Min CSP on Four Elements: Moving Beyond Submodularity

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Outline

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MAX CSP

Definition (MAX CSP(Γ))

Let Γ be a finite constraint language over a finite domain D.

Instance: A CSP(Γ)-instance I.

Goal: Find an assignment to the variables in I which maximises the number of satisfied constraints.

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Some (NP-hard) examples:

- Max cut ($\Gamma = \{\mathrm{XOR}\}$)
- Max k-cut ($\Gamma = \{\neq_k\}$)
- Max k-SAT ($\Gamma = \{\mathrm{OR}_{i,j} \mid 0 \leq j \leq i \leq k\}$)

where $\mathrm{XOR} := \{(0,1),(1,0)\}$, $\neq_k := \{(a,b) \in D^2 \mid a \neq b\}$ $(k = |D|)$, and $\mathrm{OR}_{i,j} := \{(x_1,\ldots,x_i) \in \{0,1\}^i \mid \neg x_1 \vee \cdots \vee \neg x_j \vee x_{j+1} \vee \cdots \vee x_i\}$.

PO versus NP-hard

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- We address the problem of tracing the boundary between problems inside and outside of PO.
- We find a dichotomy for constraint languages with domain size 4 between problems in PO and NP-hard problems.
- We identify a new type of tractable problems which has not previously shown up in classifications of Max CSP.

Max CSP using $\{0, 1\}$-functions

We represent a k-ary relation $R \in \Gamma$ by its characteristic function $f : D^k \to \{0, 1\}$, with $f(\mathbf{x}) = 1$ iff $\mathbf{x} \in R$.

Definition ((Weighted) Max CSP(Γ))

Let Γ be a set of $\{0, 1\}$-valued functions over D.

Instance: A formal sum $\sum_{i=1}^{n} w_i f_i(\mathbf{x_i})$, where $w_i \in \mathbb{Q}_{\geq 0}$, f_i is a k_i-ary cost function in Γ, and $\mathbf{x_i} \in V^{k_i}$.

Solution: A function $\sigma : V \to D$.

Measure: $\sum_{i=1}^{n} w_i f_i(\sigma(\mathbf{x_i}))$, where σ is applied componentwise.

MIN CSP

In order to study MAX CSP in the VCSP-framework, we choose to work with MIN CSP instead. The reason for this is to keep certain terminology consistent (multimorphisms, submodularity, etc.).

Observation

Let $\Gamma^c = \{1 - f \mid f \in \Gamma\}$. *The problems* MAX CSP$(\Gamma)$ *and* MIN CSP(Γ^c) *are polynomial-time equivalent.*

Note that the two problems may differ with respect to approximability.

Known classification results

Full classifications of $\textsc{Min CSP}(\Gamma)$ exist for the following cases:

- 2-element domains; Creignou (1995)
- 3-element domains; Jonsson, Klasson, and Krokhin (2006)
- Γ containing a single function; Jonsson and Krokhin (2007)
- Γ containing all unary functions; Deineko, Jonsson, Klasson, and Krokhin (2008)

In each of these cases, provided that Γ is a core, $\textsc{Min CSP}(\Gamma)$ is tractable if and only if Γ **is submodular with respect to some chain on** D.

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$\textsc{Min CSP}(\Gamma)$ is also tractable when

- Γ is submodular with respect to any distributive lattice.
- Γ is submodular with respect to certain non-distributive lattices; Krokhin and Larose (2007), Kuivinen (2009)

Beyond submodularity

Given the previously known evidence, a tentative conjecture has been that, for a core Γ,

- $\textsc{Min}\ \mathrm{CSP}(\Gamma)$ is tractable when Γ is submodular with respect to some lattice, and
- this is the only source of tractability for $\textsc{Min}\ \mathrm{CSP}(\Gamma)$.

Beyond submodularity

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- this is the only source of tractability for $\textsc{Min CSP}(\Gamma)$.

We show that the second part is false.

Counterexample

Let $D = \{a, b, c, d\}$, and let $\Gamma = \{\{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}, R\}$, where $R = \{(x, y) \mid x = b \vee y = c\}$. Then Γ is a core which is not submodular with respect to any lattice on D, yet $\textsc{Min CSP}(\Gamma)$ is tractable.

VCSP without mixed cost functions

It will be useful to allow "crisp" constraints in some of the reductions. There will be no "mixed" constraints, i.e., cost functions taking both a non-zero finite and an infinite value.

Definition

Let Γ be a set of finite-valued cost functions on a domain D, and Δ be a set of relations on D. $\mathrm{VCSP}(\Gamma, \Delta)$ is the following minimisation problem:

Instance: A formal sum $\sum_{i=1}^{n} w_i f_i(\mathbf{x_i})$, and a finite set of constraint applications $\{(\mathbf{y_j}; R_j)\}$, where $f_i \in \Gamma$, $R_j \in \Delta$, and $\mathbf{x_i}, \mathbf{y_j}$ are matching lists of variables from V.

Solution: A function $\sigma : V \to D$ such that $\sigma(\mathbf{y_j}) \in R_j$ for all j.

Measure: $\sum_{i=1}^{n} w_i f_i(\sigma(\mathbf{x_i}))$.

