Robust Algorithms for CSPs

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Notation

Fix a finite set D

Constraint language Γ = finite set of relations on D

- CSP(Γ): given constraints $R_1(\mathbf{x}_1), \ldots, R_q(\mathbf{x}_q)$ over a set of variables V, and such that each $R_i \in \Gamma$, decide if there is $\varphi : V \to D$ satisfying all constraints
- Max CSP(Γ): optimisation, can repeat constraints
- Want: classify the complexity and approximability
- Many algorithmic and hardness results
- Many approaches: algebra, logic, graphs, analysis etc
- Many open questions

Algorithms for (Max) CSP: Good and Bad

- Exact polynomial-time algorithm for CSP(Γ)
  - Good: correctly identifies (un)satisfiable instances
  - Bad: 0.999-satisfiable vs. (< 0.001)-satisfiable?
- Approx polynomial-time algorithm for Max CSP(Γ)
  - Good: treats all instances uniformly
  - Bad: satisfiable vs. 0.999-satisfiable?

Is there a good natural compromise between those two?

Robust Algorithms

- Call an algorithm for CSP(Γ) robust [Zwick'98] if
  - it correctly identifies satisfiable instances, and
  - for a $(1-\epsilon)$-satisfiable instance, it outputs a $(1-g(\epsilon))$-satisfying assignment, where $g() \to 0$,
  - running time is polynomial in the size of instance, does not depend on ϵ
- Robust algorithms:
  - (almost) untouched for 12 years
  - meeting point of algebra and approximation

Previous Negative Results

3-LINp: systems of linear 3-variable equations mod p

- Easy as a CSP
- Trivial $1/p$-approximation algorithm for MAX CSP: assign values to variables uniformly at random
- Best possible [Håstad'01]
- For any $\epsilon, \gamma > 0$, it is **NP**-hard to approximate within $1/p + \gamma$, even on $(1 - \epsilon)$-satisfiable instances
- Hence no robust algorithm

Previous Positive Results

Restrict MAX CSP(Γ) to $(1-\epsilon)$-satisfiable instances, algorithm finds, in expectation, $(1-g(\epsilon))$-sat assignment

- HORN-k-SAT $\quad g(\epsilon) = \frac{\log k}{\log(1/\epsilon)}$ (LP-based) [Zwick'98]
- 2-SAT
  - $g(\epsilon) = 5 \cdot \epsilon^{1/3}$ (SDP-based) [Zwick'98]
  - $g(\epsilon) = O(\epsilon^{1/2})$ [Charikar,Makarychev^2'09]
- UNIQUE GAMES-k
  - $g(\epsilon) = O(k^2\epsilon^{1/5}\sqrt{log(1/\epsilon)})$ [Khot'02]
  - $g(\epsilon) \approx 1 - O(k^{-\epsilon/2})$ [Charikar,Makarychev^2'06]

The Issue with Equality

- Let CSP(Γ) admit a robust algorithm with some $g(\epsilon)$
- Let $\Gamma_=$ denote $\Gamma \cup \{=\}$.
- Assumption (A): CSP($\Gamma_=$) admits a robust algorithm
  - need (A) to build general algebraic theory
  - current conjecture about RA does not rely on (A)
- Stronger A (SA): ditto + with $g' = \Omega(g)$
  - might need (SA) to study $g(\epsilon)$ quantitatively

Primitive Positive Definability

- Let Γ be a constraint language on D
- Let G be a sat instance of CSP(Γ) with c constraints
- Let R be a projection of $Sol(G)$ on some coordinates
- Assume CSP(Γ) admits RA with some $g(\epsilon)$
- Then CSP($\Gamma \cup \{R\}$) admits RA with $c \cdot g(c\epsilon)$
- The same if G is an instance of CSP($\Gamma_=$) + (A)/(SA)
  - logic/algebra lingo: R is pp-definable in Γ
- If Γ_1 and Γ_2 pp-define each other, treat them the same
- What controls pp-definability?

Polymorphisms by Example

Consider the relation $R = \{(0,0,0),(1,0,0),(0,0,1)\}$.

