Absorption in finitely related $SD(\wedge)$ algebras has bounded arity

Jakub Bulín

Department of Algebra, Charles University in Prague

Workshop on Algebra and CSPs

Outline

1. Introduction
2. The result
3. Proof
4. Open problems

Finitely related algebra

All algebras in this talk are finite and idempotent.

Definition

- An algebra $\mathbf{A}$ is finitely related if $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$ for some relational structure $\mathbb{A} = (A; R_1, \ldots, R_m)$.
- $\mathbf{A}$ is n-ary related if the relations of $\mathbb{A}$ are at most n-ary
- 2-ary related = binary related

Absorption

Definition

A subset $B \subseteq A$ is an absorbing subuniverse of an algebra $\mathbf{A}$ if $B \leq \mathbf{A}$ and there exists an idempotent $t \in \mathrm{Clo}(\mathbf{A})$ such that $t(a_1, a_2, \ldots, a_k) \in B$ whenever all but one of the a_i's are in B. In that case we write $B \trianglelefteq \mathbf{A}$ (or $B \overset{t}{\trianglelefteq} \mathbf{A}$).

Example: An (idempotent) algebra $\mathbf{A}$ has a near-unanimity term iff $\{a\} \trianglelefteq \mathbf{A}$ for every $a \in A$.

Absorption

Definition

A subset $B \subseteq A$ is an absorbing subuniverse of an algebra $\mathbf{A}$ if $B \leq \mathbf{A}$ and there exists an idempotent $t \in \mathrm{Clo}(\mathbf{A})$ such that $t(a_1, a_2, \ldots, a_k) \in B$ whenever all but one of the a_i's are in B. In that case we write $B \trianglelefteq \mathbf{A}$ (or $B \overset{t}{\trianglelefteq} \mathbf{A}$).

Example: An (idempotent) algebra $\mathbf{A}$ has a near-unanimity term iff $\{a\} \trianglelefteq \mathbf{A}$ for every $a \in A$.

Outline

1 Introduction

2 The result

3 Proof

4 Open problems

The result

Theorem (JB'11)

Let $\mathbf{A}$ *be a finite, n-ary related algebra in an SD(*$\wedge$*) variety. If* $B \trianglelefteq \mathbf{A}$*, then there exist a* $(4^{8^{|A|^n}} + 1)$*-ary term* $t \in \mathrm{Clo}(\mathbf{A})$ *such that* $B \overset{t}{\trianglelefteq} \mathbf{A}$*.*

Our result relies ***heavily*** on the techniques developed for the proof of the following theorem:

Theorem (Barto'09)

Every finite, finitely related algebra in a CD variety has a near unanimity term operation.

The result

Theorem (JB'11)

Let $\mathbf{A}$ be a finite, n-ary related algebra in an SD($\wedge$) variety. If $B \trianglelefteq \mathbf{A}$, then there exist a $(4^{8^{|A|^n}} + 1)$-ary term $t \in \mathrm{Clo}(\mathbf{A})$ such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.

Our result relies ***heavily*** on the techniques developed for the proof of the following theorem:

Theorem (Barto'09)

Every finite, finitely related algebra in a CD variety has a near unanimity term operation.

Outline

1 Introduction

2 The result

3 Proof

4 Open problems

Strategy of the proof

- A simple trick reduces the problem to binary related algebras: It suffices to prove that for $\mathbf{A}$ a binary related SD($\wedge$) algebra and $B \trianglelefteq \mathbf{A}$ there exists a $(4^{8^{|A|}} + 1)$-ary term t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.
- We construct an instance of $\mathrm{CSP}(\mathbf{A})$ whose solutions are precisely k-ary terms t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.
- If k is "big" ($k > 4^{8^{|A|}}$), the instance is consistent enough to have a solution (Pigeonhole principle + an algebraic lemma).
- We need the Bounded width theorem; hence the assumption that $\mathbf{A}$ is SD($\wedge$).

Strategy of the proof

- A simple trick reduces the problem to binary related algebras: It suffices to prove that for **A** a binary related SD($\wedge$) algebra and $B \trianglelefteq \mathbf{A}$ there exists a $(4^{8^{|A|}} + 1)$-ary term t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.
- We construct an instance of $\mathrm{CSP}(\mathbf{A})$ whose solutions are precisely k-ary terms t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.
- If k is "big" ($k > 4^{8^{|A|}}$), the instance is consistent enough to have a solution (Pigeonhole principle + an algebraic lemma).
- We need the Bounded width theorem; hence the assumption that **A** is SD($\wedge$).