We write $\mathrm{VCSP}(\Gamma)$ when Δ is empty and $\textsc{Min}\ \mathrm{CSP}(\Gamma, \Delta)$ when Γ consists of $\{0, 1\}$-valued constraints only.

Outline

Multimorphisms

Cohen, Cooper, Jeavons, and Krokhin (2006) introduced the first in a number of generalisations of polymorphisms for the purpose of investigating the VCSP.

Definition

A pair of functions $f, g : D^2 \to D$ is a (binary) **multimorphism** of a k-ary cost function $h : D^k \to \mathbb{Q}_{\geq 0}$ if, for all tuples $\mathbf{x}, \mathbf{y} \in D^k$,

$$h(\mathbf{x}) + h(\mathbf{y}) \geq h(f(\mathbf{x}, \mathbf{y})) + h(g(\mathbf{x}, \mathbf{y})),$$

where f and g are applied component-wise. It is a multimorphism of (Γ, Δ) if it is a multimorphism of each function in $\Gamma \cup \Delta$.

Multimorphisms; submodular cost functions

Example

A cost function $h : D^k \to \mathbb{Q}$ is **submodular** with respect to a lattice $L = (D; \wedge, \vee)$ if

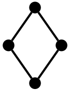
$$f(\mathbf{a}) + f(\mathbf{b}) \geq f(\mathbf{a} \wedge \mathbf{b}) + f(\mathbf{a} \vee \mathbf{b}),$$

for all $\mathbf{a}, \mathbf{b} \in D^k$.

In this case, $(\wedge, \vee)$ is a multimorphism of h.

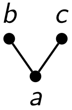
Theorem (Schrijver (2000); Iwata, Fleischer, and Fujishige (2001))

If Γ is submodular w.r.t. a distributive lattice, then $\mathrm{VCSP}(\Gamma)$ is tractable.

Examples include chains (total orders) and products of chains, e.g., 

Bisubmodular cost functions

Let $D = \{a, b, c\}$, and let $\sqcap, \sqcup : D \to D$ be the $\wedge$ and $\vee$ of the poset



on pairs for which they are defined, and $\sqcup(b, c) = \sqcup(c, b) = a$.

$\sqcap$	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c

$\sqcup$	a	b	c
a	a	b	c
b	b	b	a
c	c	a	c

Theorem (Fujishige and Iwata (2006))

If Γ has the multimorphism $(\sqcap, \sqcup)$, then $\mathrm{VCSP}(\Gamma)$ is tractable.

Congruences

Definition

Let $(D; f, g)$ be an algebra with two binary operations, f and g.
A **congruence** of $(D; f, g)$ is an equivalence relation θ such that

$$(x_1, x_2), (y_1, y_2) \in \theta \implies (f(x_1, y_1), f(x_2, y_2)), (g(x_1, y_1), g(x_2, y_2)) \in \theta,$$

for all $x_1, x_2, y_1, y_2 \in D$.

- D/θ denotes the set of equivalence classes in θ.
- $x[\theta] \in D/\theta$ denotes the equivalence class of $x \in D$.
- The following are well-defined operations on D/θ:

$$f/\theta(x[\theta], y[\theta]) := f(x, y)[\theta] \qquad g/\theta(x[\theta], y[\theta]) := g(x, y)[\theta].$$

Mal'tsev products

Definition

Let $\mathbf{V}$ and $\mathbf{W}$ be classes of algebras $(D'; f', g')$, with f' and g' binary operations. The **Mal'tsev product**, $\mathbf{V} \circ \mathbf{W}$, consists of all algebras $(D; f, g)$ such that there is a congruence θ where every class supports a subalgebra, and

1. $(x[\theta], f|_{x[\theta]}, g|_{x[\theta]}) \in \mathbf{V}$ for each $x[\theta] \in D/\theta$; and
2. $(D/\theta, f/\theta, g/\theta) \in \mathbf{W}$.

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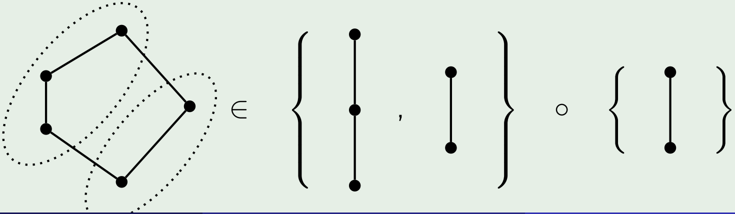
Example (The pentagon lattice)



MFM

Definition (Multimorphism Function Minimisation)

Let X be a finite set of triples $(D_i; f_i, g_i)$, where D_i is a finite set and $f_i, g_i : D_i^2 \to D_i$. $\mathrm{MFM}(X)$ is the minimisation problem:

Instance: A positive integer n, a function $j : \{1, \ldots, n\} \to \{1, \ldots, |X|\}$, and a function $h : \mathcal{D} \to \mathbb{Z}$, where $\mathcal{D} = \prod_{i=1}^{n} D_{j(i)}$. For all $\mathbf{x}, \mathbf{y} \in \mathcal{D}$,

$$h(\mathbf{x}) + h(\mathbf{y}) \geq h(f_{j(1)}(x_1, y_1), f_{j(2)}(x_2, y_2), \ldots, f_{j(n)}(x_n, y_n)) + h(g_{j(1)}(x_1, y_1), g_{j(2)}(x_2, y_2), \ldots, g_{j(n)}(x_n, y_n)).$$

The function h is assumed be supplied as an oracle.

