Homogeneous Structures and Siggers Terms

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CNRS / LIX, École Polytechnique

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Overview

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Ramsey theoretic method to study Pol(Γ)
when Γ is the reduct of a finitely bounded homogeneous structure.

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4. Plan towards a tractability conjecture

Reminder: Infinite Domain CSPs

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Γ also called the template.

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Fact (B.+Grohe'08): For every problem $\mathcal{P}$ there is a Γ such that $\mathcal{P}$ and CSP(Γ) are polynomial-time equivalent.

Homogeneous Structures

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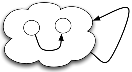
A structure is called homogeneous if isomorphisms between finite substructures can be extended to automorphisms.



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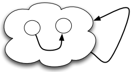

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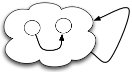

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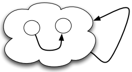

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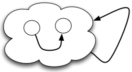

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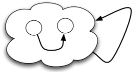

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- For reducts Γ of finitely bounded structures, CSP(Γ) is in NP.

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Conjecture 1.

If the tractability conjecture is true for finite templates,
then it is also true for reducts of finitely bounded homogeneous structures.

Set Constraints

Variables $x, y, u, v, \ldots$ denote sets.
Constraints of the form

(A) $x \cap y = \emptyset$ (x and y are disjoint),

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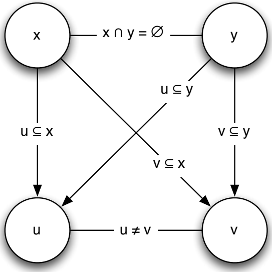

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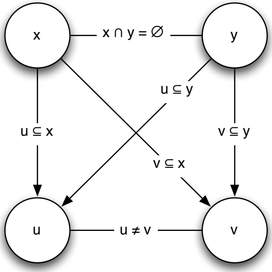

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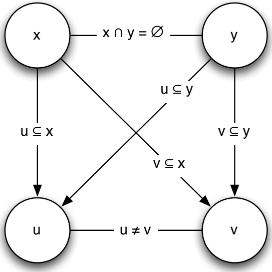

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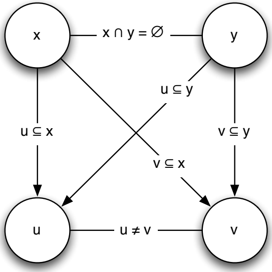

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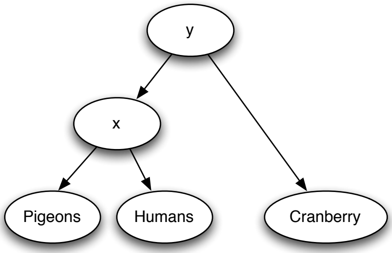
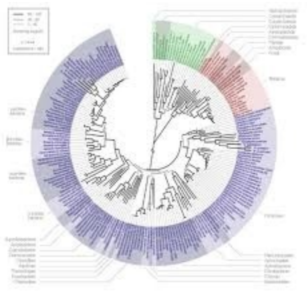
- Still tractable when we additionally allow constraints of the form $x \cap y \subseteq z$?

Phylogenetic Trees





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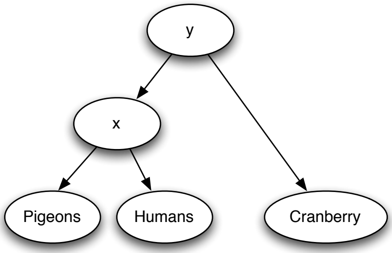

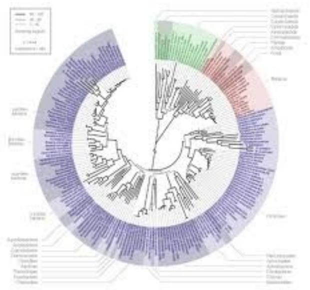

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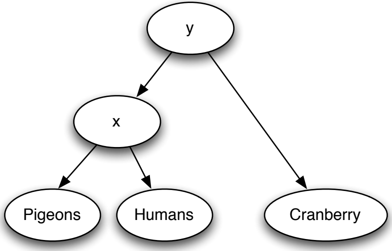

