# A short proof that 4-prismatoids have width at most 4


**Tamon Stephen**

Simon Fraser University

joint work with
Francisco Santos and Hugh Thomas


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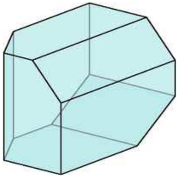
# Outline

- The Diameter of a Polytope and Hirsch's Conjecture.
- Spindles and Prismatoids.
- The Width of 4-prismatoids.
- Conclusions.

# Polytopes

Polytopes are the intersection of finitely many half-spaces in $\mathbb{R}^d$; they can also be viewed as the convex hull of finitely many extreme points or vertices.

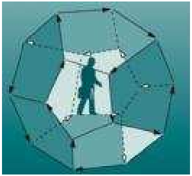


- These vertices are attached by 1-dimension edges which are the "exterior" line segments of the polytope.
- The vertices and edges of a polytope thus form a graph, which is sometimes called the skeleton of the polytope.
- Polytopes appear, for instance, as the feasible regions of linear programs. Linear programs are one of the most widely used applied mathematical models.

*Image: Wikipedia*

# The Simplex Algorithm

The most famous algorithm for linear programming is the simplex method of Dantzig (1947).



- Its outline is very simple: after finding some vertex, it tries to improve the current solution by pivoting, that is, moving along an edge to an adjacent vertex in a way that improves the objective function.
- When no improvement is possible, the optimum has been reached.
- The simplex algorithm is incredibly effective in practice.
- There are several challenges in analyzing the worst-case (theoretical) behaviour of the simplex method.

*Image: Journal of Combinatorial Optimization*

# Diameter of a Polytope

- The simplex algorithm finds a path in the skeleton graph of a polytope from the starting vertex to the optimal vertex.
- For a pivoting algorithm to solve a problem efficiently, there must be a short path available between the starting and optimal vertices.
- Given any two vertices of a polytope, we define the distance between them to be the length of the shortest path between them in the graph.
- Then we can define the diameter of a polytope to be the *maximum* distance between two of its vertices.

For example, a three dimensional cube has diameter 3.

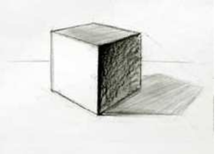


*Image: Draw23.com*

# The Hirsch conjecture

- Given a $d$-dimensional polytope, we can describe it as being bounded by a set of $(d-1)$-dimensional sides called facets. In a linear program these will be (some of) the half-spaces listed in the description of the problem.
- It is possible to build a polytope with arbitrarily large diameter by making a polytope with many facets.
- **Question:** Can we build a polytope with large diameter using only a small number of facets?
- Let $n$ be the number of facets of a $d$-dimensional polytope $P$.
- In 1957 Hirsch conjectured that the diameter of $P$ is at most $n-d$.
- Last summer Santos announced a counter-example to the Hirsch conjecture.

# Outline of Santos' construction

- Santos' proof contains two key ingredients.
- The first is the construction of a special 5-dimensional polytope called a "spindle" whose two special vertices that are relatively far apart.
- The second is to repeatedly (38 times) modify the spindle by an operation that at each step increases the dimension, number of facets and diameter of the polytope by one.
- It also roughly doubles the number of vertices.
- The result is a 43-dimensional non-Hirsch polytope with 86 facets and diameter (at least) 44. It also has around $2^{40}$ vertices.
- Subsequent refinements have produced examples in dimension as low as 20 (by Matschke, Santos and Weibel).

# Spindles

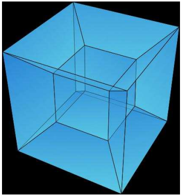
- A spindle is a polytope with two distinguished vertices $u$ and $v$ such that every facet contains exactly one of them.
- A spindle can be viewed as the result of intersecting two polyhedral cones that contain each other's origin.
- The length of a spindle is the minimum number of steps it takes to go from $u$ to $v$.
- In 3 dimensions, spindles will have length 3, unless a pair of rays of the cone collide (which would drop it to 2).
- Given a $d$-dimensional spindle of length at least $(d+1)$, Santos' methods can be used to build a counter-example to the Hirsch conjecture.
- The initial construction was a 5-dimensional spindle of length 6 with 48 facets and 322 vertices.
- Subsequent improvements give a 5-dimensional spindle of length 6 with only 25 vertices.

# Spindles and Prismatoids

- Any polytope $P$ has a dual polar polytope $P'$ where the vertices of $P$ correspond to the facets of $P'$ and vice-versa.
- Rather than work with spindles directly, it is more intuitive to work with their duals, called prismatoids.
- Examples: In dimension $d$ the cross-polytope (octahedron) is a particularly simple spindle, while the cube is a particularly simple prismatoid.
- Prismatoids have two distinguished facets $Q^+$ and $Q^-$ and each vertex lies on either $Q^+$ or $Q^-$.
- In the prismatoid setting, moving from vertex to vertex along an edge is replaced by moving between facets that share a codimension 2 boundary.
- Santos constructs a 5-prismatoid of width 6 using a pair of geodesic maps on the 3-sphere.

- The construction of the maps on 3-sphere is based on a pair of geodesic maps (graphs) that exist on the 2-torus.
- In fact, if we could get such a pair of geodesic maps on the 2-sphere, rather than the 2-torus, we could build a 4-prismatoid of length 5.
- The question of whether such a 4-prismatoid exists is a natural one that was left open in Santos' original work.
- It requires topological techniques, as the maps do exist on the torus.
- We give two proofs that such a 4-prismatoid cannot exist, one via simply-connectedness, and one via Euler characteristic.
- Both are short and fairly elementary. We present here the Euler characteristic proof.

