Robust Approximation of CSPs

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Definitions

Fix a relational structure $\mathbb{H} = (D; \Gamma)$ called the template (host structure).

An *instance* I (over $\mathbb{H}$) is a set V of variables (nodes) together with a set of constraints.

The *value* of *assignment* $s : V \to D$ is

$$I_s := \text{ fraction of constraints satisfied by } s$$

Def: $\mathrm{MAX\ CSP}(\mathbb{H})$ is the problem consisting in finding, given an instance I over $\mathbb{H}$, an assignment s with I_s maximal.

Ex. MAX CUT is $\mathrm{MAX\ CSP}(\{0, 1\}, \neq)$

Approximation Algorithms

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Let $f : [0,1] \to [0,1]$ be a decreasing function.

Alg is a f-aproximation algorithm if for every instance I

$$I_s \geq f(I_{\mathrm{OPT}})$$

where:

- $s = \mathrm{Alg}(I)$ is the output of Alg on input I
- $I_{\mathrm{OPT}} = \max_r I_r$

Many approximation results deal with functions of the form

$$f(x) = K \cdot x, \quad 0 \leq K \leq 1$$

Program: Identify, for every $\mathbb{H}$, maximum K s.t. $\mathrm{MAX\ CSP}(\mathbb{H})$ has a $(K \cdot x)$-approximation algorithm.

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- Motivation: Want an algorithm that finds a very good solution when 99% of the constraints are satisfiable

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Program: Determine, for each $\mathbb{H}$, whether $\mathrm{MAX\ CSP}(\mathbb{H})$ has a robust approximation (RA) algorithm, i.e, a f-approximation algorithm with $f_{x \to 1} \to 1$.

Previous Results

$\mathrm{MAX\ CSP}(D;\Gamma)$ has a RA algorithm if Γ consists only of:

- Horn Clauses [Zwick]
- Binary boolean clauses [Zwick]
- Binary bijective relations (aka Unique Games) [Khot]

If P$\neq$ NP then $\mathrm{MAX\ CSP}(\mathbb{Z}_q; \mathrm{3LIN\text{-}EQ}(q))$, $q > 1$ has not a RA algorithm where $\mathrm{3LIN\text{-}EQ}(q)$ contains all linear equations over $\mathbb{Z}_q$ with at most 3 variables [Hastad].

Algebraic approach

We use $\mathbb{H}' \leq_{\mathrm{RA}} \mathbb{H}$ as a shortand for
"If $\mathrm{MAX\ CSP}(\mathbb{H})$ has a RA algorithm then so has $\mathrm{MAX\ CSP}(\mathbb{H}')$"

Fact: If R is pp-definable without equality from Γ then

$$(D; \Gamma \cup \{R\}) \leq_{\mathrm{RA}} (D; \Gamma)$$

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Conjecture: [Guruswami and Zhou]
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Fact: If $\mathbb{H}'$ is compatible with some member of $\mathrm{HSP}(\mathrm{PolAlg}(\mathbb{H}))$ then $\mathbb{H}' \leq_{\mathrm{RA}} \mathbb{H}$

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Fact: For every $\mathbb{H}$, $\mathrm{MAX\ CSP}(\mathbb{H}) \equiv_{\mathrm{RA}} \mathrm{MAX\ CSP}(\mathrm{core}(\mathbb{H})^c)$

$$(D, \Gamma)^c := (D, \Gamma \cup \{C_d \mid d \in D\}),\ \ C_d = \{d\}$$

Th: [Larose & Zadori, Valeriote]
If $\mathbf{A}$ is an finite idempotent algebra that admits only a finite number of WNUs then for some $q > 1$, $(\mathbb{Z}_q; 3\mathrm{LIN\text{-}EQ}(q))$ is compatible with some member of $\mathrm{HS}(\mathbf{A})$

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It follows:

If $\mathbb{H}$ does not have bounded width then $\text{MAX CSP}(\mathbb{H})$ does not have an RA algorithm.