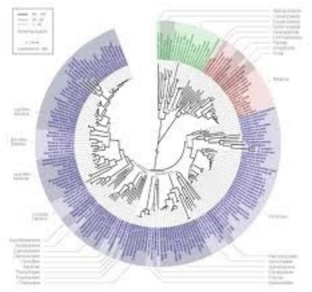

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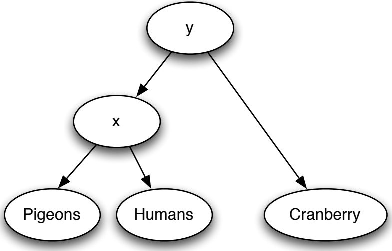

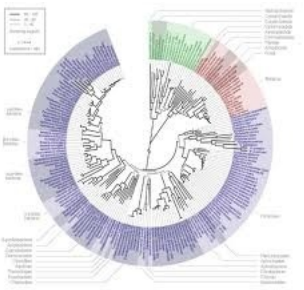

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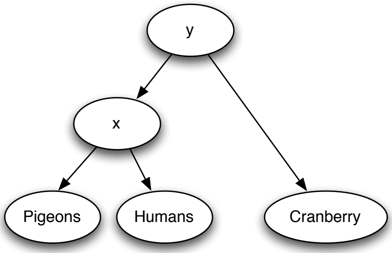

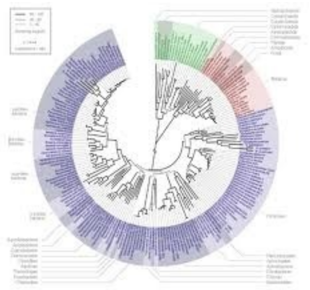

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- Example: pigeons humans | cranberries

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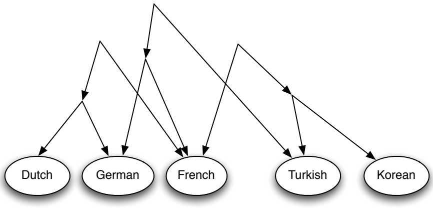

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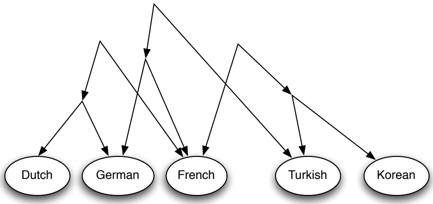

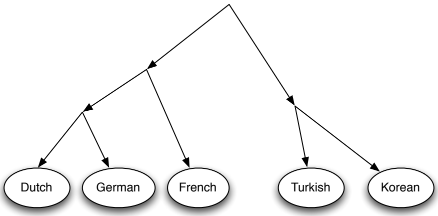

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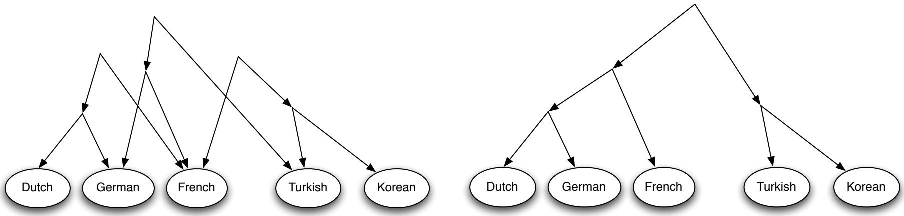

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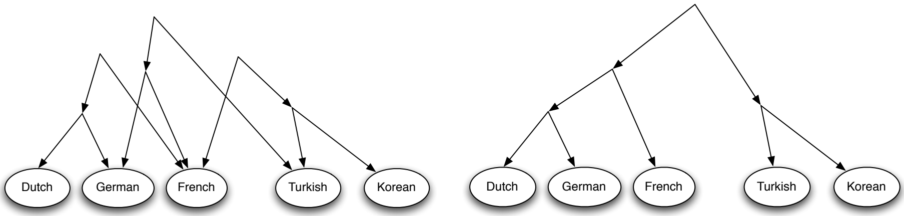