- The binary operation min is a polymorphism of R.

$$
\begin{array}{ccccccccc}
 & \min & & \min & & \min & & & \\
(& 1 & , & 0 & , & 0 &) & \in & R \\
(& 0 & , & 0 & , & 1 &) & \in & R \\
\hline
(& 0 & , & 0 & , & 0 &) & \in & R
\end{array}
$$

- The binary operation max is not.

Polymorphisms in Control

Let $\mathrm{Pol}(\Gamma) = \{f \mid f \text{ is a polymorphism of each } R \in \Gamma\}$

Theorem 1 (Geiger'68; Bodnarchuk et al'69)
Γ_1 *pp-defines* Γ_2 *iff* $\mathrm{Pol}(\Gamma_1) \subseteq \mathrm{Pol}(\Gamma_2)$.

- Need to look at $\mathrm{Pol}(\Gamma)$ rather than Γ
- "Nice" operations in $\mathrm{Pol}(\Gamma) \Rightarrow$ "easy" $\mathrm{CSP}(\Gamma)$
- Actually, algebra goes much deeper than that
- "Nice" = good identities, e.g.
  - $f(x, y) = f(y, x)$ min for Horn k-Sat
  - $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$ 2-Sat
- Good identities $\Rightarrow$ structure in relations/algebras

Important Technicalities

- Assume $f \in \mathrm{Pol}(\Gamma)$ is unary non-bijective
- Apply f to any assignment, preserve sat constraints
- So, can ignore values in $D \setminus Range(f)$
- Take f with minimum range, remove $D \setminus Range(f)$
- What remains is the core of Γ (unique up to isomorph)
- If Γ_0 is the core of Γ then $\mathrm{CSP}(\Gamma) = \mathrm{CSP}(\Gamma_0)$
- If Γ is a core (of itself), and $\Gamma^c = \Gamma \cup \{\{a\} \mid a \in D\}$
  - Lemma: If Γ admits RA, then so does Γ^c

Bounded Width

For $l \leq k$, an (l,k)-strategy $\mathcal{S}$ for I is a non-empty set of partial assignments with domain size $\leq l$ or k satisfying:

1. If $q \in \mathcal{S}$ and $p \subseteq q$ (i.e. q extends p) then $p \in \mathcal{S}$.
2. If $p \in \mathcal{S}$ and $|dom(p)| \leq l$ then, for each $u \in V$, there is $q \in \mathcal{S}$ with $|dom(q) = k|$ s.t. $p \subseteq q$ and $u \in dom(q)$.
3. If $p \in \mathcal{S}$ and C is a constraint in I whose variables are all contained in $dom(p)$ then p satisfies C.

Any satisfiable instance has an (l,k)-strategy for any l,k: restrictions of any fixed satisfying assignment

CSP(Γ) has width (l,k) if, $\forall I$, (l,k)-strategy $\Rightarrow$ satisfiable

Bounded Width Theorems

Bounded width = width (l, k) for some l, k

Wlog assume $\Gamma = \Gamma^c$.

Theorem 2 (Larose,Zádori'06; Valeriote'09)
If Γ pp-defines a constraint language whose core is 3LIN-p *for some p then* CSP(Γ) *does not have bounded width.*

The above pp-definition can be chosen without equality.

Theorem 3 (Barto,Kozik'09; Bulatov'09)
In all other cases, CSP(Γ) *has width* $(2, k)$ *where* $k = \max(3, maxar)$.

The Guruswami-Zhou Conjecture

Guruswami and Zhou observed that

- the obstructions for bounded width are the same as the known obstructions for robust algorithms: 3Lin-p
- For Boolean CSPs, BW=RA

Conjecture 1 (Guruswami,Zhou'11)
CSP(Γ) *admits a robust algorithm iff it has bounded width.*

- The $\Rightarrow$ direction holds by [Håstad] + previous slide.
- The $\Leftarrow$ direction is the conjecture.

Width 1

- CSP(Γ) has width 1 if, $\forall I$, $(1, maxar)$-strategy $\Rightarrow$ sat
- Horn k-Sat has width 1
- 2-Sat and UG-k have bounded width, but not width 1

Theorem 4 (Kun et al'11; Dalmau,AK'11)
If CSP(Γ) *has width 1 then it admits a robust algorithm.*

Reduction to (dual) Horn k'-Sat where $k' = maxar \cdot |D|$.

