On the Usefulness of Predicates

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University of Toronto

Fields Workshop on CSPs, 2011-08-15

(joint work with Johan Håstad)

MAX 4-LIN

The MAX 4-LIN problem:

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\begin{array}{lclc}
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Hardness of MAX 4-LIN

By [Hås01], it is NP-hard to

find assignment α satisfying $1/2 + \epsilon$ equations
assuming there is some α satisfying $1 - \epsilon$ equations.

NP-hard to do anything non-trivial!

Example 2: Max GLST

$$GLST(x_1, x_2, x_3, x_4) = \begin{cases} x_2 \oplus x_3 & \text{if } x_1 = \text{False} \\ x_2 \oplus x_4 & \text{if } x_1 = \text{True} \end{cases}$$

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Let us relax the conditions of the algorithm

1. The algorithm chooses $Q : \{0,1\}^k \to [0,1]$
2. Instead of maximizing

$$\sum P(\ldots)$$

   seek to maximize

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Notation

instance is pair (P, L) for

- k-ary predicate $P : \{0,1\}^k \to \{0,1\}$
- list of k-tuples of literals L

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expectation $E_P = \mathbb{E}[P(x)]$ over uniform x

Usefulness

$P : \{0,1\}^k \to \{0,1\}$ is *useful* for $Q : \{0,1\}^k \to [0,1]$ if we can

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P is *useless* if it is not useful for any Q.

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Typically Q would be relaxation of P or at least have something to do with P

- E.g., "weak majority" instead of "strong majority"

But we're generous and allow any Q

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(NAE = Not All Equal)

$GLST(x_1, x_2, x_3, x_4)$ is useful for $NAE(\cdot, x_2, x_3, x_4)$.

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What about MAX 4-LIN?

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MAX 4-LIN *is useless*
– no matter what objective we use we can't do anything useful

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- *adaptive* or *non-adaptive* usefulness:
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- *computational* or *information-theoretic* usefulness.
  – will focus on computational
- more *general classes of CSPs*
  – more on that later

Quick Conclusions

Most approximation resistance results in fact show the stronger property of uselessness

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In particular from [AM'09] follows that:

assuming UGC, *P is useless if it supports a pairwise independent distribution*.

(there exists pairwise independent distribution μ such that $\mathit{Supp}(\mu) \subseteq P^{-1}(1)$)

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Given $f : \{0, 1\}^n \to \{0, 1\}$

1. Pick random $k \times n$ matrix X over $\{0, 1\}$

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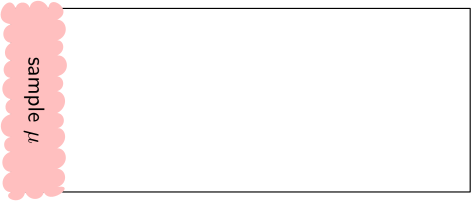

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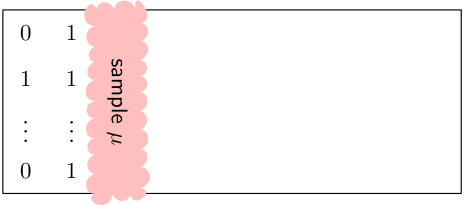

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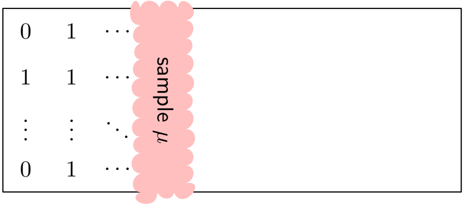

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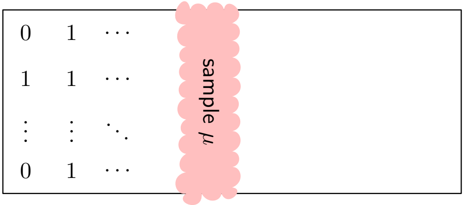

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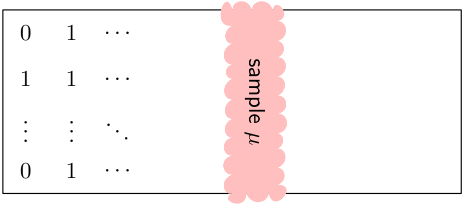

