CSPs with near-unanimity polymorphisms are solvable by linear Datalog

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joint work with Libor Barto and Ross Willard

Theoretical Computer Science
Jagiellonian University
Kraków, Poland

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Outline of the presentation

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- statement of the result;

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- systems;
- V-games;
- an unreasonable assumption;
- V-systems;
- the algebraic result.

The result

Theorem

Let $\mathbb{R}$ be a relational structure with a near-unanimity polymorphism $n(x_1, \ldots, x_{17})$

$$n(x, \ldots, x, y) \approx n(x, \ldots, x, y, x) \approx \cdots \approx n(y, x, \ldots, x) \approx x$$

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- Dalmau and Krokhin proved this result for majority,
- the V-game corresponds to a linear Datalog program,
- by Barto's result we cover all the finite relational structures in congruence distributive varieties.

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- all the relational structures are digraphs
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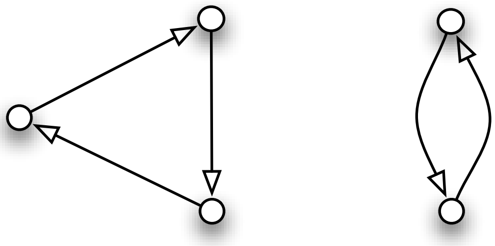
- a mysterious "unreasonable assumption".

Pebble game for bounded width problems – 2 pebbles



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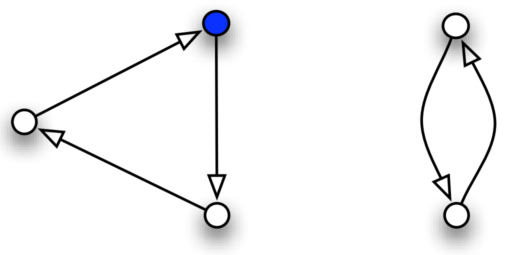

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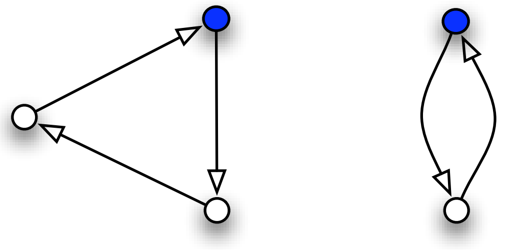

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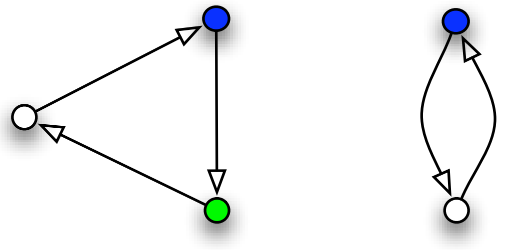

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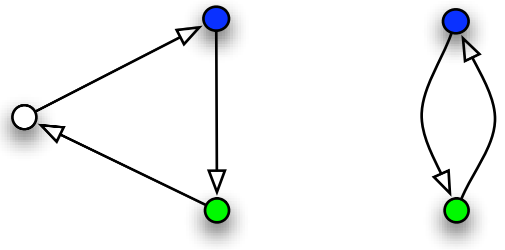

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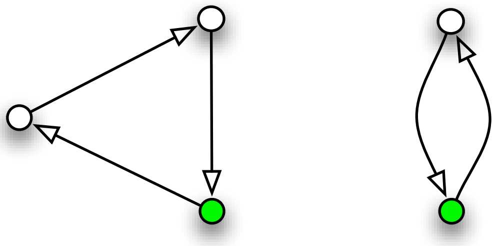

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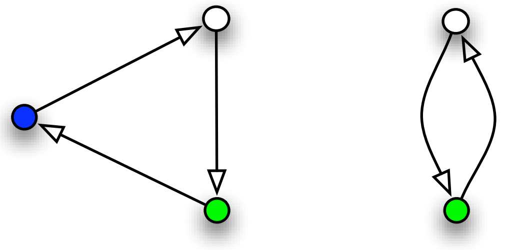

