Conservative Dichotomy Revisited

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Conservative CSP

Definition 1:

Instance: A triple $(V,\mathcal{L},\mathcal{C})$, where V is a set of variables, $\mathcal{C}$ is a set of constraints, and $\mathcal{L}$ is a set of lists L_v for each $v \in V$

Question: Is there a solution φ such that $\varphi(v) \in L_v$ for every $v \in V$

Definition 2:

$CSP(\Gamma)$ where Γ contains all the unary relations

Definition 3:

$CSP(\mathbb{A})$ where every operation of $\mathbb{A}$ is conservative

$(f(x_1,\ldots,x_n) \in \{x_1,\ldots,x_n\})$

Old Dichotomy

2003/2011 paper, about 80 pages

What has happened over the last 8 years:

- Generalized Majority-Minority algorithm (Dalmau, 2006)
- Maroti’s retraction (Maroti, this morning)
- Absorbing sets (Barto/Kozik, 2010)

Conservative Algebras: Edges

If CSP($\mathbb{A}$) is poly time, every 2-element subalgebra of A must have one of the Schaefer's operations: semilattice, majority, or affine

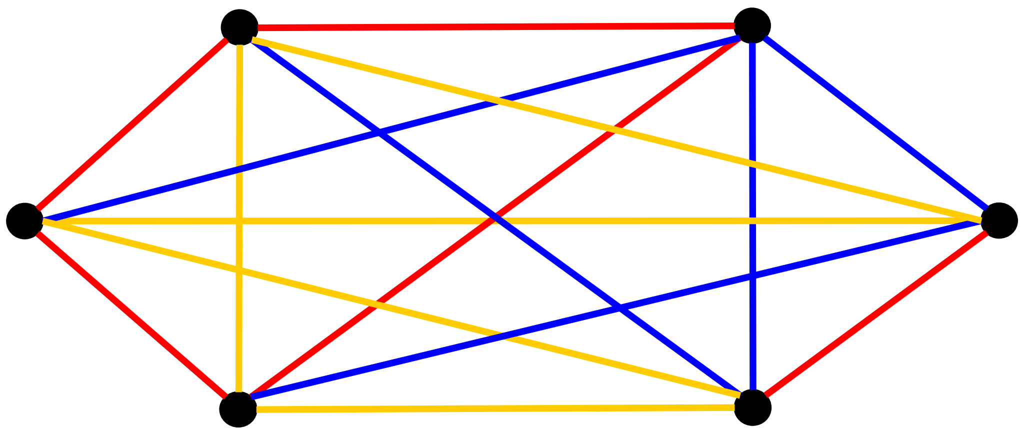
semilattice operation

majority operation, no semilattice

affine operation, no semilattice or majority

Edge Coloured Graphs

$G(\mathbb{A})$:



Theorem (B. 2003)
CSP($\mathbb{A}$) for a conservative $\mathbb{A}$ is poly time iff for any 2-element $B \subseteq A$ there is $f \in \mathrm{Term}(\mathbb{A})$, which is affine, majority, or semilattice; otherwise CSP($\mathbb{A}$) is NP-complete.

Edge Coloured Graphs II

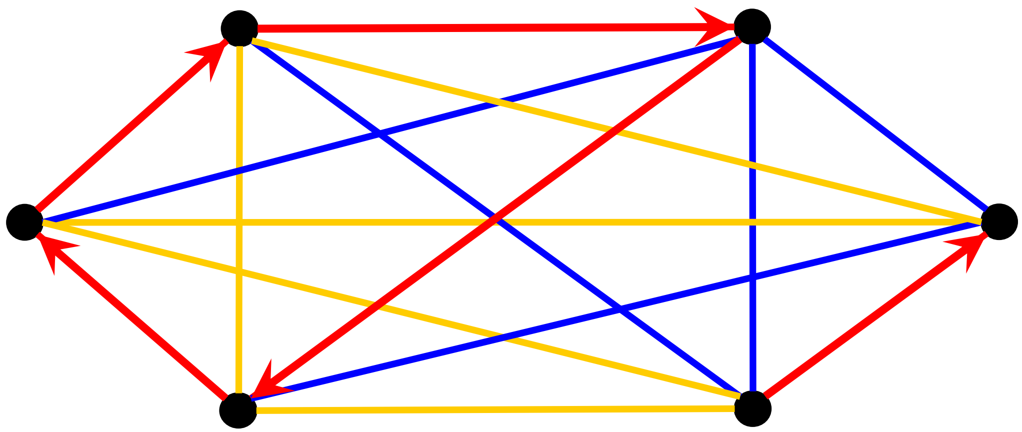
Lemma
There are f,g,h $\in$ Term($\mathbb{A}$) such that f is semilattice on every semilattice edge, g is majority on every majority edge, and h is affine on every affine edge.