Strategy of the proof

- A simple trick reduces the problem to binary related algebras: It suffices to prove that for **A** a binary related SD($\wedge$) algebra and $B \trianglelefteq \mathbf{A}$ there exists a $(4^{8^{|A|}}+1)$-ary term t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.
- We construct an instance of $\mathrm{CSP}(\mathbf{A})$ whose solutions are precisely k-ary terms t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.
- If k is "big" ($k > 4^{8^{|A|}}$), the instance is consistent enough to have a solution (Pigeonhole principle + an algebraic lemma).
- We need the Bounded width theorem; hence the assumption that **A** is SD($\wedge$).

Strategy of the proof

- A simple trick reduces the problem to binary related algebras: It suffices to prove that for **A** a binary related SD($\wedge$) algebra and $B \trianglelefteq \mathbf{A}$ there exists a $(4^{8^{|A|}} + 1)$-ary term t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.
- We construct an instance of $\mathrm{CSP}(\mathbf{A})$ whose solutions are precisely k-ary terms t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.
- If k is "big" ($k > 4^{8^{|A|}}$), the instance is consistent enough to have a solution (Pigeonhole principle + an algebraic lemma).
- We need the Bounded width theorem; hence the assumption that **A** is SD($\wedge$).

Encoding absorption as a CSP

Let $\mathrm{Clo}(\mathbf{A}, k)$ be the following instance of $\mathrm{CSP}(\mathbf{A})$:

- variables: $V = A^k$
- constraints: $\mathcal{C} = \{((\bar{a}, \bar{a}'), R_{\bar{a},\bar{a}'}) : \bar{a}, \bar{a}' \in A^k\}$, where

$$R_{\bar{a},\bar{a}'} = \{(f(\bar{a}), f(\bar{a}')) : f \in \mathrm{Clo}_k(\mathbf{A})\}$$

$\mathrm{Clo}(\mathbf{A}, k)$ is (2,3)-consistent and its solutions are precisely the k-ary term operations of $\mathbf{A}$. ($\mathbf{A}$ is binary related!)

Let $\mathcal{S} = \{((\bar{a}), S_{\bar{a}}) : \bar{a} \in A^k\}$, where $S_{\bar{a}} = B$ if $\bar{a}$ has at most one coordinate outside B, $S_{\bar{a}} = A$ else.

We construct the instance $\mathrm{Abs}(\mathbf{A}, B, k)$ of $\mathrm{CSP}(\mathbf{A})$ by adding the $S_{\bar{a}}$'s as unary constraints:

- variables: $V = A^k$
- constraints: $\mathcal{C} \cup \mathcal{S}$

The solutions to $\mathrm{Abs}(\mathbf{A}, B, k)$ are precisely k-ary t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.

Encoding absorption as a CSP

Let $\mathrm{Clo}(\mathbf{A}, k)$ be the following instance of $\mathrm{CSP}(\mathbf{A})$:

- variables: $V = A^k$
- constraints: $\mathcal{C} = \{((\bar{a}, \bar{a}'), R_{\bar{a},\bar{a}'}) : \bar{a}, \bar{a}' \in A^k\}$, where

$$R_{\bar{a},\bar{a}'} = \{(f(\bar{a}), f(\bar{a}')) : f \in \mathrm{Clo}_k(\mathbf{A})\}$$

$\mathrm{Clo}(\mathbf{A}, k)$ is (2,3)-consistent and its solutions are precisely the k-ary term operations of $\mathbf{A}$. ($\mathbf{A}$ is binary related!)

Let $\mathcal{S} = \{((\bar{a}), S_{\bar{a}}) : \bar{a} \in A^k\}$, where $S_{\bar{a}} = B$ if $\bar{a}$ has at most one coordinate outside B, $S_{\bar{a}} = A$ else.