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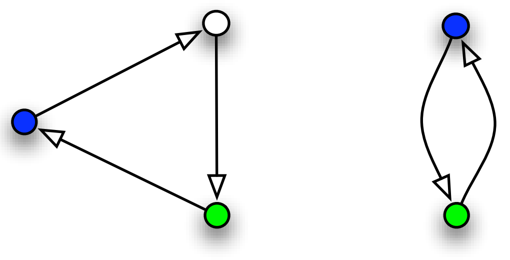

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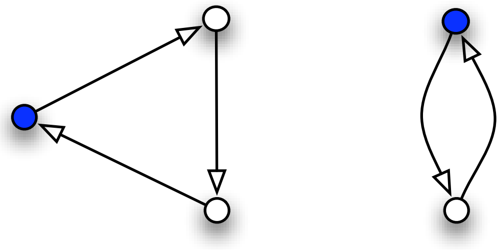

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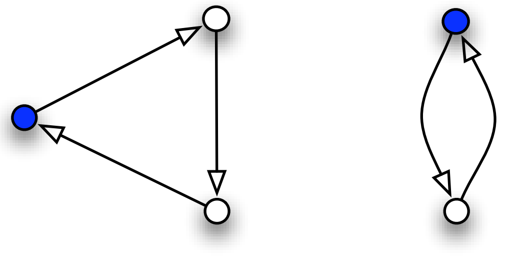

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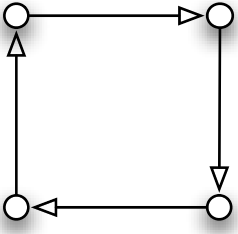

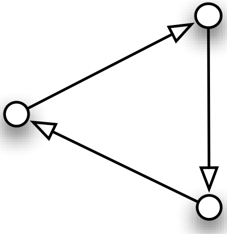

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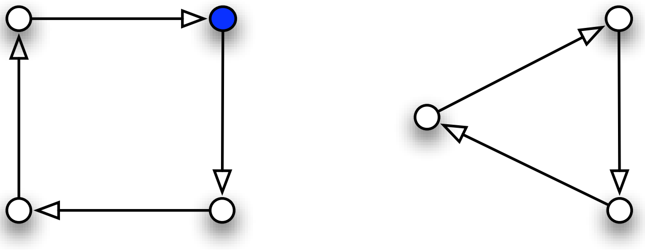

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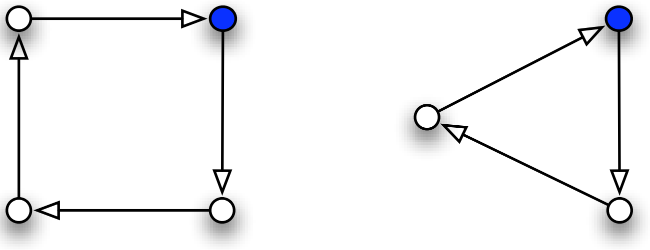

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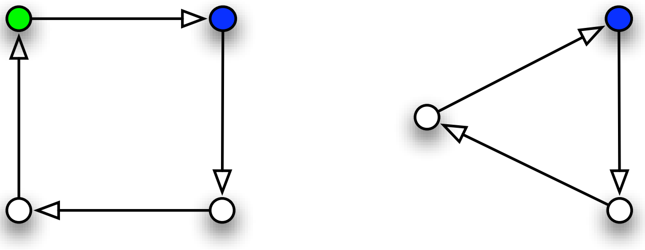

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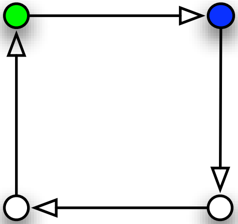

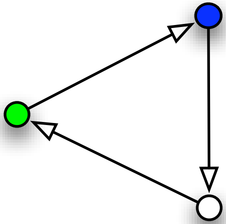

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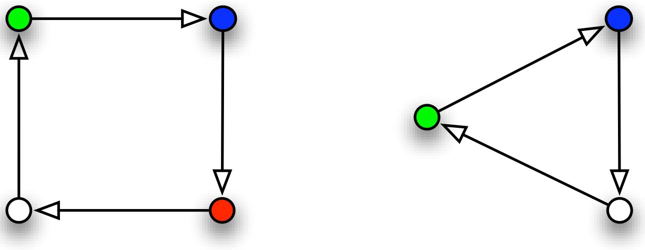