Solution: A tuple $\mathbf{x} \in \mathcal{D}$.

Measure: The value of $h(\mathbf{x})$.

MFM and Mal'tsev products

- MFM(X) is said to be **oracle-tractable** if it can be solved in polynomial time in the number of variables of the input function h.
- If Γ has the multimorphism (f, g), and MFM$(\{(D; f, g)\})$ is oracle-tractable, then VCSP(Γ) is tractable.

Theorem (cf. Krokhin and Larose (2007), Theorem 4.3)

Suppose that $\mathbf{V}$ *and* $\mathbf{W}$ *are finite sets of triples* $(D; f, g)$ *such that MFM(*$\mathbf{V}$*) and MFM(*$\mathbf{W}$*) are oracle-tractable. Then MFM(*$\mathbf{V} \circ \mathbf{W}$*) is also oracle-tractable.*

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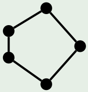
Example

MFM({  ,  }) and MFM({  }) are oracle-tractable, so

MFM({  }) is also oracle-tractable.

1-defect chains

Definition

Let $(D; <)$ be a chain, and let $b, c \in D$ be two distinct elements. Assume that $f, g : D^2 \to D$ are two commutative operations such that:

- $\{x, y\} \neq \{b, c\} \implies f(x, y) = \min_<(x, y), g(x, y) = \max_<(x, y)$;
- $\{f(b, c), g(b, c)\} \cap \{b, c\} = \emptyset$; and
- $f(b, c) < g(b, c)$.

We call $(D; f, g)$ a **1-defect chain**, and we say that (f, g) is a **1-defect chain multimorphism**.

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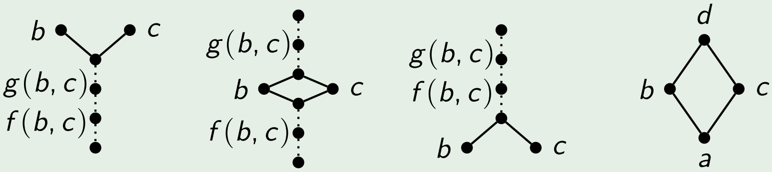
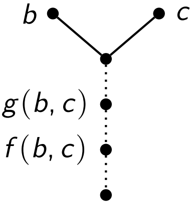


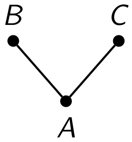
Tractability for 1-defect chain multimorphisms

Proposition

If Γ has a 1-defect chain multimorphism, then VCSP(Γ) is tractable.



$\Longrightarrow$



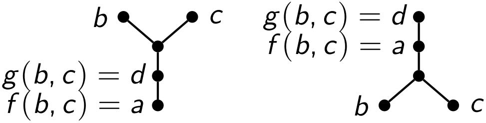
$= f/\theta(B, C) = g/\theta(B, C)$

Let $D/\theta = \{A, B, C\}$ with $A = D \setminus \{b, c\}$, $B = \{b\}$, $C = \{c\}$.

- θ is a congruence;
- $(A; f|_A, g|_A)$, $(B; f|_B, g|_B)$, and $(C; f|_C, g|_C)$ are chains; and
- $(D/\theta; f/\theta, g/\theta)$ is isomorphic to $(\{a, b, c\}; \sqcap, \sqcup)$.

Example

- $D = \{a, b, c, d\}$
- $$\Gamma = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$
- Γ is a core which is not submodular with respect to any lattice on D, but it has the following 1-defect chain multimorphisms.



Outline

Weighted pp-definitions

Let I be an instance of $\mathrm{VCSP}(\Gamma, \Delta)$ with variables $(v_1, \ldots, v_n)$.

$$\mathrm{Optsol}(I) := \{(\sigma(v_1), \ldots, \sigma(v_n)) \mid \sigma \text{ is an optimal solution to } I\}$$

A relation has a **weighted pp-definition** over (Γ, Δ) if it is equal to $\pi_{\mathbf{x}}\mathrm{Optsol}(I)$ for some I and sequence $\mathbf{x}$ of variables from I.

Let $\langle \Gamma, \Delta \rangle_w$ denote the set of all such relations.

Proposition

Let $\Delta' \subseteq \langle \Gamma, \Delta \rangle_w$ be a finite subset. Then $\mathrm{VCSP}(\Gamma, \Delta')$ is polynomial-time reducible to $\mathrm{VCSP}(\Gamma, \Delta)$.

Endomorphisms and cores for $\textsc{Min CSP}$

Let Γ be a finite set of $\{0,1\}$-valued cost functions.

- An operation $f : D \to D$ is an **endomorphism** of Γ if

$$h(\mathbf{a}) = 0 \implies h(f(\mathbf{a})) = 0.$$

- An **automorphism** is a surjective endomorphism.
- The set of endomorphisms (automorphisms) of Γ is denoted by $\mathrm{End}(\Gamma)$ $(\mathrm{Aut}(\Gamma))$.
- Γ is called a **core** if all of its endomorphisms are automorphisms.