We construct the instance $\mathrm{Abs}(\mathbf{A}, B, k)$ of $\mathrm{CSP}(\mathbf{A})$ by adding the $S_{\bar{a}}$'s as unary constraints:

- variables: $V = A^k$
- constraints: $\mathcal{C} \cup \mathcal{S}$

The solutions to $\mathrm{Abs}(\mathbf{A}, B, k)$ are precisely k-ary t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.

Encoding absorption as a CSP

Let $\mathrm{Clo}(\mathbf{A}, k)$ be the following instance of $\mathrm{CSP}(\mathbf{A})$:

- variables: $V = A^k$
- constraints: $\mathcal{C} = \{((\bar{a}, \bar{a}'), R_{\bar{a},\bar{a}'}) : \bar{a}, \bar{a}' \in A^k\}$, where

$$R_{\bar{a},\bar{a}'} = \{(f(\bar{a}), f(\bar{a}')) : f \in \mathrm{Clo}_k(\mathbf{A})\}$$

$\mathrm{Clo}(\mathbf{A}, k)$ is (2,3)-consistent and its solutions are precisely the k-ary term operations of $\mathbf{A}$. ($\mathbf{A}$ is binary related!)

Let $\mathcal{S} = \{((\bar{a}), S_{\bar{a}}) : \bar{a} \in A^k\}$, where $S_{\bar{a}} = B$ if $\bar{a}$ has at most one coordinate outside B, $S_{\bar{a}} = A$ else.

We construct the instance $\mathrm{Abs}(\mathbf{A}, B, k)$ of $\mathrm{CSP}(\mathbf{A})$ by adding the $S_{\bar{a}}$'s as unary constraints:

- variables: $V = A^k$
- constraints: $\mathcal{C} \cup \mathcal{S}$

The solutions to $\mathrm{Abs}(\mathbf{A}, B, k)$ are precisely k-ary t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.

Encoding absorption as a CSP

Let $\mathrm{Clo}(\mathbf{A}, k)$ be the following instance of $\mathrm{CSP}(\mathbf{A})$:

- variables: $V = A^k$
- constraints: $\mathcal{C} = \{((\bar{a}, \bar{a}'), R_{\bar{a},\bar{a}'}) : \bar{a}, \bar{a}' \in A^k\}$, where

$$R_{\bar{a},\bar{a}'} = \{(f(\bar{a}), f(\bar{a}')) : f \in \mathrm{Clo}_k(\mathbf{A})\}$$

$\mathrm{Clo}(\mathbf{A}, k)$ is (2,3)-consistent and its solutions are precisely the k-ary term operations of $\mathbf{A}$. ($\mathbf{A}$ is binary related!)

Let $\mathcal{S} = \{((\bar{a}), S_{\bar{a}}) : \bar{a} \in A^k\}$, where $S_{\bar{a}} = B$ if $\bar{a}$ has at most one coordinate outside B, $S_{\bar{a}} = A$ else.

We construct the instance $\mathrm{Abs}(\mathbf{A}, B, k)$ of $\mathrm{CSP}(\mathbf{A})$ by adding the $S_{\bar{a}}$'s as unary constraints:

- variables: $V = A^k$
- constraints: $\mathcal{C} \cup \mathcal{S}$

The solutions to $\mathrm{Abs}(\mathbf{A}, B, k)$ are precisely k-ary t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.

Encoding absorption as a CSP

Let $\mathrm{Clo}(\mathbf{A}, k)$ be the following instance of $\mathrm{CSP}(\mathbf{A})$:

- variables: $V = A^k$
- constraints: $\mathcal{C} = \{((\bar{a}, \bar{a}'), R_{\bar{a},\bar{a}'}) : \bar{a}, \bar{a}' \in A^k\}$, where

$$R_{\bar{a},\bar{a}'} = \{(f(\bar{a}), f(\bar{a}')) : f \in \mathrm{Clo}_k(\mathbf{A})\}$$

$\mathrm{Clo}(\mathbf{A}, k)$ is (2,3)-consistent and its solutions are precisely the k-ary term operations of $\mathbf{A}$. ($\mathbf{A}$ is binary related!)

Let $\mathcal{S} = \{((\bar{a}), S_{\bar{a}}) : \bar{a} \in A^k\}$, where $S_{\bar{a}} = B$ if $\bar{a}$ has at most one coordinate outside B, $S_{\bar{a}} = A$ else.