Extra conditions:

- $f(x,y) = x$ on every majority edge
- $g(x,y,z) = x$ on every affine edge
- $h(x,y,z) = x$ on every majority edge

Edge Coloured Graphs III

$G(\mathbb{A})$:



As semilattice operation induces an order, red edges are directed

Will use $\cdot$ instead of f

Graphs of Relations

Sometimes we use more general graphs: $G(\mathbb{A})$ where $\mathbb{A}$ is not conservative, but a subdirect product of conservative algebras.

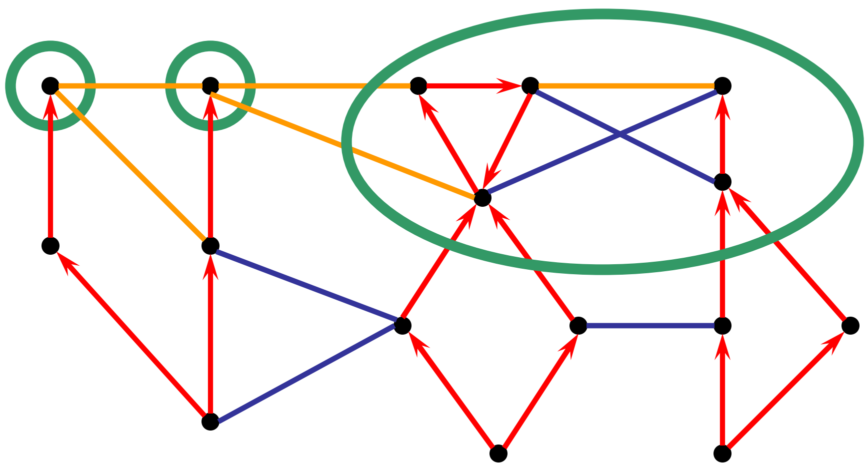
- $G(\mathbb{A})$ is constructed in the same way, edges are 2-element subalgeras.
- It is not complete, but any subalgebra induces a connected subgraph.
- All results and proofs remain the same with only minor tweaks

AS-Components

Let $\mathbb{A}$ be a conservative algebra

B $\subseteq$ A is called an as-component (affine-semilattice) if it is minimal with respect to the property:

there is no affine or semilattice (directed) edge in $G(\mathbb{A})$ sticking out of B



The remaining edges are majority

Reductions

We use two types of reductions:

- As-components exclusion

- Maroti's retractions

Linked Relations

Let $R \leq \mathbb{A} \times \mathbb{B}$.

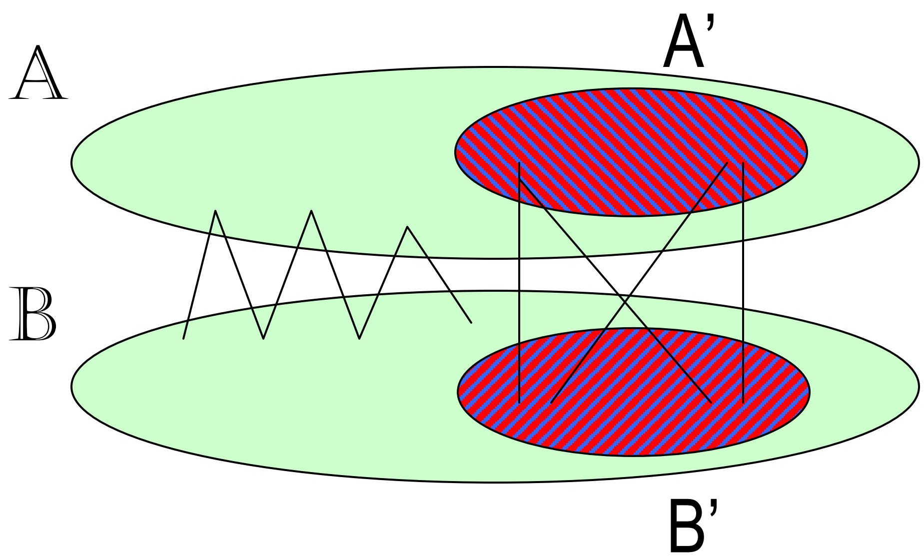
$\lambda_{\mathbb{A}}$ is a congruence on $\mathbb{A}$ defined as the transitive closure of the relation $\{(a,b) \mid (a,c),(b,c) \in R \text{ for some } c\}$;

