

STEIN-LIKE THEORY FOR BANACH-VALUED
HOLOMORPHIC FUNCTIONS ON THE MAXIMAL
IDEAL SPACE OF H^∞ AND THE OPERATOR
CORONA PROBLEM OF SZ.-NAGY

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BASIC DEFINITIONS

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For a commutative unital complex Banach algebra A with dual space A^* the *maximal ideal space* $M(A)$ of A is the set of nonzero homomorphisms $A \rightarrow \mathbb{C}$ equipped with the *Gelfand topology*, i.e., the weak* topology induced by A^* . It is a compact subset of the unit ball of A^* .

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Let $C(M(A))$ be the algebra of continuous complex-valued functions on $M(A)$ equipped with supremum norm.

The *Gelfand transform* $\hat{\cdot} : A \rightarrow C(M(A))$, $\hat{a}(\varphi) := \varphi(a)$, is a nonincreasing-norm morphism of algebras.

If the Gelfand transform is an isometry, then A is called a *uniform algebra*.

In the case of *Banach algebra H^∞ of bounded holomorphic functions on the unit disk $\mathbb{D} \subset \mathbb{C}$ with pointwise multiplication and supremum norm*, evaluation at a point of \mathbb{D} is an element of $M(H^\infty)$, so \mathbb{D} is naturally embedded into $M(H^\infty)$ as an open subset.

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The famous *Carleson corona theorem* asserts that \mathbb{D} is dense in $M(H^\infty)$.

This is equivalent to the following statement.

- For $\{f_i\}_{i=1}^n \subset H^\infty$, $n \in \mathbb{N}$, the Bezout equation $\sum_{i=1}^n g_i f_i = 1$ is solvable with $\{g_i\}_{i=1}^n \subset H^\infty$ if and only if $\max_{1 \leq i \leq n} |f_i(z)| > \delta > 0$ for every $z \in \mathbb{D}$.

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DEFINITION 1

A continuous function $f \in C(U; B)$ is said to be B -valued holomorphic if its restriction to $U \cap \mathbb{D}$ is B -valued holomorphic in the usual sense.

By $\mathcal{O}(U; B)$ we denote the vector space of B -valued holomorphic functions on U .

Next, by $\mathcal{O}_{M(H^\infty)}^B$ we denote the sheaf of B -valued holomorphic functions on $M(H^\infty)$.

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THEOREM 1

For all $k \in \mathbb{N}$

$$H^k(M(H^\infty); \mathcal{O}_{M(H^\infty)}^B) = 0.$$

Here $H^k(X; \mathcal{J})$ stands for the k th Čech cohomology group of a sheaf of abelian groups \mathcal{J} defined on a Hausdorff topological space X .

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- A function h on an open subset $U \subset M(H^\infty)$ is said to be *meromorphic* if $h = \frac{f}{g}$, where $f, g \in \mathcal{O}(U) (:= \mathcal{O}(U; \mathbb{C}))$ and g is not identically zero.

The set of meromorphic functions on U is denoted by $\mathcal{M}(U)$.

- A (Cartier) *divisor* on $M(H^\infty)$ consists of pairs (U_i, h_i) , where (U_i) is an open cover of $M(H^\infty)$ and $h_i \in \mathcal{M}(U_i)$, such that for all $U_i \cap U_j \neq \emptyset$ functions $\frac{h_i}{h_j} \in \mathcal{O}(U_i \cap U_j)$ and are nowhere zero.

As a consequence of Theorem 1 we obtain the *solution of the second Cousin problem on $M(H^\infty)$* .

THEOREM 2

For any divisor $D = \{(U_i, h_i)\}_{i \in I}$ on $M(H^\infty)$ there exist a meromorphic function $h_D \in \mathcal{M}(M(H^\infty))$ and a family of nowhere vanishing functions $c_i \in \mathcal{O}(U_i)$, $i \in I$, such that

$$h_D|_{U_i} = h_i \cdot c_i \quad \text{for all } i \in I.$$

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$$h_D|_{U_i} = h_i \cdot c_i \quad \text{for all } i \in I.$$

The statement is equivalent to the fact that **any holomorphic line bundle on $M(H^\infty)$** (i.e., a line bundle determined by a holomorphic cocycle) **is trivial**.

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THEOREM 3A

Any B -valued holomorphic function defined on a neighbourhood of a holomorphically compact set $K \subset M(H^\infty)$ can be uniformly approximated on K by functions from $\mathcal{O}(M(H^\infty); B)$.