We construct the instance $\mathrm{Abs}(\mathbf{A}, B, k)$ of $\mathrm{CSP}(\mathbf{A})$ by adding the $S_{\bar{a}}$'s as unary constraints:

- variables: $V = A^k$
- constraints: $\mathcal{C} \cup \mathcal{S}$

The solutions to $\mathrm{Abs}(\mathbf{A}, B, k)$ are precisely k-ary t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.

Encoding absorption as a CSP

Let $\mathrm{Clo}(\mathbf{A}, k)$ be the following instance of $\mathrm{CSP}(\mathbf{A})$:

- variables: $V = A^k$
- constraints: $\mathcal{C} = \{((\bar{a}, \bar{a}'), R_{\bar{a},\bar{a}'}) : \bar{a}, \bar{a}' \in A^k\}$, where

$$R_{\bar{a},\bar{a}'} = \{(f(\bar{a}), f(\bar{a}')) : f \in \mathrm{Clo}_k(\mathbf{A})\}$$

$\mathrm{Clo}(\mathbf{A}, k)$ is (2,3)-consistent and its solutions are precisely the k-ary term operations of $\mathbf{A}$. ($\mathbf{A}$ is binary related!)

Let $\mathcal{S} = \{((\bar{a}), S_{\bar{a}}) : \bar{a} \in A^k\}$, where $S_{\bar{a}} = B$ if $\bar{a}$ has at most one coordinate outside B, $S_{\bar{a}} = A$ else.

We construct the instance $\mathrm{Abs}(\mathbf{A}, B, k)$ of $\mathrm{CSP}(\mathbf{A})$ by adding the $S_{\bar{a}}$'s as unary constraints:

- variables: $V = A^k$
- constraints: $\mathcal{C} \cup \mathcal{S}$

The solutions to $\mathrm{Abs}(\mathbf{A}, B, k)$ are precisely k-ary t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.

Encoding absorption as a CSP

Let $\mathrm{Clo}(\mathbf{A}, k)$ be the following instance of $\mathrm{CSP}(\mathbf{A})$:

- variables: $V = A^k$
- constraints: $\mathcal{C} = \{((\bar{a}, \bar{a}'), R_{\bar{a},\bar{a}'}) : \bar{a}, \bar{a}' \in A^k\}$, where

$$R_{\bar{a},\bar{a}'} = \{(f(\bar{a}), f(\bar{a}')) : f \in \mathrm{Clo}_k(\mathbf{A})\}$$

$\mathrm{Clo}(\mathbf{A}, k)$ is (2,3)-consistent and its solutions are precisely the k-ary term operations of $\mathbf{A}$. ($\mathbf{A}$ is binary related!)

Let $\mathcal{S} = \{((\bar{a}), S_{\bar{a}}) : \bar{a} \in A^k\}$, where $S_{\bar{a}} = B$ if $\bar{a}$ has at most one coordinate outside B, $S_{\bar{a}} = A$ else.

We construct the instance $\mathrm{Abs}(\mathbf{A}, B, k)$ of $\mathrm{CSP}(\mathbf{A})$ by adding the $S_{\bar{a}}$'s as unary constraints:

- variables: $V = A^k$
- constraints: $\mathcal{C} \cup \mathcal{S}$

The solutions to $\mathrm{Abs}(\mathbf{A}, B, k)$ are precisely k-ary t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.

Encoding absorption as a CSP

Let $\mathrm{Clo}(\mathbf{A}, k)$ be the following instance of $\mathrm{CSP}(\mathbf{A})$:

- variables: $V = A^k$
- constraints: $\mathcal{C} = \{((\bar{a}, \bar{a}'), R_{\bar{a},\bar{a}'}) : \bar{a}, \bar{a}' \in A^k\}$, where

$$R_{\bar{a},\bar{a}'} = \{(f(\bar{a}), f(\bar{a}')) : f \in \mathrm{Clo}_k(\mathbf{A})\}$$

$\mathrm{Clo}(\mathbf{A}, k)$ is (2,3)-consistent and its solutions are precisely the k-ary term operations of $\mathbf{A}$. ($\mathbf{A}$ is binary related!)

Let $\mathcal{S} = \{((\bar{a}), S_{\bar{a}}) : \bar{a} \in A^k\}$, where $S_{\bar{a}} = B$ if $\bar{a}$ has at most one coordinate outside B, $S_{\bar{a}} = A$ else.

