \subseteq C(\mathbb{T})$ BUT ALSO $A(\mathbb{D}) \subseteq C(\text{clos}\mathbb{D})$

(GELFAND) B COMMUTATIVE BANACH ALGEBRA WITH 1
$M_B =$SPACE OF MAXIMAL IDEALS OF B WITH
WEAK*-TOPOLOGY

M_B COMPACT HAUSDORFF SPACE AND
$B \to C(M_B) : \phi \to \hat{\phi}$

THEOREM: $\phi \in B$ INVERTIBLE IN B IFF $\hat{\phi} \neq 0$ ON M_B.

I. ORIGINS OF CORONA PROBLEM

(I. J. SCHARK)
$H^\infty(\mathbb{D}) = \{\varphi : \mathbb{D} \to \mathbb{C}$ BOUNDED, HOLOMORPHIC ON $\mathbb{D}\}$.

$M_{H^\infty(\mathbb{D})}$ COMPACT HAUSDORFF SPACE

$z_0 \in \mathbb{D} \quad \{\varphi \in H^\infty(\mathbb{D}) : \ \varphi(z_0) = 0\}$ IS MAXIMAL IDEAL OF $H^\infty(\mathbb{D})$

$\mathbb{D} \to M_{H^\infty(\mathbb{D})}$ ONE-TO-ONE, CONTINUOUS.
QUESTION: IS $\mathbb{D}$ DENSE IN $M_{H^\infty(\mathbb{D})}$? IF NOT, $H^\infty(\mathbb{D})$ IS SAID TO HAVE THE CORONA $M_{H^\infty(\mathbb{D})} \setminus clos\mathbb{D}$.

I. ORIGINS OF CORONA PROBLEM

CONCRETE FORMULATION:

GIVEN $\{\varphi_\iota\}_{\iota=1}^{n} \subseteq H^\infty(\mathbb{D})$

$$\sum_{\iota=1}^{n} |\varphi_\iota(z)|^2 \geq \epsilon^2 > 0 \text{ FOR } z \in \mathbb{D}$$

DOES THERE EXIST $\{\psi_i\}_{\iota=1}^{n} \subseteq H^\infty(\mathbb{D})$ SUCH THAT

$$\sum_{\iota=1}^{n} \varphi_i(z)\psi_i(z) = 1 \text{ for } z \in \mathbb{D}?$$

(CARLESON) FOR $H^\infty(\mathbb{D})$, THE ANSWER IS AFFIRMATIVE. THAT IS, $H^\infty(\mathbb{D})$ HAS NO CORONA.

I. ORIGINS OF CORONA PROBLEM

TECHNIQUES FROM FUNCTION THEORY AND HARMONIC ANALYSIS

CORONA PROBLEM FOR EVERY APPROPRIATE ALGEBRA OF BOUNDED HOLOMORPHIC FUNCTIONS SUCH AS $H^\infty(\Omega) = \{\varphi : \Omega \to \mathbb{C} : \text{BOUNDED, HOLOMORPHIC}\}$, $\Omega \subseteq \mathbb{C}^m$, BOUNDED DOMAIN

GENERAL SETUP: $\mathcal{R}$ HILBERT SPACE COMPLETION OF $H^\infty(\Omega)$ SO THAT $\mathcal{R} \subseteq \mathcal{O}(\Omega)$, HOLOMORPHIC FUNCTIONS ON Ω AND $z_\iota \mathcal{R} \subseteq \mathcal{R}, \iota = 1 \ldots, n$.

SUBNORMAL CASE: $\mathcal{R} = L^2_a(\mu), \mu$ PROBABILITY MEASURE ON Ω.

II. SHORT EXACT SEQUENCES OF HILBERT MODULES

$\mathbb{C}[z_1, ..., z_m] \times \mathcal{R} \to \mathcal{R}$ POINTWISE MULTIPLICATION
$\mathcal{M}_\mathcal{R} = \{\psi : \Omega \to \mathbb{C} \ni \psi\mathcal{R} \subseteq \mathcal{R}\}$ MULTIPLIER ALGEBRA

$\mathcal{M}_\mathcal{R} \subseteq H^\infty(\Omega)$, EQUALS $H^\infty(\Omega)$ FOR SUBNORMAL CASE.

LET $\{e_i\}_{\iota=1}^n$ BE ORTHONORMAL BASIS FOR $\mathbb{C}^n$.

FOR $\{\varphi_i\}_{\iota=1}^n \subseteq \mathcal{M}_\mathcal{R}$ SET $M_\Phi : \mathcal{R} \to \mathcal{R} \otimes \mathbb{C}^n$ SUCH THAT

$$M_\Phi f = \sum_{\iota=1}^n \varphi_i f \otimes e_i$$

FOR $f \in \mathcal{R}$.