Lemma

Let $f \in \mathrm{End}(\Gamma)$, $D' = f(D)$, and $\Gamma' = \{h|_{D'} \mid h \in \Gamma\}$. Then $\textsc{Min CSP}(\Gamma)$ and $\textsc{Min CSP}(\Gamma')$ are polynomial-time equivalent.

So we may assume that Γ is a core.

Cores and constants

Let $\mathcal{C}_D = \{\{a\} \mid a \in D\}$.

Proposition

Let Γ be a core over D. Then $\textsc{Min CSP}(\Gamma, \mathcal{C}_D)$ is polynomial-time reducible to $\textsc{Min CSP}(\Gamma)$.

The endomorphisms of Γ are encoded as the optimal solutions to an instance I of $\textsc{Min CSP}(\Gamma)$ with

- variables $X_D = \{x_a | a \in D\}$, and
- sum $\sum_{f_i \in \Gamma} \sum_{\mathbf{a} \in f_i^{-1}(0)} f_i(\mathbf{x_a})$, where $\mathbf{x_a} = (x_{a_1}, \ldots x_{a_k}) \in X_D^k$.

When Γ is a core, and $\mathbf{d} = (d_1, \ldots, d_n)$ is an enumeration of D, we have

$$\{(f(d_1), \ldots, f(d_n)) \mid f \in \mathrm{Aut}(\Gamma)\} = \pi_{\mathbf{x_d}} \mathrm{Optsol}(I) \in \langle \Gamma \rangle_w .$$

Add a copy of I to the instance of $\textsc{Min CSP}(\Gamma, \mathcal{C}_D)$, identify variables, solve, and apply an inverse automorphism. □

Outline

Expressive power

We say that a function $h : D^k \to \mathbb{Q}_{\geq 0}$ is **expressible** over (Γ, Δ) if there is an instance I of $\mathrm{VCSP}(\Gamma, \Delta)$ and a sequence of variables $(v_1, \dots, v_k)$ such that

$$h(a_1, \dots, a_k) = \mathrm{Opt}(I \cup \{v_i = a_i\})$$

for all $(a_1, \dots, a_k) \in D^k$, and h is a total function.

Let $\langle \Gamma, \Delta \rangle_{fn}$ denote the set of all such functions.

Proposition

Let $\Gamma' \subseteq \langle \Gamma, \Delta \rangle_{fn}$ be a finite subset. Then $\mathrm{VCSP}(\Gamma', \Delta)$ is polynomial-time reducible to $\mathrm{VCSP}(\Gamma, \Delta)$.

Proposition

If (f, g) is a binary multimorphism of (Γ, Δ), then it is also a multimorphism of $\langle \Gamma, \Delta \rangle_{fn}$.

Graph of partial multimorphisms

We now proceed as follows.

- We first define a graph G which encodes the binary multimorphisms of the binary functions in $\langle \Gamma, \mathcal{C}_D \rangle_{fn}$ in certain independent sets. (This is a slight extension of a construction by Kolmogorov and Živný (2010) for conservative finite-valued VCSP.)

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- Otherwise we can argue, using the graph G, that the binary functions in $\langle\Gamma, \mathcal{C}_D\rangle_{fn}$ have one of a small number of binary mulitmorphisms which are known to imply tractability.
- Finally, we need to show that these multimorphisms are in fact multimorphisms of all of Γ.

Graph definition

Definition

Let $G = (V, E)$ be the following undirected graph. V is set of pairs of partial functions $f, g : D^2 \to D$ such that

- f and g are defined on a subset $\{a, b\} \subseteq D$;
- f and g are idempotent and commutative; and
- $\{f(a, b), g(a, b)\} = \{a, b\}$ or $\{f(a, b), g(a, b)\} \cap \{a, b\} = \emptyset$.

$\{(f_1, g_1), (f_2, g_2)\} \in E(G)$ if there is a binary function $h \in \langle \Gamma, \mathcal{C}_D \rangle_{fn}$ s.t.

$$\min\{h(a_1, a_2) + h(b_1, b_2), h(a_1, b_2) + h(b_1, a_2)\} < h(f_1(a_1, b_1), f_2(a_2, b_2)) + h(g_1(a_1, b_1), g_2(a_2, b_2))$$

Note that a and b are allowed to be equal.

Encoding of multimorphisms

For $i = 1, 2$, let $(f_i, g_i) \in V$ be defined on $\{a_i, b_i\}$. Then the edge $\{(f_1, g_1), (f_2, g_2)\} \in E$ means that any pair of operations $(f, g) : D^2 \to D^2$ such that $(f|_{\{a_i,b_i\}}, g|_{\{a_i,b_i\}}) = (f_i, g_i)$ fails to be a multimorphism of $\langle \Gamma, \mathcal{C}_D \rangle_{fn}$.