We construct the instance $\mathrm{Abs}(\mathbf{A}, B, k)$ of $\mathrm{CSP}(\mathbf{A})$ by adding the $S_{\bar{a}}$'s as unary constraints:

- variables: $V = A^k$
- constraints: $\mathcal{C} \cup \mathcal{S}$

The solutions to $\mathrm{Abs}(\mathbf{A}, B, k)$ are precisely k-ary t such that $B \overset{t}{\lhd} \mathbf{A}$.

Encoding absorption as a CSP

Let $\mathrm{Clo}(\mathbf{A}, k)$ be the following instance of $\mathrm{CSP}(\mathbf{A})$:

- variables: $V = A^k$
- constraints: $\mathcal{C} = \{((\bar{a}, \bar{a}'), R_{\bar{a},\bar{a}'}) : \bar{a}, \bar{a}' \in A^k\}$, where

$$R_{\bar{a},\bar{a}'} = \{(f(\bar{a}), f(\bar{a}')) : f \in \mathrm{Clo}_k(\mathbf{A})\}$$

$\mathrm{Clo}(\mathbf{A}, k)$ is (2,3)-consistent and its solutions are precisely the k-ary term operations of $\mathbf{A}$. ($\mathbf{A}$ is binary related!)

Let $\mathcal{S} = \{((\bar{a}), S_{\bar{a}}) : \bar{a} \in A^k\}$, where $S_{\bar{a}} = B$ if $\bar{a}$ has at most one coordinate outside B, $S_{\bar{a}} = A$ else.

We construct the instance $\mathrm{Abs}(\mathbf{A}, B, k)$ of $\mathrm{CSP}(\mathbf{A})$ by adding the $S_{\bar{a}}$'s as unary constraints:

- variables: $V = A^k$
- constraints: $\mathcal{C} \cup \mathcal{S}$

The solutions to $\mathrm{Abs}(\mathbf{A}, B, k)$ are precisely k-ary t such that $B \overset{t}{\trianglelefteq} \mathbf{A}$.

A useful fact about absorption

As $B \trianglelefteq \mathbf{A}$, the instance $\mathrm{Abs}(\mathbf{A}, B, k)$ is a restriction of $\mathrm{Clo}(\mathbf{A}, k)$ to absorbing subpotatoes. We like that! The following is a standard absorption technique (see the proof of BW theorem):

Lemma

Let $\mathcal{P}$ be a (2,3)-consistent instance of CSP of an SD($\wedge$) algebra and $\mathcal{S} = \{S_x : x \in V\}$ a family of absorbing subuniverses. If every tree is realizable inside the sets S_x, then $\mathcal{P} \restriction_{\mathcal{S}}$ has a solution.

Thus it is enough to prove that every tree is realizable inside the $S_{\bar{a}}$'s. Note that every tree with at most $k-1$ vertices is realizable (by some projection operation).

A useful fact about absorption

As $B \trianglelefteq \mathbf{A}$, the instance $\mathrm{Abs}(\mathbf{A}, B, k)$ is a restriction of $\mathrm{Clo}(\mathbf{A}, k)$ to absorbing subpotatoes. We like that! The following is a standard absorption technique (see the proof of BW theorem):

Lemma

Let $\mathcal{P}$ be a (2,3)-consistent instance of CSP of an SD($\wedge$) algebra and $\mathcal{S} = \{S_x : x \in V\}$ a family of absorbing subuniverses. If every tree is realizable inside the sets S_x, then $\mathcal{P} \restriction_{\mathcal{S}}$ has a solution.

Thus it is enough to prove that every tree is realizable inside the $S_{\bar{a}}$'s. Note that every tree with at most $k-1$ vertices is realizable (by some projection operation).

A useful fact about absorption

As $B \trianglelefteq \mathbf{A}$, the instance $\mathrm{Abs}(\mathbf{A}, B, k)$ is a restriction of $\mathrm{Clo}(\mathbf{A}, k)$ to absorbing subpotatoes. We like that! The following is a standard absorption technique (see the proof of BW theorem):

Lemma

Let $\mathcal{P}$ be a (2,3)-consistent instance of CSP of an SD($\wedge$) algebra and $\mathcal{S} = \{S_x : x \in V\}$ a family of absorbing subuniverses. If every tree is realizable inside the sets S_x, then $\mathcal{P} \restriction_{\mathcal{S}}$ has a solution.