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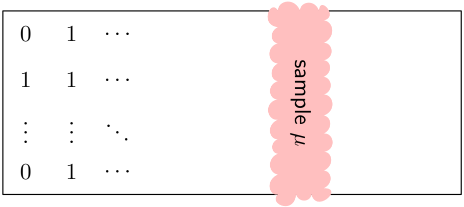

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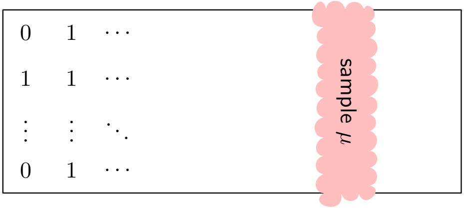

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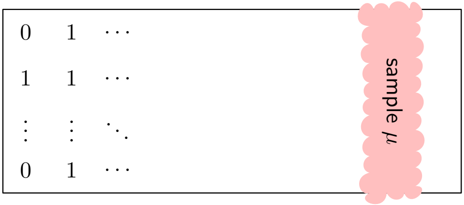

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Given $f : \{0,1\}^n \to \{0,1\}$

1. Pick random $k \times n$ matrix X over $\{0,1\}$, each column sampled from μ, independently

$$\begin{array}{|cccccc}
\hline
0 & 1 & \cdots & & \cdots \\
1 & 1 & \cdots & & \cdots \\
\vdots & \vdots & \ddots & & \ddots \\
0 & 1 & \cdots & & \cdots \\
\hline
\end{array}$$

sample μ

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$$\boxed{\begin{array}{ccccc} 0 & 1 & \cdots & \cdots & 0 \\ 1 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 1 & \cdots & \cdots & 0 \end{array}}$$

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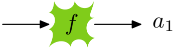

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\longrightarrow f \longrightarrow a_2
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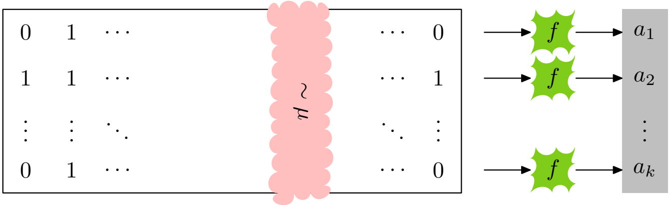
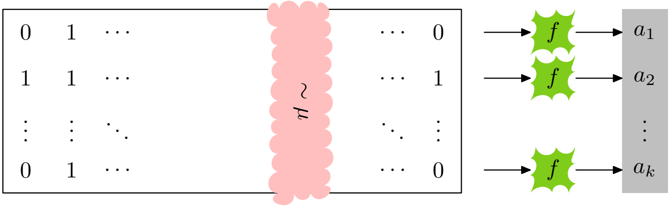
3. Output $(a_1, \ldots, a_k)$



A dictatorship test

Analysis: study the distribution η of $(a_1, \ldots, a_k)$



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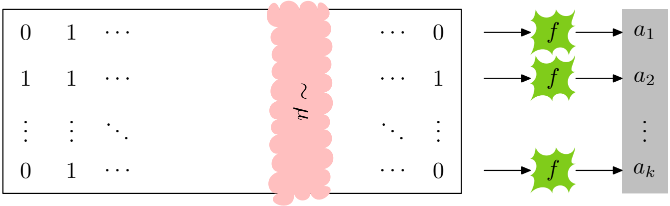

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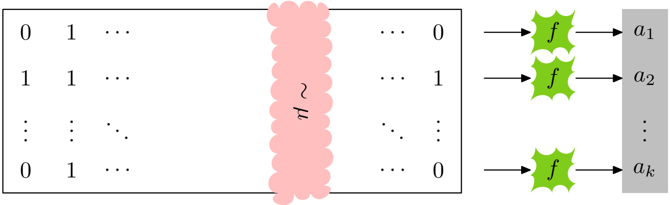
Soundness If f is far from dictator, then $\eta \approx$ uniform



Intuition behind soundness

- If f is far from dictator, can apply *invariance principle* [MOO05, Mos07]

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- Distribution of $(f(X_1), \ldots, f(X_k))$ does not change if μ *replaced by distribution* μ' with *same first and second moments*
- In particular can use μ' = *uniform*

A Converse

Theorem

If P does not support a pairwise independent distribution there is a Q for which P is useful.