Width 1

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Proof:

Th: [Feder, Vardi]
$\mathbb{H} = (D, \Gamma)$ is width 1 iff $\mathbb{H} \leftrightarrow \mathcal{P}(\mathbb{H})$ where $\mathcal{P}(\mathbb{H})$

- has universe $2^D / \emptyset$
- contains, for every $R \in \Gamma$, the relation R' (of the same arity than R) defined as:

$$R' = \{(\mathrm{pr}_1 S, \ldots, \mathrm{pr}_{\mathrm{arity}(R)} S) \mid \emptyset \neq S \subseteq R\}$$

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- $\mathcal{P}(\mathbb{H}) \leq_{\mathrm{RA}} \mathcal{P}^b(\mathbb{H})$.
  Proof: Transform instance I of $\mathrm{MAX\ CSP}(\mathcal{P}(\mathbb{H}))$ into an instance I^b of $\mathrm{MAX\ CSP}(\mathcal{P}^b(\mathbb{H}))$ by replacing every variable v in I by boolean variables v^d, $d \in D$.

  There is a one-to-one correspondence between assignments of I and assignments of I^b which preserves the # of satisfied constraints.

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  There is a one-to-one correspondence between assignments of I and assignments of I^b which preserves the # of satisfied constraints.
- $\mathcal{P}^b(\mathbb{H})$ is invariant under disjunction. Hence, all its relations can be pp-defined using (dual) horn clauses.

Beyond width 1

Consider MAX CUT=$\mathrm{MAX\ CSP}(\{0,1\},\neq)$

Remarks:

- Allow randomized approximation algorithms. Now, the value I_s where s is the output of the algorithm is a random variable. Require that the expected fraction, $\mathrm{Exp}[I_s]$, of satisfied clauses is $\geq f(I_{\mathrm{OPT}})$
- Accuracy issues

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- Accuracy issues

Trivial random algorithm: set every variable according to the flip of a $1/2$-coin.

Fact: Trivial random algorithm $\frac{1}{2}$-approximates MAX CUT.

Semidefinite Programming

[Goemans & Williamson]
Better approximation algorithm for MAX CUT

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Define a quadratic program Q:

The variables of Q are $X_u, u \in V$ and take values in $\mathbb{R}$

$$\boxed{\begin{array}{lll} Q: & \text{maximize} & \frac{1}{m}\sum_{(u,v)\in E} \frac{1-X_u \cdot X_v}{2} \\ & \text{subject to} & X_u^2 = 1, u \in V \end{array}}$$

Values $\{-1, +1\}$ correspond to the sides of the partition

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It is NP-hard to solve.

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$$\boxed{\begin{array}{lll} Q: & \text{maximize} & \frac{1}{m}\sum_{(u,v)\in E} \frac{1 - X_u \cdot X_v}{2} \\ & \text{subject to} & \|X_u\|^2 = 1, u \in V \end{array}}$$

$(\cdot)$ denotes the inner product

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Intuitively, $\frac{1-X_u\cdot X_v}{2}$ favours X_u and X_v to be opposed

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Rounding: Pick a random vector t and set a node u to

$$\begin{cases} +1 & \text{if } t \cdot X_u > 0 \\ -1 & \text{if } t \cdot X_u < 0 \end{cases}$$

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[PICTURE]

Let s be the assignement produced by the rounding.

How much is lost when rounding?

For every edge $(u, v) \in E$:

- $z_{(u,v)} := \frac{1 - X_u \cdot X_v}{2}$
- $p_{(u,v)} :=$ prob. that s cuts (u, v)

$$\mathrm{Exp}[I_s] = \frac{1}{m} \sum_{(u,v)\in E} p_{(u,v)} \qquad Q_{\mathrm{OPT}} = \frac{1}{m} \sum_{(u,v)\in E} z_{(u,v)}$$

By rotational symmetry of random vector t, $p_{(u,v)} = \Theta_{u,v}/\pi$, where $\Theta_{u,v}$ is angle formed by X_u and X_v($= \arccos(X_u \cdot X_v)$)

Analiticaly, one shows

$$\frac{\arccos x}{\pi} \geq 0.878 \cdot \frac{1-x}{2}, \quad -1 \leq x \leq 1$$

It follows

$$\mathrm{Exp}[I_S] \geq 0.878 \cdot Q_{\mathrm{OPT}} \geq 0.878 \cdot I_{\mathrm{OPT}}$$

Th: [Goemans,Williamson]
MAX CUT has a 0.878-approximation algorithm

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(and also a RA algorithm because $\frac{\arccos x}{\pi} \to 1$ whenever $\frac{1-x}{2} \to 1$)

2SAT

[Goemans & Williamson]
Let I be a 2SAT instance, namely a conjunction of m 2-clauses.