For an ideal $J \subset \mathcal{O}(M(H^\infty)) (\cong H^\infty)$ define

$$\text{hull}(J) := \{x \in M(H^\infty); f(x) = 0 \quad \forall f \in J\}.$$

THEOREM 3B

For every $g \in \mathcal{O}(N; B)$, where N is an open neighbourhood of $\text{hull}(J)$, there exists a function $\tilde{g} \in \mathcal{O}(M(H^\infty); B)$ such that $\tilde{g}|_{\text{hull}(J)} = g$.

A quantitative version of this result for $B = \mathbb{C}$ and J a principal ideal generated by a Blaschke product was proved by Carleson.

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In this part we study Banach algebras $H_{\text{comp}}^{\infty}(A)$ of holomorphic functions on \mathbb{D} with relatively compact images in a commutative complex unital Banach algebra A . (One can easily show that algebra $H_{\text{comp}}^{\infty}(A)$ is isomorphic to $\mathcal{O}(M(H^{\infty}); A)$.)

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THEOREM 4

Let $f_1, \dots, f_m, f \in \mathcal{O}(M(H^{\infty}); A)$. Then f belongs to the ideal $I \subset \mathcal{O}(M(H^{\infty}); A)$ generated by f_1, \dots, f_m if and only if there exists a finite open cover $(U_k)_{1 \leq k \leq \ell}$ of $M(H^{\infty})$ such that for every $1 \leq k \leq \ell$ the function $f|_{U_k}$ belongs to the ideal $I_k \subset \mathcal{O}(U_k; A)$ generated by functions $f_1|_{U_k}, \dots, f_m|_{U_k}$.

In the proof one uses a standard argument involving **Koszul complexes** which reduces the statement to a question on existence of bounded on the boundary solutions of certain A -valued $\bar{\partial}$ -equations on \mathbb{D} similar to those in **Wolff's proof of Carleson's corona theorem**.

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However, since the target space A may be infinite dimensional, the classical **duality method** allowing to get such solutions for scalar $\bar{\partial}$ -equations **does not work** anymore.

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However, since the target space A may be infinite dimensional, the classical **duality method** allowing to get such solutions for scalar $\bar{\partial}$ -equations **does not work** anymore.

One uses instead the fact due to Suárez that **the set of trivial Gleason parts of H^∞ is totally disconnected together with a result on existence of bounded solutions of A -valued $\bar{\partial}$ -equations with 'supports' in the set of nontrivial Gleason parts of $M(H^\infty)$** . The fact that \mathbb{D} is dense in $M(H^\infty)$ is not used in the proof; hence, from Theorem 4 one obtains yet another proof of the corona theorem for H^∞ .

As a corollary we obtain

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Theorem 5 would also follow if we knew that H^{∞} has the *Grothendieck approximation property* (which is still an open problem).

Let $S_N(H^\infty) := S(H^\infty; \dots; H^\infty)$ be the N -dimensional slice algebra on $M(H^\infty)^N$ of continuous functions f such that $f(x, \cdot, y) \in H^\infty$ for each $x \in M(H^\infty)^{k-1}$ and $y \in M(H^\infty)^{N-k}$, $k = 1, \dots, N$.

A major open problem posed in the mid of 1960s asks whether the maximal ideal space of $S_N(H^\infty)$ is $M(H^\infty)^N$.

This fact is obtained now as a corollary of Theorem 5.

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COROLLARY 1

$$M(S_N(H^\infty)) = M(H^\infty)^N.$$

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Let $E \rightarrow M(H^\infty)$ be a continuous *Banach vector bundle* with fibre X defined on an open cover $\mathcal{U} = (U_i)_{i \in I}$ of $M(H^\infty)$ by a *cocycle* $\{g_{ij} \in C(U_i \cap U_j; GL(X))\}$; here $GL(X)$ is the group of invertible elements of the Banach algebra $L(X)$ of bounded linear operators on X equipped with the operator norm.

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We say that E is *holomorphic* if all $g_{ij} \in \mathcal{O}(U_i \cap U_j; GL(X))$. In this case $E|_{\mathbb{D}}$ is a holomorphic Banach vector bundle on \mathbb{D} in the usual sense.

Recall that E is defined as the quotient space of the disjoint union $\sqcup_{i \in I} U_i \times X$ by the **equivalence relation**:

$$U_j \times X \ni u \times x \sim u \times g_{ij}(u)x \in U_i \times X.$$

The projection $p : E \rightarrow X$ is induced by natural projections $U_i \times X \rightarrow U_i$, $i \in I$.