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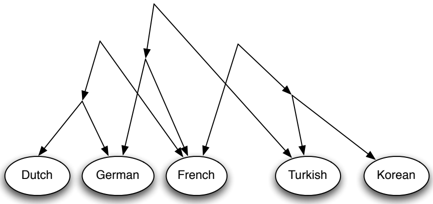

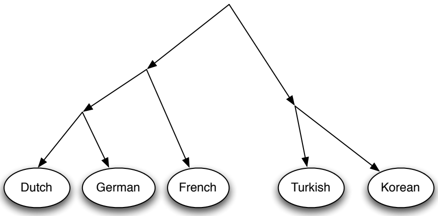

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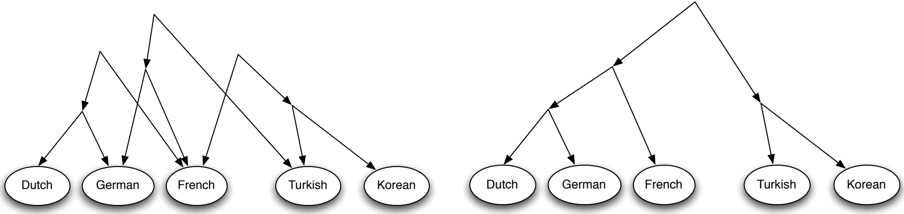

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- Cannot be solved by Datalog (B.+Mueller'09)

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All reducts of $(V; E)$ have less than $3^{\binom{n}{2}}$ many orbits of n-tuples, and hence are ω-categorical.

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Example 2: The random graph is not a core.

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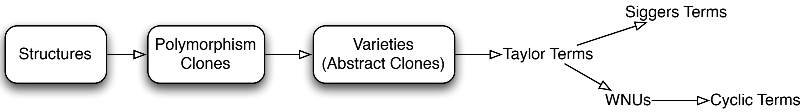
Corollary: Expansions of model-complete cores by FINITELY many constants have the same CSP complexity

The Universal-Algebraic Approach



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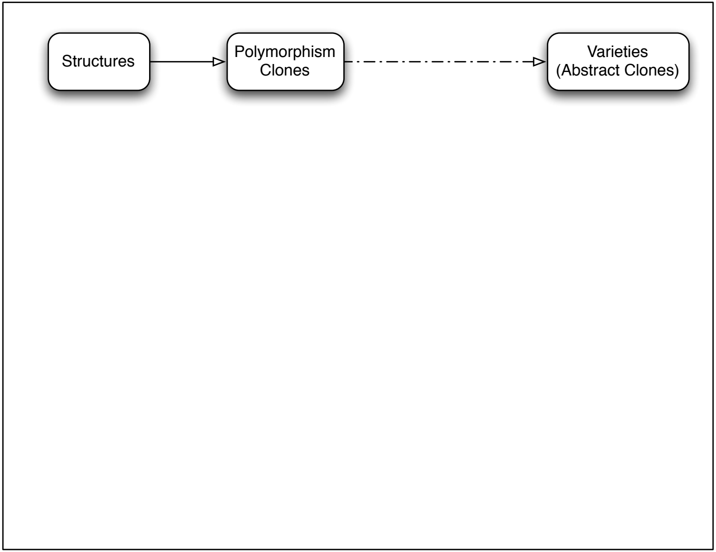