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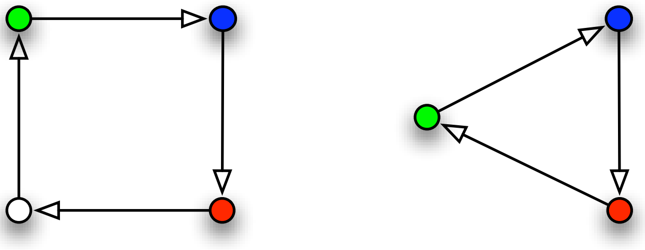

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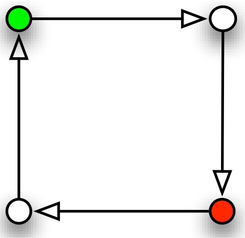

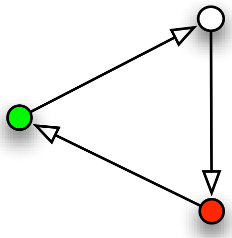

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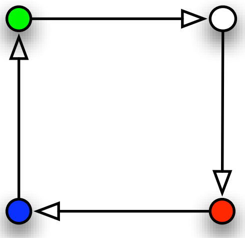

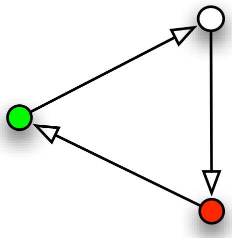

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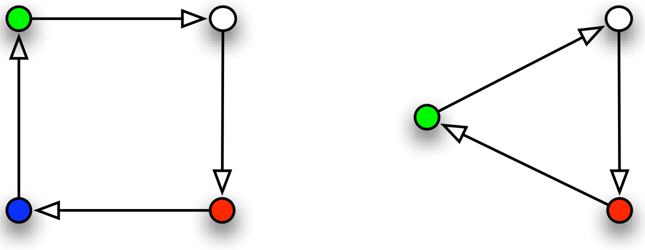

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- an existence of such a number implies that the corresponding CSP is in P.

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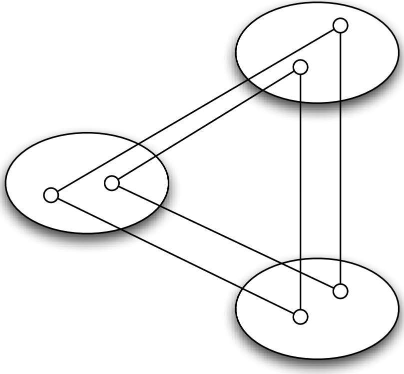

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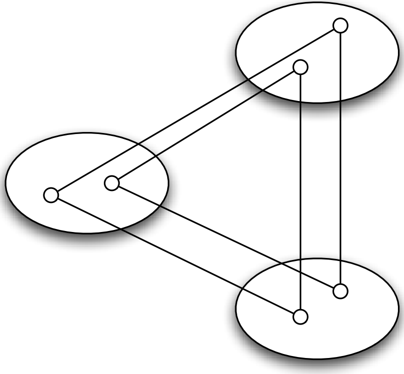

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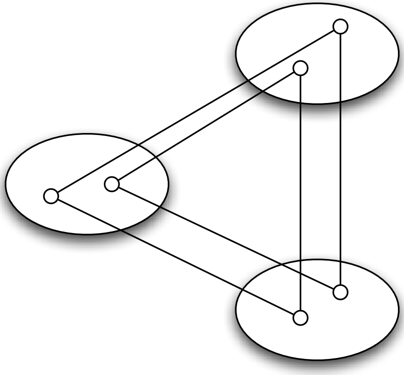

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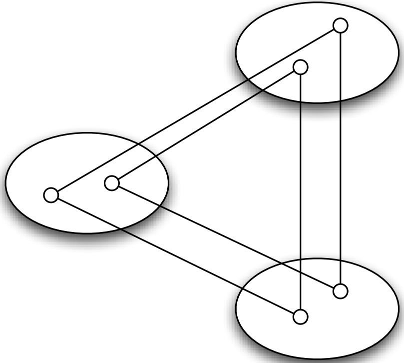