Lemma

Let $\{(f_{\{a,b\}}, g_{\{a,b\}})\} \subseteq V$ be an independent set in G containing precisely one vertex for each subset $\{a, b\} \subseteq D$. Then, every binary function in $\langle \Gamma, \mathcal{C}_D \rangle_{fn}$ has the multimorphism (f, g), where f and g are given by

$$f(x, y) = f_{\{x,y\}}(x, y), \text{ and}$$
$$g(x, y) = g_{\{x,y\}}(x, y).$$

NP-hardness

Let $\overrightarrow{xy}$ denote the vertex (f, g) for which $f(x, y) = x$ and $g(x, y) = y$. We say that $\overrightarrow{xy}$ is **conservative**.

Proposition

Let Γ be a core over D. If $\{x, y\} \in \langle \Gamma, \mathcal{C}_D \rangle_w$ and the vertex $\overrightarrow{xy}$ has a self-loop in the graph G, for some $x, y \in D$, then $\textsc{Min}\ \mathrm{CSP}(\Gamma)$ is NP-hard.

Reduce from $\textsc{Max cut}$, via $\textsc{Min}\ \mathrm{CSP}(\Gamma, \mathcal{C}_D)$, to $\textsc{Min}\ \mathrm{CSP}(\Gamma)$. $\square$

If we assume that this proposition does not apply, but $\{x, y\} \in \langle \Gamma, \mathcal{C}_D \rangle_w$, then we can exclude a self-loop on $\overrightarrow{xy}$ in G.

So which 2-element subsets $\{x, y\}$ are in $\langle \Gamma, \mathcal{C}_D \rangle_w$?

Expressible 2-element subsets

Let $e_{ab} : D \to D$ denote the operation $e_{ab}(x) = \begin{cases} b & \text{if } x = a \\ x & \text{otherwise.} \end{cases}$

Lemma

If $e_{ab} \notin \mathrm{End}(\Gamma)$, then $\langle \Gamma, \mathcal{C}_D \rangle_{fn}$ contains a unary $\{0,1\}$-valued function u such that $u(a) = 0$ and $u(b) = 1$.

- Let $h : D^k \to \{0,1\} \in \Gamma$ and $a_1, \dots, a_k \in D$ be such that $h(a_1, \dots, a_k) = 0$, but $h(e_{ab}(a_1), \dots, e_{ab}(a_k)) = 1$.
- $a_i = a$ for at least one i; replace these by a variable x.
- $u(x) = h(a_1, \dots, x, \dots, a_k)$ expresses the desired function. $\square$

Thus, when Γ is a core, we have access to a number of unary $\{0,1\}$-valued functions. We can use these functions to determine the 2-element subsets of $\langle \Gamma, \mathcal{C}_D \rangle_w$ when $D = \{a, b, c, d\}$.

Expressible 2-element subsets of $\{a, b, c, d\}$

Let Γ be a core and assume that $\{b, c\} \notin \langle \Gamma, \mathcal{C}_D \rangle_w$.

e_{ba}	e_{ca}	e_{bd}	e_{cd}
(1,0,0,0)	(1,0,0,0)	(0,0,0,1)	(0,0,0,1)
(1,0,0,1)	(1,0,0,1)	(0,0,1,1)	(0,1,0,1)
(1,0,1,0)	(1,1,0,0)	(1,0,0,1)	(1,0,0,1)
(1,0,1,1)	(1,1,0,1)	(1,0,1,1)	(1,1,0,1)

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~~(1,0,0,1)~~	~~(1,0,0,1)~~	(0,0,1,1)	(0,1,0,1)
(1,0,1,0)	(1,1,0,0)	~~(1,0,0,1)~~	~~(1,0,0,1)~~
(1,0,1,1)	(1,1,0,1)	(1,0,1,1)	(1,1,0,1)

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(1,0,1,0)	(1,1,0,0)	~~(1,0,0,1)~~	~~(1,0,0,1)~~
(1,0,1,1)	(1,1,0,1)	(1,0,1,1)	(1,1,0,1)

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(1,0,1,0)	(1,1,0,0)	~~(1,0,0,1)~~	~~(1,0,0,1)~~
(1,0,1,1)	(1,1,0,1)	(1,0,1,1)	(1,1,0,1)

$$(0,0,1,1) + (0,1,0,1) + 2 \cdot (1,0,0,0) = (2,1,1,2)$$
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(1,0,1,0)	(1,1,0,0)	~~(1,0,0,1)~~	~~(1,0,0,1)~~
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~~(1,0,0,1)~~	~~(1,0,0,1)~~	(0,0,1,1)	(0,1,0,1)
(1,0,1,0)	(1,1,0,0)	~~(1,0,0,1)~~	~~(1,0,0,1)~~
(1,0,1,1)	(1,1,0,1)	(1,0,1,1)	(1,1,0,1)

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~~(1,0,0,1)~~	~~(1,0,0,1)~~	(0,0,1,1)	(0,1,0,1)
(1,0,1,0)	(1,1,0,0)	~~(1,0,0,1)~~	~~(1,0,0,1)~~
(1,0,1,1)	~~(1,1,0,1)~~	(1,0,1,1)	~~(1,1,0,1)~~

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$$(1,0,1,1) + (1,1,0,1) = (2,1,1,2)$$
$$2 \cdot (1,0,1,1) + (1,1,0,0) + (0,1,0,1) = (3,2,2,3)$$