Thus it is enough to prove that every tree is realizable inside the $S_{\bar{a}}$'s. Note that every tree with at most $k-1$ vertices is realizable (by some projection operation).

A useful fact about absorption

As $B \trianglelefteq \mathbf{A}$, the instance $\mathrm{Abs}(\mathbf{A}, B, k)$ is a restriction of $\mathrm{Clo}(\mathbf{A}, k)$ to absorbing subpotatoes. We like that! The following is a standard absorption technique (see the proof of BW theorem):

Lemma

Let $\mathcal{P}$ be a (2,3)-consistent instance of CSP of an SD($\wedge$) algebra and $\mathcal{S} = \{S_x : x \in V\}$ a family of absorbing subuniverses. If every tree is realizable inside the sets S_x, then $\mathcal{P} \restriction_{\mathcal{S}}$ has a solution.

Thus it is enough to prove that every tree is realizable inside the $S_{\bar{a}}$'s. Note that every tree with at most $k-1$ vertices is realizable (by some projection operation).

The key step

The crucial step of the proof is to prove that realizing trees with $4^{8^{|A|}}$ vertices is enough:

Lemma

Let $\mathcal{P}$ be (2,3)-consistent and $\mathcal{S} = \{S_x : x \in V\}$ a family of absorbing subuniverses. If all trees with at most $4^{8^{|A|}}$ vertices are realizable in $\mathcal{P} \restriction_{\mathcal{S}}$, then all trees are realizable in $\mathcal{P} \restriction_{\mathcal{S}}$. (Here the algebra can be arbitrary.)

Idea of proof:

- Suppose for contradiction that there is a big tree which does not have a realization. This tree has a long path.
- Use the Pigeonhole principle to find a certain configuration.
- Prove that this configuration contradicts the fact that S_x's are absorbing subuniverses.

The key step

The crucial step of the proof is to prove that realizing trees with $4^{8^{|A|}}$ vertices is enough:

Lemma

Let $\mathcal{P}$ be (2,3)-consistent and $\mathcal{S} = \{S_x : x \in V\}$ a family of absorbing subuniverses. If all trees with at most $4^{8^{|A|}}$ vertices are realizable in $\mathcal{P} \restriction_{\mathcal{S}}$, then all trees are realizable in $\mathcal{P} \restriction_{\mathcal{S}}$. (Here the algebra can be arbitrary.)

Idea of proof:

- Suppose for contradiction that there is a big tree which does not have a realization. This tree has a long path.
- Use the Pigeonhole principle to find a certain configuration.
- Prove that this configuration contradicts the fact that S_x's are absorbing subuniverses.

The key step

The crucial step of the proof is to prove that realizing trees with $4^{8^{|A|}}$ vertices is enough:

Lemma

Let $\mathcal{P}$ be (2,3)-consistent and $\mathcal{S} = \{S_x : x \in V\}$ a family of absorbing subuniverses. If all trees with at most $4^{8^{|A|}}$ vertices are realizable in $\mathcal{P} \restriction_{\mathcal{S}}$, then all trees are realizable in $\mathcal{P} \restriction_{\mathcal{S}}$. (Here the algebra can be arbitrary.)

Idea of proof:

- Suppose for contradiction that there is a big tree which does not have a realization. This tree has a long path.
- Use the Pigeonhole principle to find a certain configuration.
- Prove that this configuration contradicts the fact that S_x's are absorbing subuniverses.

The key step

The crucial step of the proof is to prove that realizing trees with $4^{8^{|A|}}$ vertices is enough:

Lemma

Let $\mathcal{P}$ be (2,3)-consistent and $\mathcal{S} = \{S_x : x \in V\}$ a family of absorbing subuniverses. If all trees with at most $4^{8^{|A|}}$ vertices are realizable in $\mathcal{P} \restriction_{\mathcal{S}}$, then all trees are realizable in $\mathcal{P} \restriction_{\mathcal{S}}$. (Here the algebra can be arbitrary.)