II. SHORT EXACT SEQUENCES OF HILBERT MODULES

CORONA PROBLEM:

IF

$$(1) \qquad \sum_{\iota=1}^{n} |\varphi_i(z)|^2 \geq \epsilon^2 \text{ FOR } z \in \Omega, \text{ THEN DOES}$$

THERE EXIST

$$\{\Psi_i\}_{\iota=1}^{n} \subseteq \mathcal{M}_{\mathcal{R}}$$

SUCH THAT

$$(2) \qquad \sum_{\iota=1}^{n} \varphi_\iota(z)\Psi_\iota(z) = 1, z \in \Omega.$$

II. SHORT EXACT SEQUENCES OF HILBERT MODULES

Lemma

IF (1) HOLDS AND $\mathcal{R}$ IS HYPONORMAL OR (2) HOLDS, THEN M_Φ HAS CLOSED RANGE.
CONSIDER

$$(0) \qquad 0 \to \mathcal{R} \xrightarrow{M_\Phi} \mathcal{R} \otimes \mathbb{C}^n \xrightarrow{\pi_\Phi} \mathcal{R}_\Phi \to 0,$$

WHERE $\mathcal{R}_\Phi$ IS THE QUOTIENT HILBERT MODULE AND π_Φ THE QUOTIENT MODULE MAP.

QUESTION: IS HYPONORMAL NEEDED? DOES M_Φ HAVING CLOSED RANGE IMPLY (1)?

II. SHORT EXACT SEQUENCES OF HILBERT MODULES

Definition

AN OPERATOR $X : \mathcal{R} \otimes \mathbb{C}^p \to \mathcal{R} \otimes \mathbb{C}^q$, $1 \leq p, q < \infty$, IS SAID TO BE A MODULE MAP IF $X(M_\Psi \otimes I_{\mathbb{C}^p}) = (M_\Psi \otimes I_{\mathbb{C}^q})X$ FOR $\Psi \in \mathcal{M}_\mathcal{R}$.

Lemma

$X : \mathcal{R} \otimes \mathbb{C}^p \to \mathcal{R} \otimes \mathbb{C}^q$ IS A MODULE MAP IFF THERE EXISTS $\chi = \chi_{ij} : \Omega \to \mathcal{L}(\mathbb{C}^p, \mathbb{C}^q)$, BOUNDED HOLOMORPHIC, $\chi_{ij} \in \mathcal{M}_\mathcal{R}$ SUCH THAT $(Xf)(z) = \sum_{\iota=1}^q \sum_{j=1}^p \chi_{ij}(z) f_j(z) \otimes e_\iota$ FOR $f = \sum_{j=1}^p f_j \otimes e_j \in \mathcal{R} \otimes \mathbb{C}^p, z \in \Omega$.

II. SHORT EXACT SEQUENCES OF HILBERT MODULES

Proposition

ASSUME $\{\varphi_\iota\}_{\iota=1}^n \subseteq \mathcal{M}_\mathcal{R}$ *SATISFIES (1).*

THERE EXISTS $\{\Psi_\iota\}_{\iota=1}^n \subseteq \mathcal{M}_\mathcal{R}$ *SATISFYING (2) IFF*
(3) M_Φ HAS A LEFT MODULE INVERSE $N_\Psi : \mathcal{R} \otimes \mathbb{C}^n \to \mathcal{R}$ IFF

(4) THE SHORT EXACT SEQUENCE (0) SPLITS OR THERE EXISTS A MODULE MAP $\sigma_\Phi : \mathcal{R}_\Phi \to \mathcal{R} \otimes \mathbb{C}^n$ SUCH THAT $\sigma_\Phi \pi_\Phi = I_{\mathcal{R}\otimes\mathbb{C}^n}$ OR A RIGHT INVERSE FOR π_Φ.

NOTE: M_Φ ALWAYS HAS A LEFT OPERATOR INVERSE.

II. SHORT EXACT SEQUENCES OF HILBERT MODULES

PROOF: BY THE LEMMA, A LEFT MODULE INVERSE FOR M_Φ HAS THE FORM N_Ψ FOR $\Psi = \{\Psi_\iota\}_{\iota=1}^n \subseteq \mathcal{M}_\mathcal{R}$ WITH $N_\Psi(\sum_{\iota=1}^n f_\iota \otimes e_\iota) = \sum_{\iota=1}^n \Psi_\iota f_\iota, \{f_\iota\}_{\iota=1}^n \subseteq \mathcal{R}$.