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Let Γ be a core and assume that $\{b, c\} \notin \langle \Gamma, \mathcal{C}_D \rangle_w$.

e_{ba}	e_{ca}	e_{bd}	e_{cd}
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~~(1,0,0,1)~~	~~(1,0,0,1)~~	(0,0,1,1)	(0,1,0,1)
(1,0,1,0)	(1,1,0,0)	~~(1,0,0,1)~~	~~(1,0,0,1)~~
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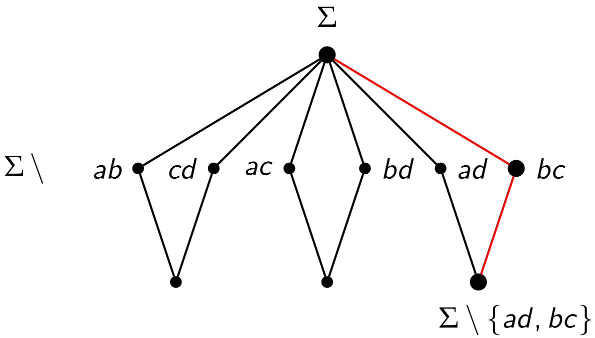
Expressible 2-element subsets of $\{a, b, c, d\}$

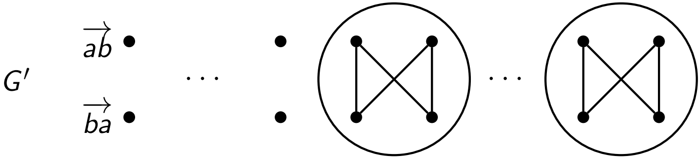
Let Σ be the set of all 2-subsets of $\{a, b, c, d\}$ and let $\Sigma_\Gamma := \Sigma \cap \langle \Gamma, \mathcal{C}_D \rangle_w$.



We will assume that $\Sigma_\Gamma = \Sigma \setminus \{ad, bc\}$, $\Sigma \setminus \{bc\}$, or Σ.

G' bipartite implies submodularity w.r.t. a chain

Let G' be the subgraph of G induced by all conservative vertices $\overrightarrow{xy}$.



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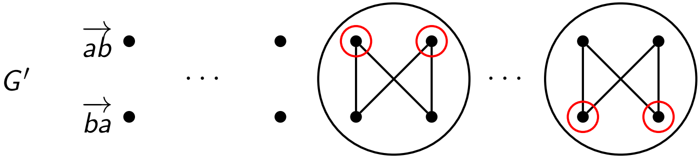

- Let I be the union of one part from each non-trivial component.

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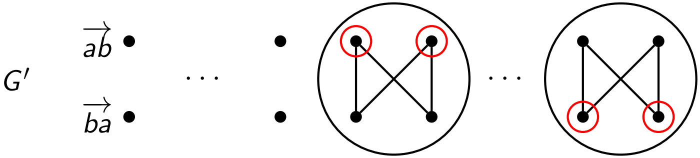

- Let I be the union of one part from each non-trivial component.
- We can show that $\overrightarrow{xy}, \overrightarrow{yz} \in I$ implies $\overrightarrow{xz} \in I$, so I induces a partial order $\prec := \{(x, y) \mid \overrightarrow{xy} \in I\}$.

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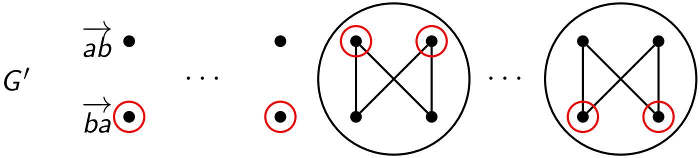

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- Extend $\prec$ to a linear order, and add the corresponding vertices to I.

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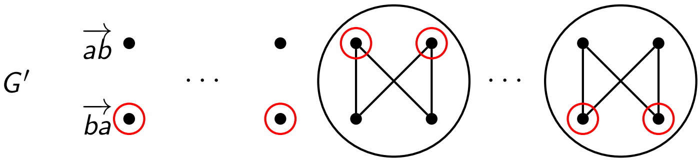

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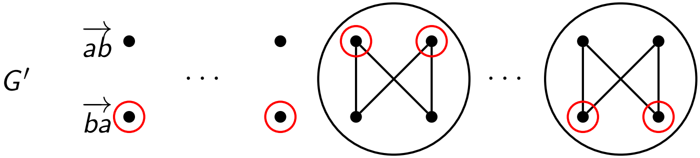