$\lambda_{\mathbb{B}}$ is defined on $\mathbb{B}$ in the same way

R is linked if $\lambda_{\mathbb{A}}, \lambda_{\mathbb{B}}$ are total relations

Binary Rectangularity

Binary Rectangularity Lemma
Let $R \leq \mathbb{A} \times \mathbb{B}$ be linked, and let A', B' be as-components of $\mathbb{A}$ and $\mathbb{B}$, respectively, such that $(a,b) \in R$ for some $a \in A'$ and $b \in B'$. Then $A' \times B' \subseteq R$.



The Connectivity Lemma

Connectivity Lemma

Let $R \le \mathbb{A}_1 \times \ldots \times \mathbb{A}_n$, and let A'_i be an as-component of $\mathbb{A}_i$, $i \in [n]$, such that $(a_1, \ldots, a_n)$ for some $a_i \in A'_i$, $i \in [n]$. Then $R' = R \cap (A'_1 \times \ldots \times A'_n)$ is a subdirect product of the A'_i, and R' is an as-component of R.

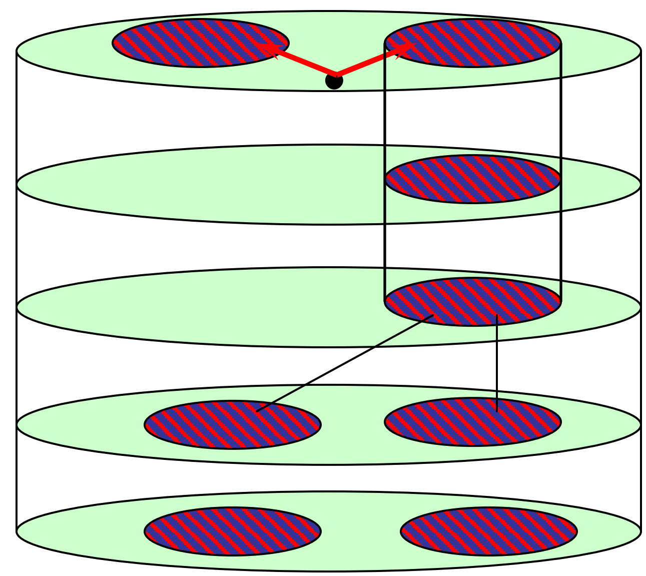
Strands

Let $R \leq \mathbb{A}_1 \times \ldots \times \mathbb{A}_n$ let $A_i \subseteq \mathbb{A}_i, A_j \subseteq \mathbb{A}_j$ be as-components

Positions i and j are A_i, A_j-related if for any $(a_1, \ldots, a_k) \in R$

$a_i \in A_i$ iff $a_j \in A_j$

$I \in \{1,...,k\}$ is a strand w.r.t. as-components $A_1, \ldots, A_k$ if any $i,j \in I$ are A_i, A_j-related