Define Γ' on domain $2^D \setminus \{\emptyset\}$ as follows [Feder,Vardi'98]

- $\Gamma' = \{R' \mid R \in \Gamma\}$
- $R' = \{(A_1, \dots, A_r) \mid \exists S \subseteq R \text{ s.t. } \forall i \ A_i = pr_i S\}$

Width 1 Cont'd

Facts about Γ' [Feder,Vardi'98]:

- Function $a \to \{a\}$ maps tuples in R to tuples in R'
- If CSP(Γ) has width 1 then there exists $f : 2^D \to D$ (aka set function) mapping tuples in R' to tuples in R.
- Can transform an instance I of CSP(Γ) to an instance I' of Max CSP(Γ') by replacing each R with R'.
- Can move between assignments for I and I', preserving satisfied constraints
- Hence can replace Γ by Γ'

Width 1 Cont'd

- Obtain Γ'' by Booleanising Γ'
  - the domain of Γ'' is $\{0,1\}$
  - $ar(R) = ar(R') = r \Rightarrow ar(R'') = r|D|$.
  - replace each $A \subseteq D$ by its characteristic vector
- Max CSP(Γ') and Max CSP(Γ'') are the same
- Trivially, $\cup$ is a polymorphism of Γ'
- So, max is a polymorphism of Γ''
- Hence Max CSP(Γ'') is dual Horn k'-Sat where $k' = maxar \cdot |D|$
- Apply [Zwick'98], use set function f to go back to I

Beyond Width 1

Theorem 5 (Dalmau,AK'11) *If* $d \in \mathrm{Pol}(\Gamma)$ *then* $\mathrm{CSP}(\Gamma)$ *admits a robust algorithm with* $g(\epsilon) = O(\epsilon^{1/10})$*.*

The operation d (dual discriminator) is defined as

$$d(x, y, z) = \begin{cases} x & \text{if } x = y \\ z & \text{otherwise} \end{cases}$$

Can assume that relations in Γ are of three types [JCG'96]:

1. all unary relations (wlog of the form $D \setminus \{a\}$)
2. $(x = a \vee y = b)$ where $a, b \in D$ — 2-Sat-like
3. $\{(a, \pi(a) \mid a \in D\}$ where π is a permutation — UG

SDP relaxation

Basic SDP for an instance I of CSP(Γ) with vars V.

- Vector variables X_u^i, $u \in V$, $i \in D = \{1, \ldots, k\}$.
  (Rough intuition: $||X_u^i|| \approx 1 \Leftrightarrow u$ is assigned i)
- SDP constraints

$$\begin{array}{ll}
||X_u^1||^2 + \cdots + ||X_u^k||^2 = 1, & u \in V \\
X_u^i \cdot X_u^j = 0, & u \in V,\ 1 \le i \ne j \le k \\
X_u^i \cdot X_v^j \ge 0, & u, v \in V,\ 1 \le i, j \le k \\
\sum_{1 \le i,j \le k} X_u^i \cdot X_v^j = 1, & u, v \in V
\end{array}$$

Goal Function

- The goal function is $\sum_{C\in I} z_C$ where $z_C =$

$$\begin{cases} \sum_{1\le i\le k} X_u^i \cdot X_v^{\pi(i)} & \text{if } C=((u,v),\pi) \\ 1 - X_u^i \cdot X_u^i & \text{if } C=(u, D\setminus\{i\}) \\ 1 - \overline{X}_u^i \cdot \overline{X}_v^j & \text{if } C=(u=i \vee v=j) \end{cases}$$

where $\overline{X}_u^i$ is a shortand for $\sum_{1\le j\neq i\le k} X_u^j$

2-Step Rounding

Combination of Zwick's 2-Sat and Khot's UG-k ideas.