Idea of proof:

- Suppose for contradiction that there is a big tree which does not have a realization. This tree has a long path.
- Use the Pigeonhole principle to find a certain configuration.
- Prove that this configuration contradicts the fact that S_x's are absorbing subuniverses.

The key step

The crucial step of the proof is to prove that realizing trees with $4^{8^{|A|}}$ vertices is enough:

Lemma

Let $\mathcal{P}$ be (2,3)-consistent and $\mathcal{S} = \{S_x : x \in V\}$ a family of absorbing subuniverses. If all trees with at most $4^{8^{|A|}}$ vertices are realizable in $\mathcal{P} \restriction_{\mathcal{S}}$, then all trees are realizable in $\mathcal{P} \restriction_{\mathcal{S}}$. (Here the algebra can be arbitrary.)

Idea of proof:

- Suppose for contradiction that there is a big tree which does not have a realization. This tree has a long path.
- Use the Pigeonhole principle to find a certain configuration.
- Prove that this configuration contradicts the fact that S_x's are absorbing subuniverses.

Outline

1 Introduction

2 The result

3 Proof

4 Open problems

Libor has some problems...

Problem

Given a finite relational structure $\mathbb{A}$ and a subset $B \subseteq A$, is it decidable if B is an absorbing subuniverse?

YES, if the algebra of polymorphisms of $\mathbb{A}$ is SD($\wedge$) (for example, if $\mathbb{A}$ is a core and has bounded width).

Problem

Given a finite (finitely presented) algebra $\mathbf{A}$ and $B \subseteq A$, is it decidable if $B \trianglelefteq \mathbf{A}$?

Problem (This is my problem.)

Our proof provides an absorbing term of arity double exponential in the size of $\mathbf{A}$. Is there a better bound?

Libor has some problems...

Problem

Given a finite relational structure $\mathbb{A}$ and a subset $B \subseteq A$, is it decidable if B is an absorbing subuniverse?

YES, if the algebra of polymorphisms of $\mathbb{A}$ is SD($\wedge$) (for example, if $\mathbb{A}$ is a core and has bounded width).

Problem

Given a finite (finitely presented) algebra $\mathbf{A}$ and $B \subseteq A$, is it decidable if $B \trianglelefteq \mathbf{A}$?

Problem (This is my problem.)

Our proof provides an absorbing term of arity double exponential in the size of $\mathbf{A}$. Is there a better bound?

Libor has some problems...

Problem

Given a finite relational structure $\mathbb{A}$ and a subset $B \subseteq A$, is it decidable if B is an absorbing subuniverse?

YES, if the algebra of polymorphisms of $\mathbb{A}$ is SD($\wedge$) (for example, if $\mathbb{A}$ is a core and has bounded width).

Problem

Given a finite (finitely presented) algebra $\mathbf{A}$ and $B \subseteq A$, is it decidable if $B \trianglelefteq \mathbf{A}$?

Problem (This is my problem.)

Our proof provides an absorbing term of arity double exponential in the size of $\mathbf{A}$. Is there a better bound?

Libor has some problems...

Problem

Given a finite relational structure $\mathbb{A}$ and a subset $B \subseteq A$, is it decidable if B is an absorbing subuniverse?

YES, if the algebra of polymorphisms of $\mathbb{A}$ is SD($\wedge$) (for example, if $\mathbb{A}$ is a core and has bounded width).

Problem

Given a finite (finitely presented) algebra $\mathbf{A}$ and $B \subseteq A$, is it decidable if $B \trianglelefteq \mathbf{A}$?

Problem (This is my problem.)

Our proof provides an absorbing term of arity double exponential in the size of $\mathbf{A}$. Is there a better bound?

Libor has some problems...

Problem

Given a finite relational structure $\mathbb{A}$ and a subset $B \subseteq A$, is it decidable if B is an absorbing subuniverse?

YES, if the algebra of polymorphisms of $\mathbb{A}$ is SD($\wedge$) (for example, if $\mathbb{A}$ is a core and has bounded width).

Problem

Given a finite (finitely presented) algebra $\mathbf{A}$ and $B \subseteq A$, is it decidable if $B \trianglelefteq \mathbf{A}$?

Problem (This is my problem.)

Our proof provides an absorbing term of arity double exponential in the size of $\mathbf{A}$. Is there a better bound?

Thanks

Thank you for your attention!