$E = N_\Psi M_\Phi$ IS A MODULE IDEMPOTENT AND $\sigma_\Phi = E\pi_\Phi^{-1}$ IS WELL-DEFINED AND A RIGHT MODULE INVERSE FOR π_Φ.

CONVERSELY, IF E' IS A MODULE IDEMPOTENT ON $\mathcal{R} \otimes \mathbb{C}^n$ WITH RANGE OF THE COMPLEMENTARY IDEMPOTENT, $I - E'$, EQUAL THE RANGE OF M_Φ, ONE DEFINES A LEFT MODULE INVERSE FOR M_Φ BY $M_\phi^{-1}(I - E')$ WHICH IS WELL-DEFINED.

II. SHORT EXACT SEQUENCES OF HILBERT MODULES

ASSUME (1) HOLDS AND LET $P(z)$ BE THE ORTHOGONAL PROJECTION OF $\mathbb{C}^n$ ONTO RANGE $\Phi(z)$. IF (3) HOLDS, AND $E = N_\Psi M_\Phi$, THEN RANGE $E(z)$ IS BOUNDED HOLOMORPHIC IN $\mathcal{M}_\mathcal{R} \otimes \mathcal{L}(\mathbb{C}^n)$ AND

(5) $\qquad E(z) = P(z) + V(z)$ WITH
$V(z) = P(z)V(z)(I - P(z)), \quad z \in \Omega.$

THIS IS THE ANALYSIS OF TREIL-WICK IN CONNECTION WITH NIKOLSKI'S LEMMA TO OBTAIN:

II. SHORT EXACT SEQUENCES OF HILBERT MODULES

Theorem

ASSUME (1). THEN (2), (3) OR (4) HOLD IFF THERE EXISTS $V : \Omega \to \mathcal{L}(\mathbb{C}^n)$ SATISFYING (5) SO THAT $E(z) = P(z) + V(z)$ IS BOUNDED AND HOLOMORPHIC IN $\mathcal{M}_{\mathcal{R}}$.

PROOF: TO SHOW (5) IMPLIES (2), WE NOTE THAT $E(z)$ IS A MODULE IDEMPOTENT DEFINED SUCH THAT RANGE $E =$ RANGE M_Φ.

II. SHORT EXACT SEQUENCES OF HILBERT MODULES

IT IS NOT HARD TO SEE THAT SUCH A MODULE IDEMPOTENT E EXISTS IFF THERE EXISTS A COMPLEMENTARY SUBMODULE $\mathcal{S} \subseteq \mathcal{R} \otimes \mathbb{C}^n$ SUCH THAT RANGE $M_\Phi \dot{+} \mathcal{S} = \mathcal{R} \otimes \mathbb{C}^n$ IFF THE HOLOMORPHIC BUNDLE

$$\amalg_{z \in \Omega} \mathrm{RAN}\Phi(z)$$

HAS A COMPLEMENTARY HOLOMORPHIC SUB-BUNDLE OF $\Omega \times \mathbb{C}^n$ WITH A FRAME IN $\mathcal{M}_{\mathcal{R}} \otimes \mathbb{C}^n$ WITH ANGLES UNIFORMLY BOUNDED AWAY FROM ZERO.

II. SHORT EXACT SEQUENCES OF HILBERT MODULES

TREIL-WICK PROCEED TO SHOW FOR $\mathcal{R} = H^2(\mathbb{D})$ THAT (1) IMPLIES THE EXISTENCE OF SUCH A $V(z)$. APPROACH FORMALLY SIMILAR TO THAT OF HÖRMANDER, BUT MORE IS TRUE. FOR $\mathcal{R} = L^2_a(\mu)$ GIVEN $\{\varphi_\iota\}_{\iota=1}^n \subseteq H^\infty(\Omega)$

SATISFYING (1), IF

$$\{\alpha_\iota\}_{\iota=1}^n \subseteq L^\infty(\mu), \alpha_\iota(z) = \overline{\varphi_\iota}(z) / \sum_{j=1}^n |\varphi_j(z)|^2,$$

THEN

$$\sum_{\iota=1}^n \varphi_\iota(z)\alpha_\iota(z) = 1, \quad z \in \Omega \text{ BUT } \{\alpha_\iota\}_{\iota=1}^n \not\subseteq H^\infty(\Omega).$$

II. SHORT EXACT SEQUENCES OF HILBERT MODULES

HÖRMANDER SEEKS $\{\theta_\iota(z)\}_{\iota=1}^n \subseteq C^2(\Omega)$ SUCH THAT

$$\sum_{\iota=1}^{n} \varphi_\iota(z)\theta_\iota(z) = 0, \quad z \in \Omega, \quad \{\alpha_\iota + \theta_\iota\}_{\iota=1}^n \subseteq H^\infty(\Omega). \tag{6}$$