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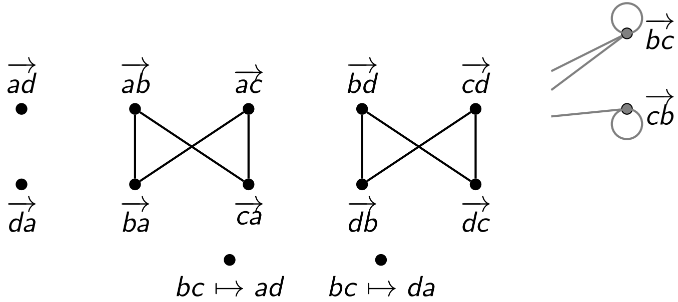
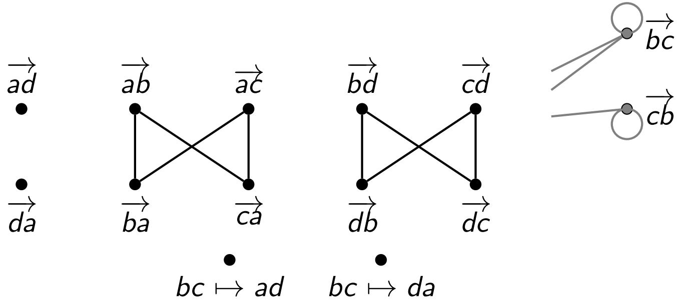
- Extend $\prec$ to a linear order, and add the corresponding vertices to I.
- Add the vertex on $\{x\}$ to I, for each $x \in D$.
- Otherwise $\Sigma_\Gamma \subseteq \Sigma \setminus \{bc\}$.

$\Sigma_\Gamma \subseteq \Sigma \setminus \{bc\}$, 1-defect chains



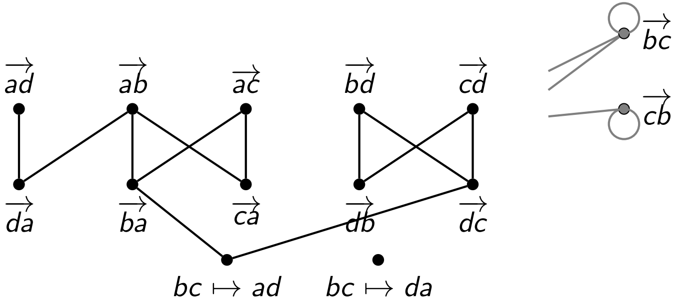
- $G' \setminus \{\overrightarrow{bc}, \overrightarrow{cb}\}$ is bipartite.

$\Sigma_\Gamma \subseteq \Sigma \setminus \{bc\}$, 1-defect chains



- $G' \setminus \{\overrightarrow{bc}, \overrightarrow{cb}\}$ is bipartite.
- The vertices of a connected component again induce a partial order.

$\Sigma_\Gamma \subseteq \Sigma \setminus \{bc\}$, 1-defect chains



- $G' \setminus \{\overrightarrow{bc}, \overrightarrow{cb}\}$ is bipartite.
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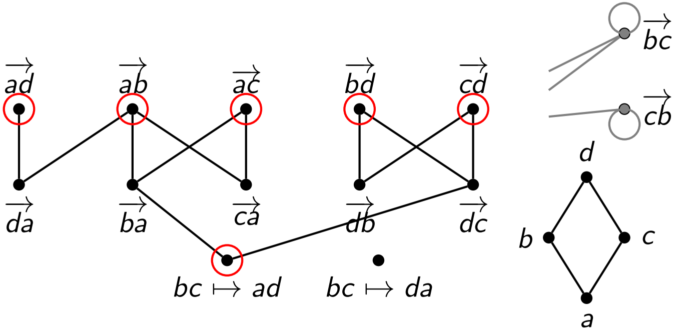

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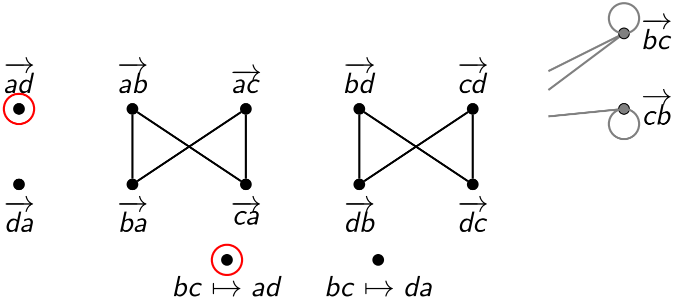

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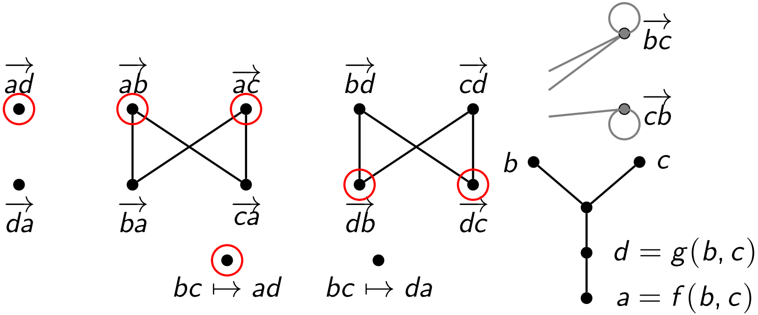

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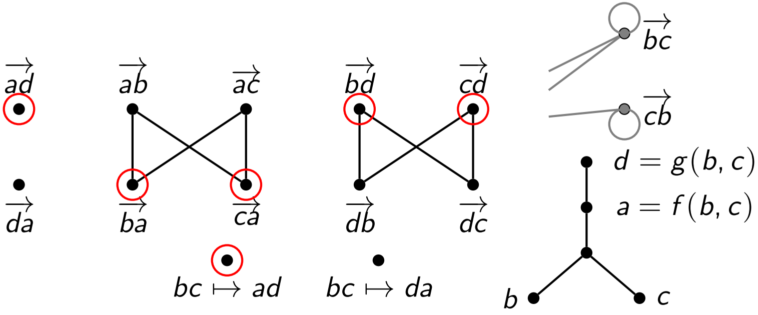

