Beyond bounded width and few subpowers

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Two powerful algorithms

Constraint Satisfaction Problem

Definition

Template: a finite set $\mathcal{B}$ of similar finite idempotent algebras closed under taking subuniverses.

Definition

Binary instance: a set

$$\mathcal{A} = \{\, \mathbf{B}_i, \mathbf{R}_{ij} \mid i, j \in V \,\}$$

of universes $\mathbf{B}_i \in \mathcal{B}$ and relations $\mathbf{R}_{ij} \leq \mathbf{B}_i \times \mathbf{B}_j$ indexed by variables in V, such that $R_{ii} = \{\, (b, b) \mid b \in B_i \,\}$ and $R_{ij} = R_{ji}^{-1}$.

Definition

Solution: a map $f \in \prod_{i \in V} B_i$ such that $(f(i), f(j)) \in R_{ij}$ for all $i, j \in V$.

Local Consistency Algorithm

Definition

An instance $\mathcal{A} = \{\, \mathbf{B}_i, \mathbf{R}_{ij} \mid i, j \in V \,\}$ is

- $(1,2)$-**consistent**: if $R_{ij} \leq B_i \times B_j$ is subdirect
- $(2,3)$-**consistent**: if $R_{ik} \subseteq R_{ij} \circ R_{jk}$

for all $i, j, k \in V$.

Theorem

Every instance $\mathcal{A}$ can be turned into a $(2,3)$-consistent instance $\mathcal{A}'$ in polynomial time such that they have the same set of solutions.

Proof.

Reduce the instance until it becomes consistent:

$$R'_{ik} = R_{ik} \cap (R_{ij} \circ R_{jk}). \qquad \square$$

Bounded Width Theorem

Theorem (Barto, Kozik)

If $\mathcal{B}$ generates a congruence meet-semidistributive variety, then every nonempty $(2,3)$*-consistent instance has a solution.*

Proof Overview.

- If $|B_i| = 1$ for all $i \in V$, then this is a solution
- If the instance is nonempty and nontrivial, then find a smaller instance with the same consistency property
- We need absorbtion theory to get smaller instance
- Use new consistency: $(1,2) <$ Prague strategy $< (2,3)$ $\square$

Few Subpowers Theorem

Definition

An algebra **B** has **few subpowers**, if there is a polynomial $p(n)$ such that $|\mathrm{S}(\mathbf{P}^n)| \leq 2^{p(n)}$ for all integer n.

Theorem (Berman, Idziak, Marković, McKenzie, Valeriote, Willard)

An algebra **B** *has few subpowers iff it has an* **edge term** *t*

$$\begin{aligned}
t(y,y,x,x,x,\dots,x,x) &\approx x \\
t(x,y,y,x,x,\dots,x,x) &\approx x \\
t(x,x,x,y,x,\dots,x,x) &\approx x \\
\ddots \quad & \quad \vdots \\
t(x,x,x,x,x,\dots,x,y) &\approx x.
\end{aligned}$$

Edge Term Algorithm

Definition

A **compact representation** of a subuniverse $\mathbf{S} \leq \prod_{i \in V} \mathbf{B}_i$ is

- a subset $T \subseteq S$ that generates $\mathbf{S}$ and is small $|T| \leq p(|V|)$,
- every "minority fork" and "small projection" is represented

Theorem (Idziak, Marković, McKenzie, Valeriote, Willard)

If the variety generated by $\mathcal{B}$ has an edge term, then the compact representation of the solution set is computable in polynomial time.

Proof Overview.

- Take the compact representation of $\prod_{i \in V} \mathbf{B}_i$.
- From the compact representation of $\mathbf{S}$ and $\mathbf{R}_{ij} \leq \mathbf{B}_i \times \mathbf{B}_j$ compute the compact representation of

$$\mathbf{S}' = \{\, f \in S \mid (f(i), f(j)) \in R_{ij} \,\}. \qquad \square$$

Maltsev on Top

Maltsev on Top Algorithm

Theorem (Maróti)

Suppose, that each algebra $\mathbf{B} \in \mathcal{B}$ *has a congruence* $\beta \in \mathrm{Con}(\mathbf{B})$ *such that* $\mathbf{B}/\beta$ *has few subpowers and each* β *block has bounded width. Then we can solve the constraint satisfaction problem over* $\mathcal{B}$ *in polynomial time.*

Proof Overview.