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A *morphism* $\varphi : (E_1, X_1, p_1) \rightarrow (E_2, X_2, p_2)$ of holomorphic Banach vector bundles on $M(H^\infty)$ is a continuous map which maps each vector space $p_1^{-1}(w) \cong X_1$ linearly to vector space $p_2^{-1}(w) \cong X_2$, $w \in M(H^\infty)$, and such that $\varphi|_{\mathbb{D}} : E_1|_{\mathbb{D}} \rightarrow E_2|_{\mathbb{D}}$ is a holomorphic map of complex Banach manifolds. If, in addition, φ is bijective, then it is called an **isomorphism**.

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We say that a holomorphic Banach vector bundle (E, X, p) on $M(H^\infty)$ is **holomorphically trivial** if it is isomorphic to the trivial bundle $M(H^\infty) \times X$.

Let $GL_0(X)$ be the connected component of $GL(X)$ containing the identity map $I_X := \text{id}_X : X \rightarrow X$. Then $GL_0(X)$ is a clopen normal subgroup of $GL(X)$. By $q : GL(X) \rightarrow GL(X)/GL_0(X) := C(GL(X))$ we denote the continuous quotient homomorphism onto the discrete group of connected components of $GL(X)$.

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Let $E \rightarrow M(H^\infty)$ be a holomorphic Banach vector bundle with fibre X defined on a finite open cover $\mathcal{U} = (U_i)_{i \in I}$ of $M(H^\infty)$ by a cocycle $g = \{g_{ij} \in \mathcal{O}(U_i \cap U_j; GL(X))\}$. By $E_{C(GL(X))}$ we denote the principal bundle on $M(H^\infty)$ with fibre $C(GL(X))$ defined on \mathcal{U} by the locally constant cocycle $q(g) = \{q(g_{ij}) \in C(U_i \cap U_j; C(GL(X)))\}$.

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THEOREM 6 (*Oka Principle*)

E is holomorphically trivial if and only if the associated bundle $E_{C(GL(X))}$ is trivial in the category of principal bundles with discrete fibres.

COROLLARY 2

E is holomorphically trivial in one of the following cases:

- (1) The image of each function g_{ij} in the definition of E belongs to $GL_0(X)$ (e.g., this is true if $GL(X)$ is connected);
- (2) E is trivial in the category of continuous Banach vector bundles.

In particular, the result is valid for spaces X with contractible group $GL(X)$. The class of such spaces include infinite-dimensional Hilbert spaces, spaces ℓ^p and $L^p[0, 1]$, $1 \leq p \leq \infty$, c_0 and $C[0, 1]$, spaces $L_p(\Omega, \mu)$, $1 < p < \infty$, of p -integrable measurable functions on an arbitrary measure space Ω , and some classes of reflexive symmetric function spaces; the class of spaces X with connected but not simply connected group $GL(X)$ include finite dimensional Banach spaces, finite direct products of James spaces etc. There are also Banach spaces X whose linear groups $GL(X)$ are not connected. E.g., the groups of connected components of spaces $\ell^p \times \ell^q$, $1 \leq p < q < \infty$, are isomorphic to \mathbb{Z} .

SZ.-NAGY OPERATOR CORONA PROBLEM

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We apply Theorem 5 to the **Sz.-Nagy operator corona problem posed in 1978**. In its formulation $H^\infty(L(X, Y))$ stands for the Banach space of holomorphic functions F on \mathbb{D} with values in the space of bounded linear operators $X \rightarrow Y$ of complex Banach spaces X, Y with norm $\|F\| := \sup_{z \in \mathbb{D}} \|F(z)\|_{L(X, Y)}$.

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PROBLEM (SZ.-NAGY)

Let $F \in H^\infty(L(H_1, H_2))$, where H_i , $i = 1, 2$, are separable Hilbert spaces, satisfy $\|F(z)x\| \geq \delta\|x\|$ for every $x \in H_1$ and every $z \in \mathbb{D}$, where $\delta > 0$ is a constant. Does there exist $G \in H^\infty(L(H_2, H_1))$ such that $G(z)F(z) = I_{H_1}$ for every $z \in \mathbb{D}$?

This problem is of great interest in operator theory (angles between invariant subspaces, unconditionally convergent spectral decompositions), as well as in control theory. It is also related to the study of submodules of H^∞ and to many other subjects of analysis.

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Obviously, the condition imposed on F is necessary. It implies existence of a uniformly bounded family of left inverses of $F(z)$, $z \in \mathbb{D}$.

The question is whether this condition is sufficient for the existence of a bounded analytic left inverse of F .

In general, the answer is known to be negative (S. Treil). But in some specific cases it is positive. In particular,

- Carleson's theorem stating that a Bezout equation $\sum_{i=1}^n g_i f_i = 1$ is solvable with $\{g_i\}_{i=1}^n \subset H^\infty$ as soon as $\{f_i\}_{i=1}^n \subset H^\infty$ satisfies $\max_{1 \leq i \leq n} |f_i(z)| > \delta > 0$ for every $z \in \mathbb{D}$ means that the answer is positive when $\dim H_1 = 1$, $\dim H_2 = n < \infty$.
- More generally, the answer is positive as soon as $\dim H_1 < \infty$ (Fuhrmann, Vasjunin, Tolokonnikov).