A well known construction in polytopes is the Schlegel diagram: one face of the polytope is enlarged and the remainder of the polytope is projected onto that face.



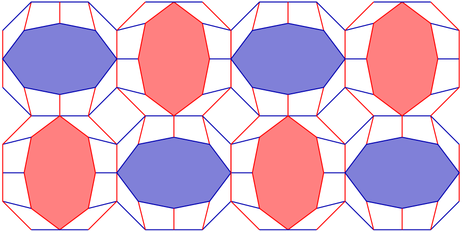
- This has the effect of replacing a $d$-dimensional polytope by a $(d-1)$-dimensional polytope that is subdivided into $(d-1)$-dimensional cells representing the facets of the original polytope.
- In this case we will project onto the facet $Q^+$, with $Q^-$ becoming an interior cell.
- The question then is how many $(d-2)$-dimensional facets of the resulting complex must be crossed to go from the outside $Q^+$ to the inside $Q^-$.

*Image: Wikipedia*

# Further processing

- We observe that it is enough to prove it for triangulated cell complexes. Since this is 3-dimensional, the "triangles" are tetrahedra (simplices).
- Since all vertices lie on $Q^+$ or $Q^-$, the tetrahedra come in 3 flavours: they can have either one, two or three vertices on $Q^+$. We think of colouring these blue, white and red respectively.
- All we have to prove is that there is some white tetrahedron that shares a 2-dimensional face with both a red and a blue tetrahedron.
- As a preprocessing step, we glob together groups of adjacent blue or red tetrahedra.

- Now we look at a "middle slice" of the complex, given by a (topological) sphere that lies strictly between $Q^+$ and $Q^-$.
- This sphere is divided in 2-dimensional cells by the tetrahedra.
- White tetrahedra become quadrilaterals, while blue and red ones become triangles. However, they may have an arbitrary shape after globbing.
- Here is a possible portion of a middle slice:



# A pair of graphs

- Each edge in the middle slice is a section of a face of a white tetrahedron that has an edge on either $Q^+$ or $Q^-$, but not both.
- By colouring these edges red and blue depending on whether the face has an edge on $Q^+$ or $Q^-$, we see that the graph formed by the vertices and edges on the slice is the refinement of transversal red and blue graphs embedded in the sphere.
- This gives us a purely combinatorial question about embedding pairs of transversal graphs.
- Each white face alternates between red and blue edges as we go around its perimeter.

# The Euler characteristic

- We proceed by evaluating the Euler characteristic locally at each vertex.
- Recall the Euler characteristic in 2 dimensions is V-E+F, i.e. the number of vertices $-$ the number of edges $+$ the number of faces.
- The Euler characteristic of any polytope is 2.
- We compute at each vertex $v$, the quantity $f(v)$ by summing: $+1$, $(-\frac{1}{2})\times$ the number of edges incident on the vertex, and $\frac{1}{4}\times$ the number of white faces incident on the vertex.
- Observe that $\sum_v f(v) + b = \chi(\mathbb{S})$ where $b$ is the number of red and blue faces and $\chi(\mathbb{S}) = 2$ is the Euler characteristic of the sphere: vertices and white face contribute $+1$ to the sum, while edges contribute $-1$.

# Non-positivity of $f$

- Claim: For any $v$, we have that $f(v) \leq 0$.
- Proof: Walk around $v$ alternating between edges contributing $-\frac{1}{2}$ and faces contributing at most $\frac{1}{4}$.
- If we see at least 4 faces, this is sufficient to cancel the $+1$ from the vertex.
- We might see only 3 faces, in which case one of them is not white and contributes 0 rather than $+\frac{1}{4}$. This follows from the alternation of the edges, and again allows us to cancel the $+1$ from the vertex.
- We still need to show that there are sufficient $v$ with $f(v) < 0$ to cancel the $b$ term.

# Special vertices

- To do this, we show that each red or blue face must contain at least two *special* vertices which either lie on a second red or blue face, or lie on at least 4 white faces.
- Assume not. Then decomposing the red or blue face into its constituent triangles, we see they all intersect $Q^-$ in the same line segment. But this line segment then loops around and intersects itself.
- Each special vertex contributes at most $-1/2$ to $\sum_v f(v)$.
- We assign special vertices to blue and red faces to show that if we have $b$ blue and red faces, we have a total contribution of $-b$ from special vertices.
- The illustrated complex from the torus has $\sum_v f(v) = -b$.

# Second proof

- There is a second proof that uses simply connectedness directly.
- The idea is to find a cycle in the refined graph that does not bound a face.
- This proof is more fully topological, i.e. it works for any pair of transversal graphs embedded in the sphere that avoid some degeneracies.

# Conclusions and Remarks

- We prove that 4-prismatoids have length at most 4.
- This rules out one very particular avenue for possible low-dimensional non-Hirsch polytopes.
- Constructing low-dimensional non-Hirsch polytopes remains an interesting open question.
- An important outstanding question is the Polynomial Hirsch Conjecture: whether there is any polynomial upper bound (in $n$ and $d$) for the diameter of a polytope.
- Kalai and Santos propose adapting this approach to get a polynomial upper bound for the diameter of a polytope by looking at pairs of maps on $\mathbb{S}^d$ and their common refinement.

Thank you!