- Take an instance $\mathcal{A} = \{\, \mathbf{B}_i, \mathbf{R}_{ij} \mid i, j \in V \,\}$ and $\beta_i \in \mathrm{Con}(\mathbf{B}_i)$
- Consider **extended constraints** that not only limit the projection of the solution set to the $\{i, j\}$ coordinates, but also to $\prod_{v \in V} \mathbf{B}_v/\beta_v$
- Use extended (2,3)-consistency algorithm
- Obtain a solution modulo the β congruences so that the restriction of the problem to the selected congruence blocks is (2,3)-consistent.
- By the bounded width theorem there exists a solution. □

Edge Term Observations

Lemma

Given the compact representations of subproducts $\mathbf{S}$ *and* $\mathbf{P}$ *over* $\mathcal{B}$*, then the compact representations of* $\mathbf{S} \times \mathbf{P}$ *and* $\mathbf{S} \cap \mathbf{P}$ *can be computed in polynomial time.*

Lemma

Given the compact representations of $\mathbf{S}_1, \dots, \mathbf{S}_k$ *and assume that* $\mathbf{S} = \bigcup_{i=1}^{k} \mathbf{S}_i$ *is a subuniverse, then the compact representation of* $\mathbf{S}$ *can be computed in polynomial time.*

Corollary

Given the compact representations of a set $\mathcal{R}$ *of subproducts over* $\mathcal{B}$*, then the compact representation of any subproduct defined by a primitive positive formula over* $\mathcal{R}$ *can be computed in polynomial time.*

Extended Constraints

Definition

An **extended instance** is $\mathcal{E} = \{\mathbf{B}_i, \mathbf{S}_{ij} \mid i, j \in V\}$ where

- $\mathbf{S}_{ij} \leq \mathbf{B}_i \times \mathbf{B}_j \times \prod_{v \in V} \mathbf{B}_v/\beta_v$,
- if $(x, y, \bar{u}) \in S_{ij}$ then $x/\beta_i = u_i$ and $y/\beta_j = u_j$,
- if $(x, y, \bar{u}) \in S_{ii}$ then $x = y$, and
- $(x, y, \bar{u}) \in S_{ij}$ if and only if $(y, x, \bar{u}) \in S_{ji}$.

A map $f \in \prod_{v \in V} B_v$ is a **solution** if for all $i, j \in V$

$$(f(i), f(j), f(v)/\beta_v : v \in V) \in S_{ij}.$$

Definition

The extended instance $\mathcal{E}$ is $(2,3)$-**consistent** if

$$S_{ik} \subseteq \underbrace{\{(x, z, \bar{u}) \mid \exists y \in B_j \text{ such that } (x, y, \bar{u}) \in S_{ij}, (y, z, \bar{u}) \in S_{jk}\}}_{S_{ij} \circ S_{jk}}$$

Extended Consistency Algorithm

Lemma

Every instance $\mathcal{A} = \{ \mathbf{B}_i, \mathbf{R}_{ij} \mid i, j \in V \}$ can be turned into a $(2,3)$-consistent extended instance $\mathcal{E} = \{\mathbf{B}_i, \mathbf{S}_{ij} \mid i, j \in V \}$ in polynomial time such that they have the same set of solutions.

Proof Overview.

- Start with $\mathbf{S}_{ij} = \{ (x, y, \bar{u}) \mid (x, y) \in R_{i,j},\ x/\beta_i = u_i,\ y/\beta_j = u_j \}$
- $\mathbf{S}_{ij}$ has a compact representation (one for each $(x, y) \in B_i \times B_j$)
- If $\mathcal{E}$ is not $(2,3)$-consistent, then take $S'_{ik} = S_{ik} \cap (S_{ij} \circ S_{jk})$
- Stops in polynomial time: the number of witnessed indices are decreasing $\square$

Lemma

If a $(2,3)$-consistent extended instance is nonempty, then it has a solution.

