Near Unanimity and Total Symmetry for Reflexive Digraphs

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Valeriote's conjecture

Conjecture

For structures of finite signature:
Gumm operations $\Rightarrow$ *edge operation*

Reduction

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The conjecture is open even for digraphs.

The case of reflexive digraphs

Theorem (Maroti, Zadori)

For reflexive digraphs:
Gumm operations $\Rightarrow$ *near unanimity operation*

A consequence

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Theorem (Maroti, Zadori)

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Theorem (equivalent formulation)

Over retraction problems for reflexive digraphs:
bounded strict width $\Rightarrow$ *width 1*

The sketch of the proof of the theorem

Special operations

We call an n-ary operation f a *near unanimity operation*, if $n \geq 3$ and f satisfies the identities

$$f(y, x, \dots, x) = f(x, y, x, \dots) = \cdots = f(x, \dots, x, y) = x$$

in two variables x and y.

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f is a *cyclic operation* if it satisfies the identity

$$f(x_1, x_2, \dots, x_n) = f(x_2, \dots, x_n, x_1).$$

f is a *totally symmetric operation* if

$$\{x_1, x_2, \dots, x_n\} = \{y_1, y_2, \dots, y_n\} \Rightarrow f(x_1, x_2, \dots, x_n) = f(y_1, y_2, \dots, y_n).$$

Dismantlability

A *homomorphism* between two digraphs is an edge preserving map. A *polymorphism* is a homomorphism from G^n to G, and when $n = 1$ we call it an *endomorphism*. For $f, g : H \to G$ we write $f \to g$ if whenever $a \to b$ in H, then $f(a) \to f(b)$ in G.

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Digraph G is *dismantlable*, if there is a sequence G_i, $i = 0, \ldots, n$, of digraphs such that $G_0 = G$, G_n is a singleton, and for each $i = 1, 2, \ldots, n$, G_i is the image of G_{i-1} under some one point elementary retraction of G_{i-1}.

Near unanimity implies dismantlability for certain digraphs

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Theorem (Larose, Loten, Zadori)

For connected symmetric *reflexive digraphs:*
near unanimity $\Rightarrow$ *dismantlability*

Fixed clique and fixed point properties

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A digraph has the *fixed point property*, if every endomorphism has a fixed vertex.

Dismantlability implies fixed points or fixed cliques

Theorem (Duffus, Rival)

For acyclic reflexive digraphs:
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Theorem (Duffus, Rival)

For acyclic reflexive digraphs:
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Theorem (Bandelt)

For symmetric reflexive digraphs:
dismantlability $\Rightarrow$ *fixed clique property*

Near unanimity implies fixed cliques for reflexive digraphs

Corollary

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Proof of the third corollary

Suppose G is connected and admits an nu operation and $f \in \mathrm{End}G$. Let θ be the *strong connectivity* equivalence of G. Then θ is a congruence and f induces a map $f_\theta \in \mathrm{End}(G/\theta)$.

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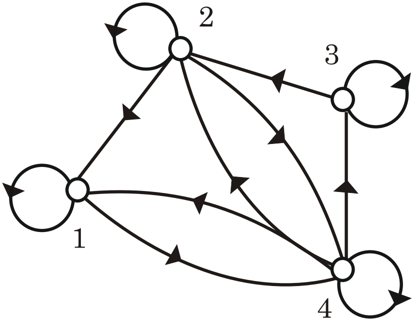
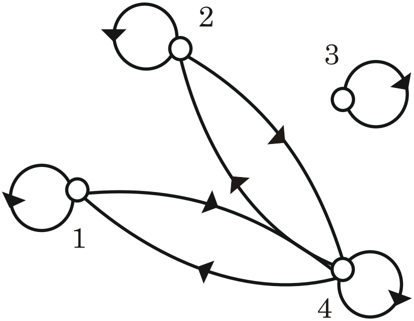
G/θ is connected, acyclic and admits an nu operation, so, by the 1st corollary, f_θ fixes a strong component $B \in G/\theta$. So $f(B) \subseteq B$.

Example: a digraph and its symmetric skeleton





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As B admits an nu, its *symmetric skeleton* is connected. The symmetric skeleton is preserved by $f|_B$ and, by the 2nd corollary, has a fixed clique.

NU implies cyclic idempotent

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Proof of that NU implies cyclic idempotent

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Define a digraph:
vertex set: $G = \{g : H^2 \to H | \ g$ is an idempotent polymorphism$\}$
edge relation: $f \to g$

Fact: G is a connected reflexive digraph that admits a near unanimity operation.

Proof continued

Then $\alpha : G \to G$, $f(x_1, x_2) \mapsto f(x_2, x_1)$ is a endomorphism of G.
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Let $f(x_1, x_2) \in C$. Then $f(x_2, x_1) \in C$. Since C is a clique $f(x_1, x_2) \leftrightarrow f(x_2, x_1)$.

Proof continued

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Let then $g : H^2 \to H$ be defined by $f(x_1, x_2)$ on $\{(x_1, x_2), (x_2, x_1)\}$. Clearly, g is commutative and idempotent. Because of $f(x_1, x_2) \leftrightarrow f(x_2, x_1)$, g is a polymorphism of H.

G-colored digraphs

A pair (H, f) where H is a finite digraph and f is a partial map from H to G is called a *G-colored digraph*. The elements in the domain of f are called *colored elements*

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Map g is a *homomorphism* from (H, f) to (H', f'), if g is a homomorphism from H to H' and $f = f'g$.

G-obstructions

A G-colored digraph *contains* an other, if its vertex set, edge set and partial coloring contains the vertex set, edge set and partial coloring of the other, respectively.

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A G-colored digraph (H, f) is called a *G-obstruction*,
if (H, f) is non-extendible, but
any (H', f') properly contained in (H, f) is extendible.

Characterizations of NU and TSI via obstructions

Theorem

Let G be any finite digraph. TFAE:

1. *G admits a near unanimity operation.*
2. *The numbers of colored elements of G-obstructions have a finite upper bound.*

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Theorem (Feder and Vardi, Dalmau and Pearson)

Let G be a finite digraph. TFAE:

1. *G admits totally symmetric idempotent operations of all arities.*
2. *Every G-obstruction is a homomorphic image of a tree obstruction.*

NU implies TSI for all arities

Theorem (Maroti, Zadori)

Every finite reflexive digraph that admits a near unanimity operation, admits totally symmetric idempotent operations of all arities.