THE OPERATOR $N_A(z) : \mathbb{C}^n \to \mathbb{C}, z \in \Omega$, SUCH THAT $N_A(z)(\underline{a}) = \sum_{\iota=1}^n \alpha_\iota(z)a_\iota$, $\underline{a} = (a_1, ..., a_n) \in \mathbb{C}^n$, DEFINES $N_A \in \mathcal{L}(L^2(\mu) \otimes \mathbb{C}^n, L^2(\mu))$ AND $\widetilde{E} = M_\Phi N_A$ IS MODULE MAP ON $L^2(\mu) \otimes \mathbb{C}^n$ SUCH THAT $\widetilde{E}(z)$ NON-NEGATIVE MULTIPLE OF PROJECTION $P(z)$, RANGE $\widetilde{E}(z) =$ RANGE $P(z), z \in \Omega$. MODIFICATION $N_A + \Theta$ TO BE BOUNDED HOLOMORPHIC EQUIVALENT TO MODIFICATION $P + V$.

III. TOEPLITZ CORONA PROBLEM

$\Psi \in \mathcal{M}_{\mathcal{R}}$ TOEPLITZ OPERATOR $T_{\Psi}^{\mathcal{R}} \in \mathcal{L}(\mathcal{R})$, $T_{\Psi}^{\mathcal{R}} f = \Psi f, f \in \mathcal{R}$.

$\mathcal{R}$-TOEPLITZ CORONA PROBLEM:

(7) THERE EXISTS $\{f_\iota\}_{\iota=1}^{n} \subseteq \mathcal{R}$ SUCH THAT $\sum_{\iota=1}^{n} \varphi_\iota(z) f_\iota(z) = 1, z \in \Omega$.

(2) IMPLIES A STRONGER RESULT. FOR $f \in \mathcal{R}$ THERE EXISTS $\{f_\iota\}_{\iota=1}^{n} \subseteq \mathcal{R}$ SUCH THAT

(8) $$\sum_{\iota=1}^{n} \varphi_\iota(z) f_\iota(z) = f(z), z \in \Omega.$$

III. TOEPLITZ CORONA PROBLEM

Lemma

(8) IFF N_{Φ}^{*} IS ONTO OR, EQUIVALENTLY,

(9) $\sum_{\iota=1}^{n} ||T_{\Phi_\iota}^{\mathcal{R}*} f||^2 \geq \epsilon^2 ||f||^2, f \in \mathcal{R}$.

FOR $\omega \in \Omega$, THERE EXISTS $0 \neq k_\omega \in \mathcal{R}$ SUCH THAT $T_{\Psi}^{\mathcal{R}*} k_\omega = \overline{\Psi(\omega)} k_\omega, \Psi \in \mathcal{M}_{\mathcal{R}}$.

III. TOEPLITZ CORONA PROBLEM

Proposition

(8) OR (9) IMPLIES (1)

PROOF: USING k_ω WE HAVE

$$\sum_{\iota=1}^{n} |\varphi_\iota(\omega)|^2 ||k_\omega||^2 = \langle \sum_{\iota=1}^{n} T^{\mathcal{R}}_{\varphi_\iota} T^{\mathcal{R}*}_{\varphi_\iota} k_\omega, k_\omega \rangle =$$

$$= \sum_{\iota=1}^{n} ||T^{\mathcal{R}*}_{\varphi_\iota} k_\omega||^2 \geq \epsilon^2 ||k_\omega||^2.$$

III. TOEPLITZ CORONA PROBLEM

QUESTION: DOES (8) OR (9) IMPLY (2)?

> SINCE (2) IMPLIES (8) AND (9) FOR ALL $\mathcal{R}'$ WITH $\mathcal{M}_{\mathcal{R}'} = \mathcal{M}_{\mathcal{R}}$, WHAT ABOUT STRONGER ASSUMPTION?

QUESTION: DOES (8) OR (9) FOR ALL $\mathcal{R}'$ WITH $\mathcal{M}_{\mathcal{R}'} = \mathcal{M}_{\mathcal{R}}$, SAME LOWER BOUND, IMPLY (2)?

NOTE: (2) DOES NOT INVOLVE ANY $\mathcal{R}$.

III. TOEPLITZ CORONA PROBLEM

AFFIRMATIVE IF WE ASSUME COMMUTANT LIFTING THEOREM HOLDS FOR $\mathcal{R}$.