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- More generally, the answer is positive as soon as $\dim H_1 < \infty$ (Fuhrmann, Vasjunin, Tolokonnikov).

For a long time there were no positive results in the case $\dim H_1 = \infty$. The first positive results in this case were obtained by P. Vitse. Following P. Vitse we consider a more general

PROBLEM 1

Let X_1, X_2 be complex Banach spaces and $F \in H^\infty(L(X_1, X_2))$ be such that for each $z \in \mathbb{D}$ there exists a left inverse G_z of $F(z)$ satisfying $\sup_{z \in \mathbb{D}} \|G_z\| < \infty$. Does there exist $G \in H^\infty(L(X_2, X_1))$ such that $G(z)F(z) = I_{X_1}$ for every $z \in \mathbb{D}$?

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Since in this general setting the answer is negative, we restrict ourselves to the case of $F \in H_{\text{comp}}^\infty(L(X_1, X_2))$, the space of holomorphic functions on \mathbb{D} with relatively compact images in $L(X_1, X_2)$. In this case P. Vitse proved that

- the answer is positive for F that can be uniformly approximated by finite sums $\sum f_k(z)L_k$, where $f_k \in H^\infty$ and $L_k \in L(X_1, X_2)$.

The question of whether each $F \in H_{\text{comp}}^\infty(L(X_1, X_2))$ can be obtained in that form is closely related to the still open problem about the *Grothendieck approximation property* for H^∞ .

THEOREM 7 (COMPLEMENT PROBLEM FOR $H_{\text{comp}}^\infty(L(X_1, X_2))$)

Let $F \in H_{\text{comp}}^\infty(L(X_1, X_2))$, where X_i , $i = 1, 2$, are complex Banach spaces, be such that for every $z \in \mathbb{D}$ there exists a left inverse G_z of $F(z)$ satisfying $\sup_{z \in \mathbb{D}} \|G_z\| < \infty$. Let $Y := \text{Ker } G_0$. Assume that $GL(Y)$ is connected. Then there exist functions $H \in H_{\text{comp}}^\infty(L(X_1 \oplus Y, X_2))$ and $G \in H_{\text{comp}}^\infty(L(X_2, X_1 \oplus Y))$ such that for all $z \in \mathbb{D}$

$$H(z)G(z) = I_{X_2}, \quad G(z)H(z) = I_{X_1 \oplus Y} \quad \text{and}$$

$$H(z)|_{X_1} = F(z).$$

COROLLARY 2

Let $F \in H_{\text{comp}}^\infty(L(H_1, H_2))$, where H_i , $i = 1, 2$, are Hilbert spaces, satisfy $\|F(z)x\| \geq \delta\|x\|$ for every $x \in H_1$ and every $z \in \mathbb{D}$, where $\delta > 0$ is a constant. Let $Y := (F(0)(H_1))^\perp$. Then there exist functions $H \in H_{\text{comp}}^\infty(L(H_1 \oplus Y, H_2))$, $G \in H_{\text{comp}}^\infty(L(H_2, H_1 \oplus Y))$ such that for all $z \in \mathbb{D}$

$$H(z)G(z) = I_{H_2}, \quad G(z)H(z) = I_{H_1 \oplus Y} \quad \text{and}$$

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Also, we obtain a positive answer in Problem 1 for spaces H_{comp}^∞ .

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THEOREM 8 (GENERALIZED SZ.-NAGY PROBLEM FOR H_{comp}^∞)

Let X_1, X_2 be complex Banach spaces and $F \in H_{\text{comp}}^\infty(L(X_1, X_2))$ be such that for each $z \in \mathbb{D}$ there exists a left inverse G_z of $F(z)$ satisfying $\sup_{z \in \mathbb{D}} \|G_z\| < \infty$. Then there exist $G \in H_{\text{comp}}^\infty(L(X_2, X_1))$ such that for all $z \in \mathbb{D}$

$$G(z)F(z) = I_{X_1}.$$

For an ideal $J \subset H^\infty$ by $\hat{J} \subset H^\infty$ we denote the ideal of functions vanishing on $\text{hull}(J) \subset M(H^\infty)$. Consider the Banach algebra $H_J^\infty \subset H^\infty$ generated by constant functions and functions from \hat{J} . Then all the results presented in the talk are valid also if one replaces in their formulations H^∞ by H_J^∞ .