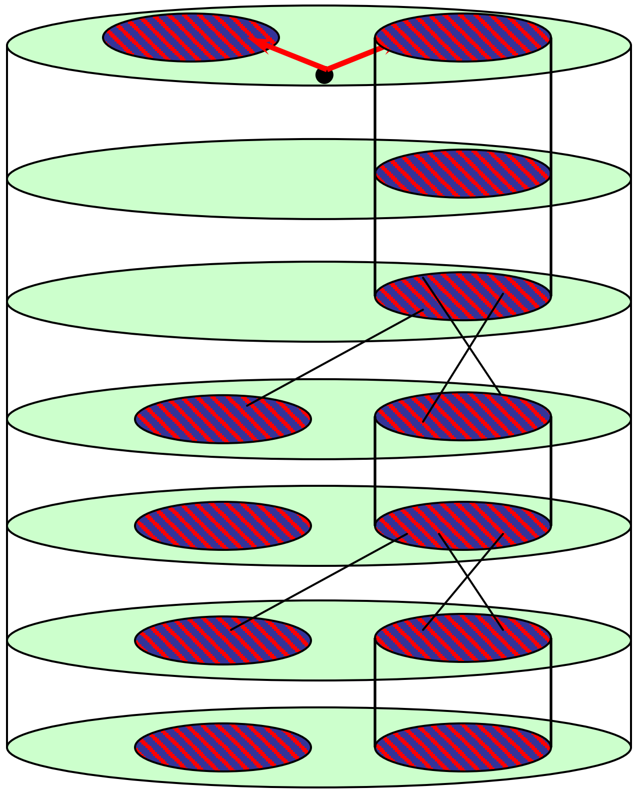


Rectangularity

Rectangularity Lemma

Let $R \leq \mathbb{A}_1 \times \ldots \times \mathbb{A}_n$ and $A_1, \ldots, A_k$ as-components such that $R \cap (A_1 \times \cdots \times A_k) \neq \varnothing$. Let also $I_1, \ldots, I_k$ be the partition of {1,...,n} into strands w.r.t. $A_1, \ldots, A_s$ and $R_i = pr_{I_i} R \cap \prod_{j \in I_i} A_j$.

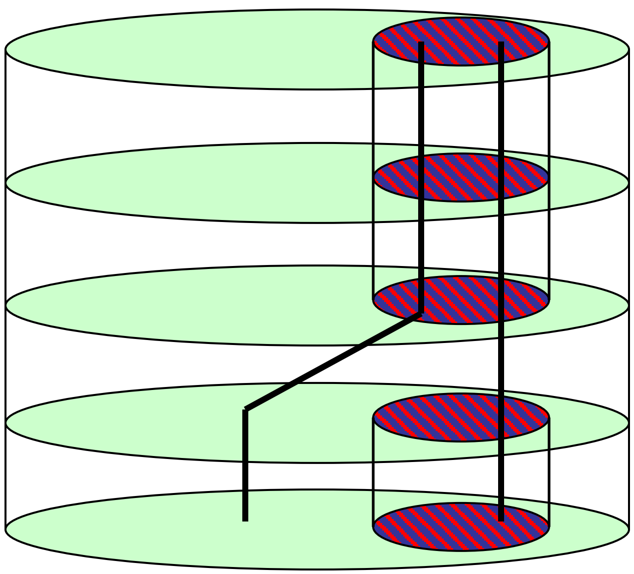
Then $R_1 \times \cdots \times R_s \subseteq R$.



Rectangularity: to Binary Relations

Let $R \le \mathbb{A}_1 \times \ldots \times \mathbb{A}_n$ and $A_1, \ldots, A_n$ be as-components such that there is $(a_1, \ldots, a_n) \in R$ with $a_i \in A'_i$
Suppose 1 and n belong to different strands. Then there are $(a,b),(a',b') \in pr_{1,n} R$ with $a, a' \in A'_1$, $b \in A'_n$, $b' \in A_n - A'_n$.

Take $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in R$
with $a_1 = a, b_1 = a', \ a_n = b, \ b_n = b'$
Let $a_i \in A'_i$ for $i \in [m]$ and
$a_i \in A_i - A'_i$ otherwise
Consider R as a binary relation on
$pr_{[m]}R$ and $pr_{\{m+1,\ldots,n\}}R$

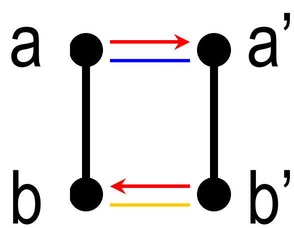


Rectangularity: Binary Relations

Let $R \le \mathbb{A} \times \mathbb{B}$, and A',B' as components of A,B such that there are $(a,b),(a',b') \in R$ with $a,a' \in A'$, $b \in B'$, $b' \in B - B'$.

We prove that the pairs can be chosen such that $(a',b) \in R$.