Global Considerations

Lemma (McKenzie)

If two finite algebras generate $\mathrm{SD}(\wedge)$ *varietes (or varieties with edge terms), then the variety generated by their product has the same property.*

Corollary

Let $\mathcal{V}$ *be an idempotent variety generated by finite algebras, each of which has either bounded width or few subpowers. Then for each template* $\mathcal{B} \subset \mathcal{V}$ *the constraint satisfaction problem is solvable in polynomial time.*

Problem

Given compact representations of relations **S** and **P** in the few subpower case, is it possible to find the compact representation of $\mathrm{Sg}(S \cup P)$?

Problem

What goes wrong, if the quotient $\mathbf{B}/\beta$ has bounded width?

Consistent Maps

Consistent Maps

Definition

Let $\mathcal{A} = \{ \mathbf{B}_i, \mathbf{R}_{ij} \mid i, j \in V \}$ be a binary instance. A collection of maps

$$\mathcal{P} = \{ p_i : B_i \to B_i \mid i \in V \}$$

is **consistent**, if $p_i \times p_j$ preserves $\mathbf{R}_{ij}$ for all $i, j \in V$, i.e.

$$(a, b) \in \mathbf{R}_{ij} \implies (p_i(a), p_j(b)) \in \mathbf{R}_{ij}.$$

- The identity maps $p_i(x) = x$ are always consistent.
- If f is a solution, then the constant maps $p_i(x) = f(i)$ are consistent.
- Consistent maps map solutions to solutions.
- Consistent sets of maps can be composed pointwise.

Using Consistent Maps

Definition

A consistent set $\mathcal{P} = \{ p_i \mid i \in V \}$ of maps is

- **idempotent** if $p_i(p_i(x)) = p_i(x)$,
- **permutational** if $p_i(x)$ is a permutation of $\mathbf{B}_i$, for all $i \in V$.

Lemma

Let $\mathcal{P}$ be a non-permutational consistent set of maps for an instance $\mathcal{A}$. Then a smaller instance $\mathcal{A}'$ can be constructed in polynomial time such that $\mathcal{A}$ has a solution if and only if $\mathcal{A}'$ does.

- Consistent sets of maps can be iterated to get idempotency.
- Take idempotent images of the universes and relations (this is smaller)
- Larger instance has a solution if and only if the smaller does.
- We step outside of the variety (we use an idempotent image of an algebra), but linear equations are preserved.

Finding Consistent Maps

Lemma

Let $\mathcal{A} = \{ \mathbf{B}_i, \mathbf{R}_{ij} \mid i, j \in V \}$ be a binary instance. A collection of maps $\mathcal{P} = \{ p_i \mid i \in V \}$ is consistent if and only if the binary instance $\mathcal{A}'$ with

- *variables $V' = \{ (i, a) \mid i \in V, a \in B_i \}$, domains $\mathbf{B}'_{ia} = \mathbf{B}_i$, and*
- *relations*

$$\mathbf{R}'_{iajb} = \begin{cases} \mathbf{R}_{ij}, & \text{if } (a, b) \in \mathbf{R}_{ij}, \\ \mathbf{B}_i \times \mathbf{B}_j, & \text{otherwise} \end{cases}$$

has the function $f'(ia) = p_i(a)$ as a solution.

Lemma

For any binary term $t(x, y)$ and a solution f of the binary instance $\mathcal{A}$ the maps

$$\mathcal{P} = \{ p_i \mid i \in V \}, \qquad p_i(x) = t(x, f(i))$$

are consistent.