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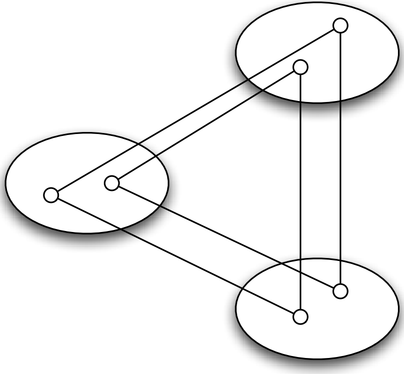

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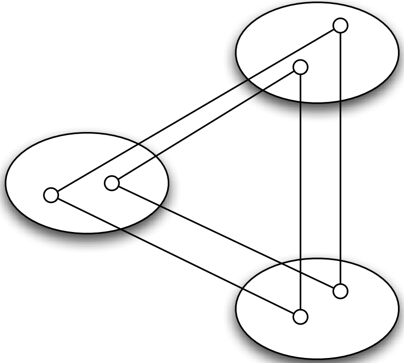

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- a $|V(\mathbb{H})|$-element clique in the system (which is a graph).

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then $(f(a), f(b)) \in E(\mathbb{G})$.

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This V-game can be played with 17 pebbles.

V-game and an unreasonable assumption

Fix a vertex $v \in V(\mathbb{H})$, let $\mathcal{T}_0^v, \mathcal{T}_1^v \ldots$ be a list of all the tree patterns such that:

- each tree has 16 leaves;
- each leaf is indexed by v and
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The unreasonable assumption

Let assume that in all the games the duplicator, for any $v \in V(\mathbb{H})$, is always choosing an element of P_v.

The system

For every $v, w \in V(\mathbb{H})$ let $E_{vw} \subseteq P_v \times P_w$ be:

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A $|V(\mathbb{H})|$-element clique in the system defines a homomorphism from $\mathbb{H}$ to $\mathbb{G}$.

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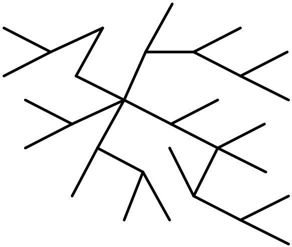
We call such a system a V-system. And the goal is to show that every V-system has a solution.

All the trees

Claim

Let $v \in V(\mathbb{H})$, $a \in P_v$ and $\mathcal{T}$ be any pattern with all the leaves indexed by v. There exists a realization of $\mathcal{T}$ in the V-system sending all the leaves to a.



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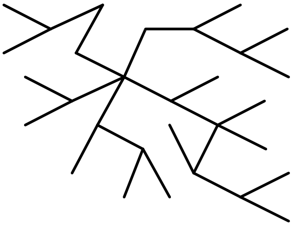

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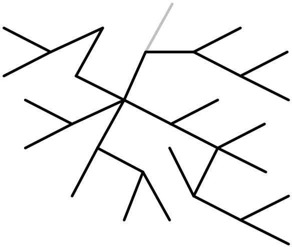

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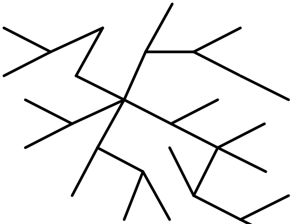

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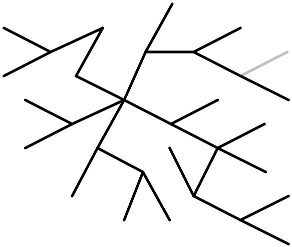

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There exist (and we take maximal) sets $P''_w \subseteq P'_w$ such that the restriction of the V-system to these sets is a non-empty (1,2)-system.

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Definition

Let $\mathbf{A}$ *be an algebra, a subuniverse* B *of* $\mathbf{A}$ *is absorbing if there exists a term* $t(x_1, \ldots, x_{17})$ *such that:*

$$t(a_1, \ldots, a_{17}) \in B \text{ whenever } |\{i : a_i \notin B\}| \leq 1.$$

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Then:

- S is a subuniverse of B;
- S is subdirect in P''^{16}_w;
- S absorbs B.

The algebraic theorem

Theorem

Let $\mathbf{A}$ *be an algebra. There exists* $a \in A$ *such that, for any* $\mathbf{S}, \mathbf{B} \leq_s A^n$ *if*

- *all the constants belong to* $\mathbf{B}$*;*
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