Define SDP relaxation Q for I. The variables of Q are $X_v, v \in V$ and a (reference) variable X_1.

$$\boxed{\begin{array}{lll} Q: & \text{maximize} & \frac{1}{m}\sum_{C\in I} z_C \\ & \text{subject to} & \|X_u\|^2 = 1, u \in V \end{array}}$$

$$\text{where} \quad \begin{aligned} z_{\overline{u}\vee\overline{v}} &= \frac{3 - X_u \cdot X_1 - X_v \cdot X_1 - X_u \cdot X_v}{4} \\ z_{u\vee\overline{v}} &= \frac{3 + X_u \cdot X_1 - X_v \cdot X_1 + X_u \cdot X_v}{4} \\ &\vdots \end{aligned}$$

Intuitively, u is close to true if X_u is close to X_1 and is close to false if X_u is close to $-X_1$.

Let $X_u, u \in V \cup \{1\}$ be an optimal solution of Q

GW Rounding for 2SAT:

Pick random vector t. Replace t by $-t$ if $X_1 \cdot t < 0$. Set a node u to

$$\begin{cases} 1 & \text{if } t \cdot X_u > 0 \\ 0 & \text{if } t \cdot X_u < 0 \end{cases}$$

The GW rounding gives a 0.878-approximation algorithm for 2SAT but not a RA algorithm.

Analysis

Let s be the assignment produced by the GW rounding.

$$\mathrm{Exp}[I_s] = \frac{1}{m}\sum_{C\in I} p_C \qquad Q_{\mathrm{OPT}} = \frac{1}{m}\sum_{C\in I} z_C$$

where p_C and z_C are (say, for clause $C = \overline{u} \vee \overline{v}$):

$$\begin{aligned} z_C &= \frac{3 - X_u\cdot X_1 - X_v\cdot X_1 - X_u\cdot X_u}{4} \\ &= \frac{1}{2}\left(\frac{1 - X_u\cdot X_1}{2} + \frac{1 - X_v\cdot X_1}{2} + \frac{1 - X_u\cdot X_v}{2}\right) \end{aligned}$$

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We have $p_C \geq 0.878 \cdot Z_C$ but $z_C \to 1 \not\Rightarrow p_C \to 1$ (for example if $\Theta_{u,1} = \Theta_{v,1} = \Theta_{u,v} = \arccos(-1/3)$)

Zwick's RA algorithm for 2SAT

- Add to SDP Q the restrictions

$$X_u \cdot X_1 + X_v \cdot X_1 + X_u \cdot X_v \geq -1, u, v \in V$$

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- Change rounding. Two steps:

1. "Rotate" each vector X_u obtaining a new vector X'_u
2. Apply GW rounding to X'_u

Rotation

Simple rotation: works if $Q_{\mathrm{OPT}} = 1$ (decision problem)

$$\forall u \in V \cup \{1\},\ X'_u := \begin{cases} X_1 & \text{if } X_u \cdot X_1 > 0 \\ X_u & \text{if } X_u \cdot X_1 = 0 \\ -X_1 & \text{if } X_u \cdot X_1 < 0 \end{cases}$$

Apply GW rounding to $X'_u, u \in V$.

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Assume $Q_{\mathrm{OPT}} = 1$. Then $Z_C \leq 1, C \in I \Rightarrow Z_C = 1, C \in I$.