There is a sequence in R



Since bb' is a majority or semilattice edge, either

$$\genfrac{}{}{0pt}{}{a'}{b} = \genfrac{}{}{0pt}{}{a}{b} \cdot \genfrac{}{}{0pt}{}{a'}{b'} \in R \quad \text{or} \quad \genfrac{}{}{0pt}{}{a'}{b} = h\left(\genfrac{}{}{0pt}{}{a}{b}, \genfrac{}{}{0pt}{}{a}{b}, \genfrac{}{}{0pt}{}{a'}{b'}\right) \in R$$

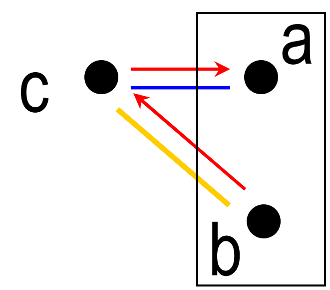
Rectangularity: Congruences of AS-Components

Let A' be an as-component of $\mathbb{A}$, and λ a congruence such that $(a,b) \in \lambda$ for some $a \in A'$ and $b \in A - A'$. Then A' is in a λ-block.

If λ is nontrivial on A', choose $c \in A'$ with (a,c) not in λ and such that ca is either semilattice or affine. Since $b \in A - A'$, bc is either semilattice or majority.

Then $(b \cdot c, b) \in \lambda$, so $b \cdot c = b$, and bc is majority.

Then $\begin{matrix} a \\ c \end{matrix} = g\left(\begin{matrix} a \\ b \end{matrix}, \begin{matrix} c \\ c \end{matrix}, \begin{matrix} c \\ c \end{matrix}\right) \in \lambda$



Rectangularity: Binary Relations II

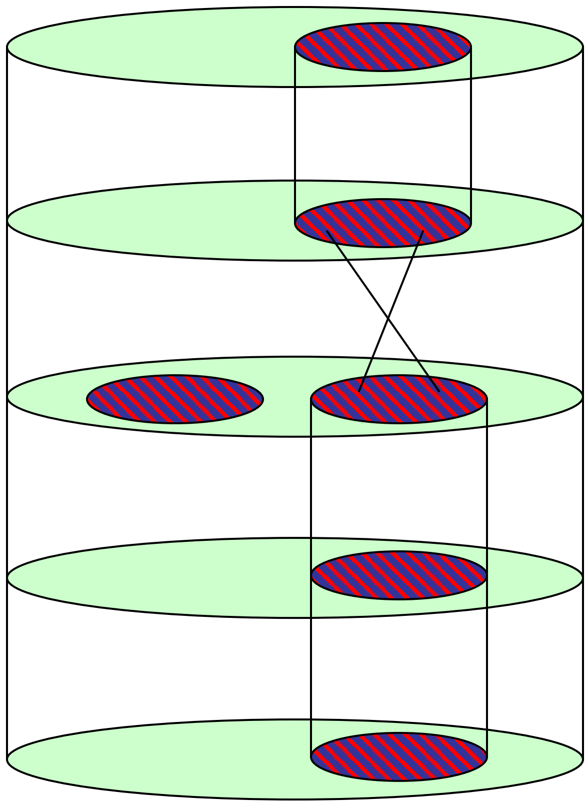
- Let $R \leq A \times B$, and A', B' as-components of $\mathbb{A}, \mathbb{B}$ such that the coordinate positions of R are not A', B'-related. W.l.o.g. there are $(a,b),(a,b') \in R$ with $a \in A'$, $b \in B'$, $b' \in B - B'$.

- Let λ be the link congruence on B. B' is inside a λ-block.

- Therefore if R' denotes the restriction of R on the link congruences blocks containing A' and B', R' is linked.