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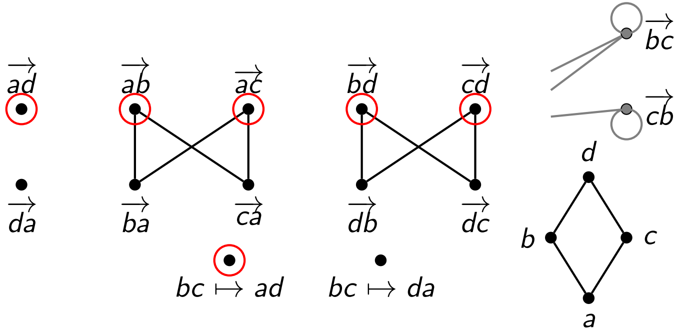

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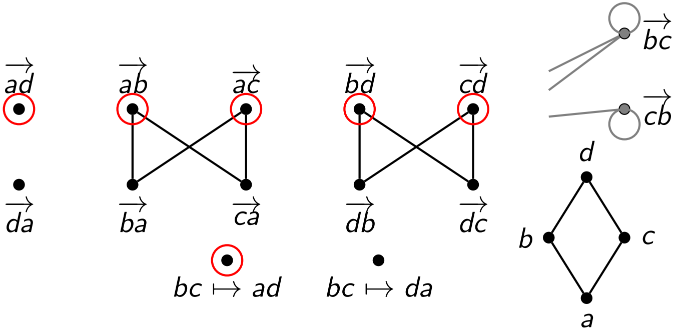

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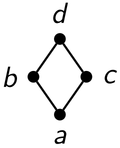
- Add the vertex on $\{x\}$ to I, for each $x \in D$.

Tractable cases, $D = \{a, b, c, d\}$

Assume that for each $x \neq y \in D$, either $\{x, y\} \notin \langle \Gamma, \mathcal{C}_D \rangle_w$ or the vertex $\overrightarrow{xy}$ has no self-loop. Let Γ_{bin} denote the set of at most binary functions in $\langle \Gamma, \mathcal{C}_D \rangle_{fn}$.

Up to a permutation of D we have that

- If G' is bipartite, then Γ_{bin} is submodular w.r.t. a chain.
- Otherwise, either Γ_{bin} has a 1-defect chain multimorphism; or
- Γ_{bin} is submodular w.r.t.



Outline

From binary to arbitrary arity

Lemma (Topkis, 1978)

A function $h : D^k \to \mathbb{Q}_{\geq 0}$ is submodular w.r.t. a chain $(D; \wedge, \vee)$ if and only if every binary function obtained from h by replacing any given $k-2$ arguments by constants is submodular on this chain.

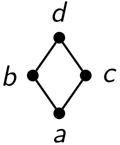
If every binary function in $\langle \Gamma, \mathcal{C}_D \rangle_{fn}$ is submodular w.r.t. a chain, then in particular, every $h \in \Gamma$ fulfils the second part of the lemma, so Γ is submodular w.r.t. this chain.

From binary to arbitrary arity

We can show that the same holds for 1-defect chains.

Lemma

A function $h : D^k \to \mathbb{Q}_{\geq 0}$ has a 1-defect chain multimorphism $\langle f, g \rangle$ if and only if every binary function obtained from h by replacing any given $k-2$ arguments by constants has the multimorphism $\langle f, g \rangle$.

Recall that 

is a 1-defect chain.

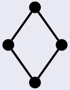
One can also derive the property for this lattice by regarding h as a $2k$-ary function over $\{ac, bd\}$ which is submodular with respect to the chain $ac < bd$.

Classification

To summarise, we have the following.

Theorem

Let Γ be a core with domain $D = \{a, b, c, d\}$.

- *If Γ is submodular w.r.t. a chain on D;*
- *if Γ has a 1-defect chain multimorphism; or*
- *if Γ is submodular w.r.t. a lattice isomorphic to*  ,

then $\textsc{Min CSP}(\Gamma)$ *is tractable.*

Otherwise $\textsc{Min CSP}(\Gamma)$ *is NP-hard.*

Outline

Open problems

Problem

It now seems reasonable to expect that binary idempotent commutative multimorphisms will play a big role in the classification of $\textsc{Min CSP}$.

- *Which of these actually show up?*
  (bisubmodularity does not show up for 3-element $\textsc{Min CSP}$*)*
- *Algorithms!*

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Problem

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- *Which of these actually show up?* *(bisubmodularity does not show up for 3-element* $\textsc{Min CSP}$*)*
- *Algorithms!*

Problem

Find a general class $\mathcal{C}$ *of binary idempotent commutative multimorphisms such that: a* k*-ary cost function* h *has the multimorphism* $(f, g) \in \mathcal{C}$ *iff every binary function obtained from* h *by replacing any given* $k - 2$ *arguments by constants has the multimorphism* (f, g).