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Then $p_C = \frac{\Theta'_{u,1} + \Theta'_{v,1} + \Theta'_{u,v}}{2\pi} = 1$, $\Theta'_{i,j} := \arccos(X'_i, X'_j)$

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However, the rounding is too insensitive if $Q_{\mathrm{OPT}} < 1$

General rotation. Let $r : [0, \pi] \to [0, \pi]$. The rotation of X_u (wrt. r), X'_u, is obtained as follows:

If Θ is the angle formed by X_u and X_1, then rotate X_u (in the plane spawned by X_u and X_1) until forms an angle of $r(\Theta)$ with X_1

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The simple rotation is when r is r_0 where:

$$r_0 = \begin{cases} 0 & \text{if } \Theta < \pi/2 \\ \pi/2 & \text{if } \Theta = \pi/2 \\ \pi & \text{if } \Theta > \pi/2 \end{cases}$$

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Zwick proves that by using r_ϵ with $\epsilon = (1 - Q_{\mathrm{OPT}})^{1/3}$ one obtains $\mathrm{Exp}[I_S] \geq 1 - 5\epsilon$.

Th: [D., Krokhin] If $\mathbb{H}$ is invariant under the dual discriminator then $\mathrm{MAX\ CSP}(\mathbb{H})$ has a RA algorithm.

Dual discriminator is the ternary operation

$$m(x,y,z) = \begin{cases} x & \text{if } x = y \\ z & \text{otherwise} \end{cases}$$

Common generalization of 2SAT and Unique games.

Th: [D., Krokhin] If $\mathbb{H}$ is invariant under the dual discriminator then $\mathrm{MAX\ CSP}(\mathbb{H})$ has a RA algorithm.

Dual discriminator is the ternary operation

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Common generalization of 2SAT and Unique games.

Proof: Assume $D = \{1, \ldots, k\}$. Every relation inv. under m can be pp-defined using relations of the following types:

- $\{(i, \pi(i)) \mid i \in D\}$, where $\pi : D \to D$ is a bijection
- $D/\{i\}$, $i \in D$
- $(i \times D) \cup (D \times j)$, $i, j \in D$

We use mainly Khot's SDP relaxation for unique games

Let I be an instance of $\mathrm{MAX\ CSP}(\mathbb{H})$ with nodes V

- Variables X_u^i, $u \in V$, $i \in D$.
  (Intuitively, $\|X_u^i\| = 1$ means that u is set to i)
- Restrictions

$$
\begin{array}{ll}
\|X_u^1\|^2 + \cdots + \|X_u^k\|^2 = 1, & u \in V \\
X_u^i \cdot X_u^j = 0, & u \in V,\ 1 \le i \ne j \le k \\
X_u^i \cdot X_v^j \ge 0, & u, v \in V,\ 1 \le i, j \le k \\
\sum_{1 \le i,j \le k} X_u^i \cdot X_v^j = 1, & u, v \in V
\end{array}
$$

- Goal function $\sum_{C \in I} z_C$ where z_C is

$$\begin{cases} \sum_{1 \le i \le k} X_u^i \cdot X_v^{\pi(i)} & \text{if } C = ((u,v),\pi) \\ 1 - \|\overline{X_u^i}\|^2 & \text{if } C = (u, D \setminus \{i\}) \\ 1 - \overline{X_u^i} \cdot \overline{X_v^j} & \text{if } C = ((u,v),(i \times D) \cup (D \times j)) \end{cases}$$

where $\overline{X_u^i}$ is a shortand for $\sum_{1 \le j \ne i \le k} X_u^j$

Rounding

Step 1: Set $Y_1 := \sum_{1 \leq i \leq k} X_u^i$ for some $u \in V$ (invariant of the choice of u)

Apply Zwick's rounding with appropiate ϵ to $Y_u^i = 2X_u^i - Y_1$, $u \in V, i \in D$ (follows from the restrictions that $\|Y_u^i\| = 1$) obtaining $u^i \in \{0, 1\}, u \in V, i \in D$.

For every u, if there exists exactly one i with $u^i = 1$ then set u to i otherwise leave u undefined.

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One shows that step one satisfyies most constraints of type 2 or 3 and falsifies or partially assigns only a small fraction of constraints of type 1.

Mixed Constraints

In a *mixed* intance I constraints are divided in two sets:

- *Hard* constraints that must be satisfied
- *Soft* constraints that can be falsified.