Elimination Theorem

Definition

A **template** is a finite set $\mathcal{B}$ of idempotent algebras closed under taking subalgebras and idempotent images. We say that an algebra **B can be eliminated** if $\mathrm{CSP}(\mathcal{B})$ is tractable for all templates $\mathcal{B}$ for which $\mathcal{B} \setminus \{\mathbf{B}\}$ is also a template and $\mathrm{CSP}(\mathcal{B} \setminus \{\mathbf{B}\})$ is tractable.

Theorem (Maróti)

Let $\mathbf{B} \in \mathcal{B}$ be an algebra and $t(x, y)$ be a binary term such that the unary maps $y \mapsto t(a, y)$, $b \in B$, are idempotent and not surjective. Let C be the set of elements $c \in B$ for which $x \mapsto t(x, c)$ is a permutation. If C generates a proper subuniverse of $\mathbf{B}$, then $\mathbf{B}$ can be eliminated.

Proof of Elimination Theorem

- Take a template $\mathcal{B}$, an algebra $\mathbf{B} \in \mathcal{B}$ such that $\mathcal{B} \setminus \{\mathbf{B}\}$ is also a template, and an instance $\mathcal{A} = \{\, \mathbf{B}_i, \mathbf{R}_{ij} \mid i, j \in V \,\}$.
- First check if there is a solution f for which $f(i) \in \mathrm{Sg}(C)$ for all $i \in V$ for which $\mathbf{B}_i = \mathbf{B}$. This is a smaller instance.
- If we have a solution, then we are done. Otherwise if there exists a solution f at all, then we have a consistent set of maps of the form $p_i(x) = t(x, f(i))$ which is not permutational.
- We find a non-permutational consistent set of maps for which $p_i(a) \in t(a, B_i) = \{\, t(a, y) \mid y \in B_i \,\}$ for all $(i, a) \in V'$.
- The maps $y \mapsto t(a, y)$ are idempotent and not surjective, so in our instance we can take $\mathbf{B}'_{ia} = t(a, B_i)$.
- For each choice of $i \in V$, $a, b \in B_i$ we create an instance $\mathcal{A}'_{iab}$ with an extra equality constraint between the variables (i, a) and (i, b).
- These are a smaller instances that we can solve. If one of them has a solution, then it is non-permutational, and we can reduce $\mathcal{A}$.
- Otherwise $\mathcal{A}$ has no solution.

Applications

Lemma

Let **B** *be a finite idempotent algebra,* $\beta \in \mathrm{Con}(\mathbf{B})$ *such that* $\mathbf{B}/\beta$ *is a semilattice (with extra operations) having more than one maximal elements. Then* **B** *can be eliminated.*

Proof.

Take a binary term t of **B** that is the semilattice term on $\mathbf{B}/\beta$. We can iterate, so we can assume that $t(x, t(x, y)) = t(x, y)$. Now the maps $x \mapsto t(x, b)$ and $y \mapsto t(a, y)$ are not permutations, so we can apply the elimination theorem. $\square$

Corollary (Using Marković, McKenzie)

Let $\mathcal{B}$ *be a template and assume that each algebra* $\mathbf{B} \in \mathcal{B}$ *has a congruence* $\beta \in \mathrm{Con}(\mathbf{B})$ *such that* $\mathbf{B}/\beta$ *is a semilattice with a rooted tree order, and each* β *block is Maltsev. Then* $\mathrm{CSP}(\mathcal{B})$ *can be solved in polynomial time.*

Applications Cont.

Theorem (Bulatov)

If $|\mathbf{B}| = 3$ *and* $\mathbf{B}$ *has a Taylor term, then* $\mathrm{CSP}(\mathbf{B})$ *is tractable.*

Theorem (Marković)

If $|\mathbf{B}| = 4$ *and* $\mathbf{B}$ *has a Taylor term, then* $\mathrm{CSP}(\mathbf{B})$ *is tractable.*

Theorem (Bulatov)

If $\mathbf{B}$ *is conservative (every subset is a subuniverse) and has a Taylor term, then* $\mathrm{CSP}(\mathbf{B})$ *is tractable.*

Problem

Can you avoid the condition $\mathrm{Sg}(C) \neq B$ in the elimination theorem?

Thank you!