FOR $\mathcal{R}$, GIVEN SUBMODULES $\mathcal{S}_1 \subseteq \mathcal{R} \otimes \mathbb{C}^p, \mathcal{S}_2 \subseteq \mathcal{R} \otimes \mathbb{C}^q$ AND BOUNDED MODULE MAP $X : \mathcal{L}(\mathcal{R} \otimes \mathbb{C}^p, \mathcal{R} \otimes \mathbb{C}^q)$, THERE EXISTS MODULE MAP $\hat{X} \in \mathcal{L}(\mathcal{R} \otimes \mathbb{C}^p, \mathcal{R} \otimes \mathbb{C}^q)$ SUCH THAT

$$\pi_{\mathcal{S}_2} \hat{X} = X \pi_{\mathcal{S}_1} \text{ AND } ||\hat{X}|| = ||X||.$$

THEOREM: IF $\mathcal{R}$ SATISFIES *CLT*, THEN (8) OR (9) IMPLIES (2).

PROOF: LET $\mathcal{K} =$ KERNEL $N_\Phi : \mathcal{R} \otimes \mathbb{C}^n \to \mathcal{R}$. CONSIDER $Y : \mathcal{R} \otimes \mathbb{C}^n/\mathcal{K} \to \mathcal{R}$ SO THAT $N_\Phi = Y\pi_{\mathcal{K}}$. (8) IMPLIES THAT $X = Y^{-1}$ IS BOUNDED MODULE MAP. *CLT* IMPLIES THERE EXISTS $\hat{X} : \mathcal{R} \to \mathcal{R} \otimes \mathbb{C}^n$ SO THAT $X = \pi_{\mathcal{K}}\hat{X}$. SINCE $\pi_{\mathcal{K}} = XN_\Phi$, ONE HAS $X = \pi_{\mathcal{K}}\hat{X} = XN_\Phi\hat{X}$, WHICH IMPLIES $N_\Phi\hat{X} = I_{\mathcal{R}}$ SINCE X IS INERTIBLE. BY EARLIER LEMMA, $\hat{X} = M_\Psi$ FOR $\Psi = \{\Psi_\iota\}_{\iota=1}^n \subseteq \mathcal{M}_{\mathcal{R}}$ OR $\sum_{\iota=1}^n \varphi_\iota(z)\Psi_\iota(z) = 1, z \in \Omega$.

III. TOEPLITZ CORONA PROBLEM

CLT HOLDS ONLY FOR VERY SPECIAL $\mathcal{R}$. AMAR, TRENT-WICK SHOW (9) FOR ALL SUBMODULES OF $H^2(\Omega)$, SAME LOWER BOUND, IMPLIES (2) FOR $\Omega = \mathbb{B}^m, \mathbb{D}^m$.

(D-SARKAR) THEOREM: SUPPOSE $\mathcal{R} = L^2_a(\mu), \mathcal{R} \subseteq \mathcal{O}(\Omega)$

(ι) $\{k_\omega\}_{\omega \in \Omega} \subseteq \mathcal{M}_\mathcal{R}$ AND RANGE $T_{\mathcal{R}_\omega}$ CLOSED, $z \in \Omega$.

$(\iota\iota)$ FOR $\{\omega_j\}_{j=1}^N \subseteq \Omega, \{\lambda_j\} \in \mathbb{C}^N, \mathrm{N} \in \mathbb{N}$,

THERE EXISTS $g \in \mathcal{R}$ SUCH THAT

$$|g(z)|^2 = \sum_{j=1}^N |\lambda_j|^2 |k_{\omega_j}(z)|^2, \mu \quad a.e.$$

THEN ASSUMING (8) OR (9)HOLDS FOR ALL CYCLIC SUBMODULES OF $\mathcal{R}$, SAME LOWERBOUND IMPLIES (2).

III. TOEPLITZ CORONA PROBLEM

QUESTION: DOES (ι) HOLD FOR HARDY MODULE $H^2(\Omega)$ FOR A STRONGLY PSEUDO-CONVEX DOMAIN Ω IN $\mathbb{C}^m$? WHAT ABOUT $\iota\iota$?

KOSZUL COMPLEX FOR $\{T_{\varphi_\iota}^{\mathcal{R}}\}_{\iota=1}^{n}$ ON $\mathcal{R}$

$$0 \to \mathcal{R} \to \mathcal{R} \otimes \mathbb{C}^2 \to \mathcal{R} \to 0$$

(10) $\underline{0} \notin$ TAYLOR SPECTRUM IF EXACT

PROPOSITION: (10) IMPLIES (8) AND (9).
QUESTION: DOES (10) IMPLY (2)?