Def: $\mathrm{MIXED\ CSP}(\mathbb{H})$ is the problem consisting in finding, given a mixed instance I over $\mathbb{H}$, an assignment s with I_s maximal where

$$I_s = \begin{cases} 0 \text{ if } s \text{ falsifies some hard constraint} \\ \text{fraction of soft constraints satisfied by } s \text{ otherwise} \end{cases}$$

Observation: If $\mathbb{H}$ has bounded width then

$$\mathrm{MAX\ CSP}(\mathbb{H}) \leq_{\mathrm{RA}} \mathrm{MIXED\ CSP}(\mathrm{HORN}^r)$$

for some $r > 0$, where where HORN^r is the boolean structure containing of all horn clauses of arity up to r.

Proof: Let $l = 3\lceil \frac{\text{arity}(\mathbb{H})}{2} \rceil$ and let I be an instance $\mathrm{MAX\ CSP}(\mathbb{H})$ with variable set V.

A partial assignment is any mapping $p : U \rightarrow D$ with $U \subseteq V$.

If p, q are partial assignments, $p \subseteq q$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and p and q coincide over $\mathrm{dom}(p)$.

A $(l-1,l)$-strategy $\mathcal{K}$ for I is a nonempty collection of partial assignments of domain size at most l satisfying:

1. If $q \in \mathcal{K}$ and $p \subseteq q$ then $p \in \mathcal{K}$.
2. If $p \in S$ and $|\mathrm{dom}(p)| < l$ then for every $u \in V$ there exists some $q \in \mathcal{K}$ with $p \subseteq q$ and $u \in \mathrm{dom}(q)$.
3. If $p \in \mathcal{K}$ and C is a constraint in I whose scope is entirely contained in $\mathrm{dom}(p)$ then p satisfies C

If $\mathbb{H}$ has bounded width and I has a $(l-1,l)$-strategy then I has a solution.

The existence of a $(l-1,l)$-strategy can be formulated as a horn formula, I', by introducing one boolean variable X_p for every p with $|\operatorname{dom}(p)| \leq l$ stating "p is not in the strategy".

Mark every clause in I' arising from conditions (1) or (2) as hard and every clause arising from condition (3) as soft.

Observe that the number of soft constraints of I' is in $[m, m \cdot |D|^l]$ where m is the number of constraints of I.

For every assignment s' of I' let $\mathcal{K}_{s'}$ be the set containing precisely all those p such that X_p is false.

Algorithm for $\mathrm{MAX\ CSP}(\mathbb{H})$:

Let s' be the assignment returned with input I' by the hypothetical RA algorithm (say, with function f) for $\mathrm{MIXED\ CSP(HORN)}$.

$\mathcal{K}_{s'}$ is an strategy for the subinstance $J \subseteq I$ obtained by removing all constraints associated to clauses violated by s'.

Output a solution s of J (obtained by applying iteratively the consistency algorithm)

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We have

$$1 - I_s \leq |D|^k(1 - I'_{s'}) \leq |D|^k(1 - f(I'_{\mathrm{OPT}})) \leq |D|^k(1 - f(I_{\mathrm{OPT}}))$$

Open Problems

- Does $\mathrm{MIXED\ CSP(HORN)}$ have a RA algorithm? Does it have a K-approximation algorithm for some $0 < K$? More generally, determine, for every $\mathbb{H}$, whether there is a RA algorithm for $\mathrm{MIXED\ CSP}(\mathbb{H})$.
- Consider more general polymorphisms. For example, does $\mathrm{MAX\ CSP}(\mathbb{H})$ have a RA algorithm if $\mathbb{H}$ admits a majority (near-unanimity) polymorphism?
- Consider consistency algorithms more powerful than 1-minimality but still less powerfull than 3-minimality. For example, is it true that if $\mathrm{CSP}(\mathbb{H})$ solvable by peek-arc-consistency (or singleton arc-consistency) then $\mathrm{MAX\ CSP}(\mathbb{H})$ has a RA algorithm?

THANKS FOR YOUR ATTENTION!!!!