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If P does not support a pairwise independent distribution there is a Q for which P is useful.

Gives complete characterization, assuming UGC.

Proof Sketch

Fact

If P does not support pairwise independence then there is a quadratic function $Q : \{0,1\}^k \to [0,1]$ such that $Q(x) > E_Q$ for all $x \in P^{-1}(1)$.

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Q is *quadratic* so we can use *standard SDP techniques* to find assignment with value $E_Q + \epsilon'$

Without negations

What if we don't have negations?

E.g., MAX CUT, MAX 4-LIN$^+$

$$
\begin{array}{llll}
(x_1 \oplus x_2 \oplus x_3 \oplus x_4) & \wedge & (x_1 \oplus x_2 \oplus x_3 \oplus x_7) & \wedge \\
(x_1 \oplus x_2 \oplus x_6 \oplus x_7) & \wedge & (x_1 \oplus x_3 \oplus x_4 \oplus x_6) & \wedge \\
(x_1 \oplus x_3 \oplus x_5 \oplus x_7) & \wedge & (x_1 \oplus x_3 \oplus x_6 \oplus x_7) & \wedge \\
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\end{array}
$$

$$
\begin{array}{lcl}
x_1 & = & \text{FALSE} \\
x_2 & = & \text{FALSE} \\
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x_5 & = & \text{FALSE} \\
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A distribution μ over $\{0,1\}^k$ is *pairwise symmetric* if there is b, ρ such that $\mathbb{E}_\mu[x_i] = b$ for all i and $\mathbb{E}_\mu[x_i x_j] = \rho$ for all $i \neq j$.

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E.g., (possibly biased) pairwise independent distribution

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Remark: *checkable in time $2^{O(k)}$* by convex optimization.

Algorithm

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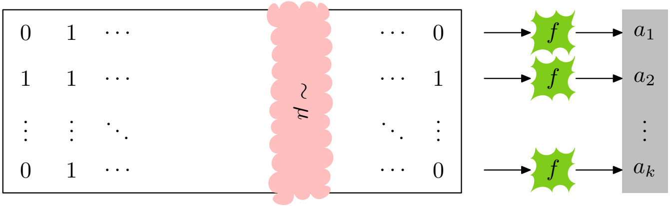
Caveat: need to be careful about Q where E_Q^+ attained by $p = 0$ or $p = 1$.

Hardness Without Negations



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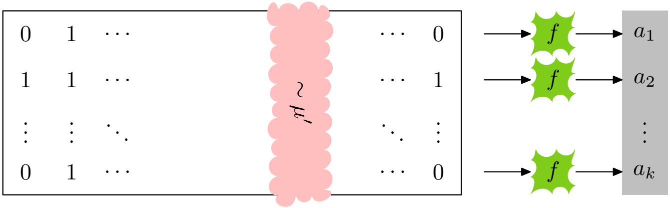

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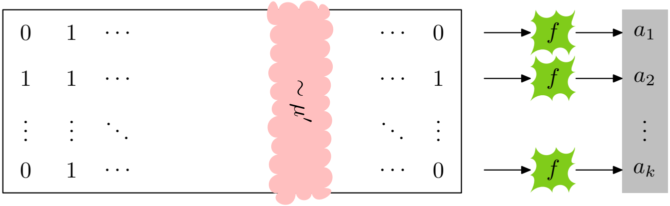

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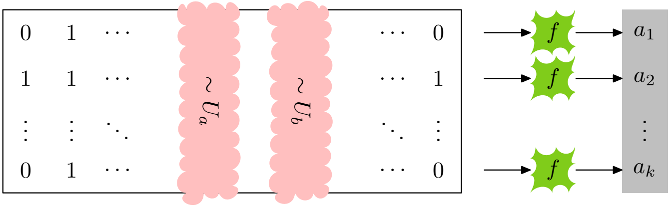
- Change each column to either U_a or U_b without losing value.



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3. Some compromise?

Some Final Comments

New natural relaxation of Max-CSPs

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Would be interesting to consider other settings

- Satisfiability
- Robust satisfiability

Thank you!