- Set $S = \sum_{1 \le i \le k} X_u^i$ (independent of u, $||S|| = 1$)
- Set $Y_u^i = 2X_u^i - S$, $u \in V, i \in D$ (again, $||Y_u^i|| = 1$)
- Apply Zwick's rounding with appropriate ϵ to Y_u^i
  - rotate vectors Y_u^i towards the line given by S
  - take vector N_1 uniformly generated from the unit sphere, replace by $-N$ if necess to have $N_1 \cdot S > 0$
  - if there is a unique Y_u^i with $N_1 \cdot Y_u^i > 0$, set $u = i$
- Choose vector N_2 from distribution $N(0,1)$, $N_2 \cdot S > 0$
- Set each undefined u to i such that $N_2 \cdot X_u^i$ is max

Mixed MAX CSP

In a mixed MAX CSP instance, constraints are of two sorts:

- Hard constraints that must be satisfied
- Soft constraints that can be falsified.

The case when all soft constraints are unary was studied:

- as MINCOSTCSP, complexity [Takhanov'10]
- as STRICT CSP, approximability [Kumar et al'11]

Robust Algorithms for MIXED CSP

- Can study robust algorithms for Mixed CSP
- Compute all fractions of the number of soft constraints
- Non-empty notion: MIXED UG-k admits RA
  - reduction to ordinary (soft) UG-k
  - eliminate all hard constraints
  - if $R_1(x, y)$ is hard, then, for each soft $R_2(y, z)$, replace $R_2(y, z)$ by $R_3(x, z)$ where $R_3 = R_1 \circ R_2$
  - eventually no soft constraint will touch y
  - now eliminate all hard constraints involving y

Mixed Horn k-Sat

Theorem 6 (Dalmau,AK'11)
If, for each fixed k', Mixed Horn k'-Sat *admits RA then the Guruswami-Zhou conjecture holds.*

Proof Idea: Fix Γ such that CSP(Γ) has bounded width.
Express existence of $(2, maxar)$-strategy for an instance I of CSP(Γ) as an instance I' of Mixed Horn k'-Sat with $k' = \sum_{i=1}^{maxar} |D|^i$.

For each p with $|dom(p)| \leq maxar$, introduce a Boolean variable x_p stating "p is not in the strategy".
Since $maxar$ is constant, polynomially many variables.

Clauses in I'

1. If $q \in \mathcal{S}$ and $p \subseteq q$ (i.e. q extends p) then $p \in \mathcal{S}$.
   - add hard $x_p \to x_q$
2. If $p \in \mathcal{S}$ and $|dom(p)| < k$ then for every $u \in V$ there exists some $q \in \mathcal{S}$ with $p \subseteq q$ and $u \in dom(q)$.
   - add hard $\bigvee x_q \to x_p$
     where $\bigvee$ is over all q with $u \in dom(q) \supseteq dom(p)$
3. If $p \in \mathcal{S}$ and C is a constraint in I whose scope is entirely contained in $dom(p)$ then p satisfies C
   - add soft x_p, for each C and p falsifying the above
4. non-empty strategy: add hard $\bigvee_p \overline{x}_p$

More about I'

- Since k is constant, I' has polynomially many constraints
- There are at most M different soft constraints in I' where $M = \sum_{i=1}^{k} |D|^i$
- Each soft constraint appears at most m times where m is the number of constraints in I
- Satisfying assignment φ' for $I' \Rightarrow$ strategy $\mathcal{S}_{\varphi'}$ for I consisting of all p with $x_p = 0$

Almost Satisfiable I'

Assume φ' fails to satisfy ϵ-fraction of soft constraints in I'.

- If a soft clause (x_p) is falsified then remove from I all constraints corresponding to it
- These form at most $(M \cdot \epsilon)$-fraction of constraints in I
- Now $\mathcal{S}_{\varphi'}$ is a strategy for the remaining instance
- Bounded width algorithm finds a solution for it

Open Problems

- Settle the Guruswami-Zhou conjecture
- Study quantitative dependence of $g(\epsilon)$ on Γ
  - determined by $\mathrm{Pol}(\Gamma)$, modulo (SA)
- Study Mixed CSPs
  - robust algorithm for Mixed Horn k-Sat ?
  - r-approx algorithm for Mixed Horn k-Sat ?
- Study CSPs with global constraints
  - Max Bisection = Max Cut into equal parts
    robust algorithm with $g(\epsilon) = O(\epsilon^{1/3} \log(1/\epsilon))$
    (via expander decomposition) [Guruswami et al'11]