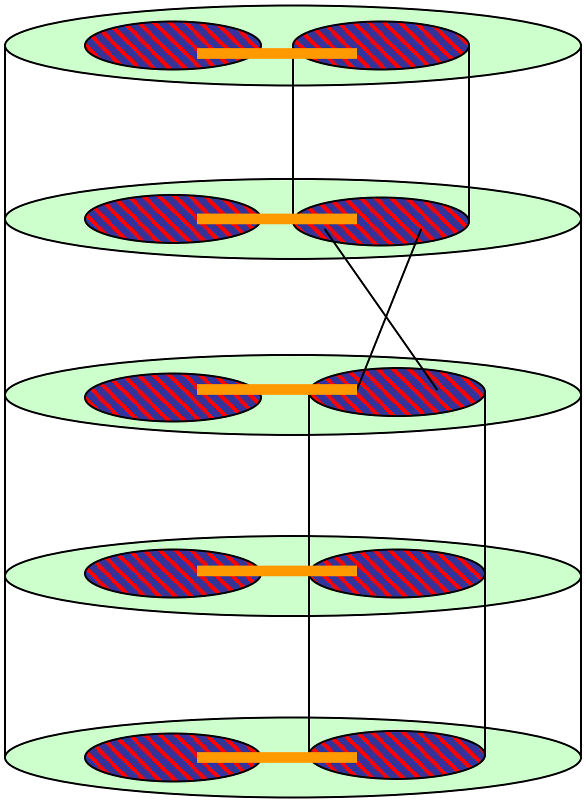
- By the Binary Rectangularity Lemma $A' \times B' \subseteq R$

AS-Components Exclusion



1. find as-components $A_v, v \in V$, such that for any $v,w \in V$ as-components A_v, A_w are consistent
2. find the strands
3. for each strand W solve the problem restricted to W and $A_w, w \in W$
4. if every such problem has a solution, by the Rectangularity Lemma any combination of such solutions gives a solution to the problem
5. otherwise remove elements of the failed as-components

Finding Strands and Components



2. To find a strand take a variable v and an as-component A of A_v and find all variables w and as-components B of A_w such that v,w are A,B-related

1.

Chinese Remainder Theorem

Let $R \leq \mathbb{A}_1 \times \ldots \times \mathbb{A}_n$ and let $A'_1, \ldots, A'_n$ be as-components of $\mathbb{A}_1, \ldots, \mathbb{A}_n$ such that for any $i,j \in \{1,...,n\}$ there is a tuple $(a_1, \ldots, a_k) \in R$ such that $a_i \in A'_i, a_j \in A'_j$.
Then $R \cap (A'_1 \times \ldots \times A'_n)$ is a sudirect product of $A'_1, \ldots, A'_n$

Maroti's Retractions

Let $\mathcal{A}$ be a class of algebras of similar type closed under taking subalgebras and retracts via idempotent unary polynomials.

Let $\mathbb{A} \in \mathcal{A}$ and f a binary term operation of $\mathbb{A}$ such that

a. $f(x,f(x,y)) = f(x,y)$;

b. for each $a \in \mathbb{A}$ the map $x \mapsto f(x,a)$ is not surjective

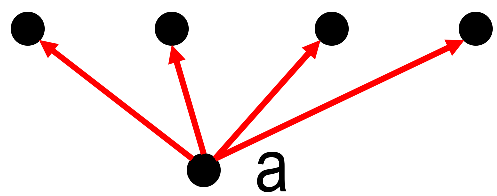
c. the set C of $a \in \mathbb{A}$ such that $x \to f(x,a)$ is surjective generates a proper subalgebra of $\mathbb{A}$

Then $\mathrm{CSP}(\mathcal{A})$ is poly time reducible to $\mathrm{CSP}(\mathcal{A} - \{\mathbb{A}\})$.

Reductions II

a. $x \cdot (x \cdot y) = x \cdot y$

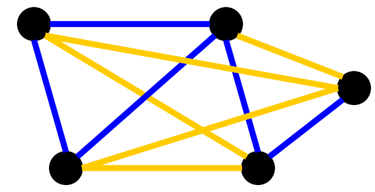
b. If $a \in \mathbb{A}$ is such that $x \longrightarrow a \cdot x$ is surjective, then $a \cdot x = x$



a does not belong to any as-component, and we can use as-components exclusion

Reductions III

c. If $x \to x \cdot a$ is surjective for each a, then $x \cdot a = x$, and $\mathbb{A}$ has no semilattice edges.



An operation $m(x,y,z)$ on A is called Generalized Majority Minority if for any $a,b \in A$,
either $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$ for $x,y \in \{a,b\}$,
or $m(x,x,y) = m(y,x,x) = y$
Operation $g(h(x,y,z),y,z)$ is majority on each majority edge, and is affine on each affine edge

Then there is a ternary GMM operation on $\mathbb{A}$.

Question

Are as-components and minimal absorbing subuniverses the same??