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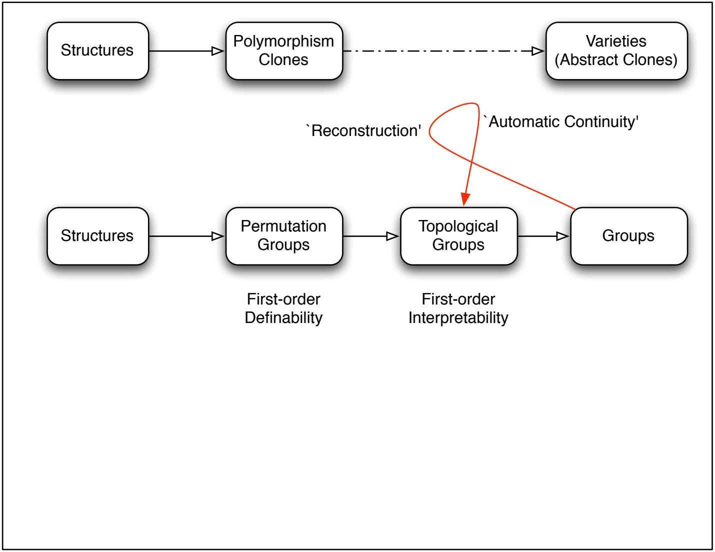

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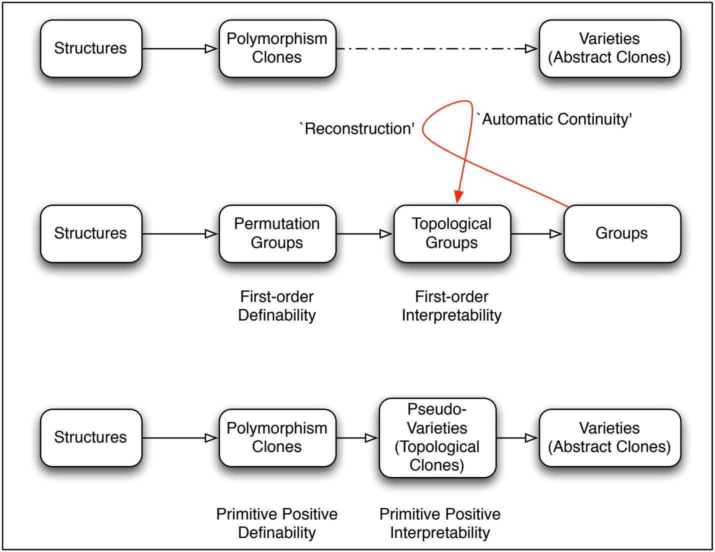

Primitive Positive Interpretations

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A σ-structure Γ has an interpretation in a τ-structure Δ if there is a $d \geq 1$, and

- a τ-formula $\delta_I(x_1, \ldots, x_d)$,
- for each atomic σ-formula $\phi(y_1, \ldots, y_k)$ a τ-formula $\phi_I(\overline{x}_1, \ldots, \overline{x}_k)$,
- a surjective map $h\colon \delta_I(\Delta^d) \to \Gamma$,

such that for all atomic σ-formulas ϕ and all $\overline{a}_i \in \delta_I(\Delta^d)$

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Fact: When there is a primitive positive interpretation of Γ in Δ, then there is a polynomial-time reduction from $\mathrm{CSP}(\Gamma)$ to $\mathrm{CSP}(\Delta)$.

PP Interpretations: Example

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- One dimension does not suffice
- This is unlike the situation for finite structures with idempotent polymorphism clone!

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Empirical: All hardness proofs for reducts Γ of finitely bounded homogenous templates are via pp interpretations of

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Question: Does the complexity of $\mathrm{CSP}(\Gamma)$ only depend on the variety generated by $\mathrm{Pol}(\Gamma)$?

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Two ω-categorical structures have isomorphic topological automorphism groups if and only if they are first-order bi-interpretable.

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- Every Baire measurable homomorphism between Polish groups is continuous. And there exists a model of ZF+DC where every set is Baire measurable (Shelah'84).

Topological Clones

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A **topological clone** is an abstract clone with a topology on the elements so that composition is continuos.

For polymorphism clones: use the topology of pointwise convergence.
So for $\text{Pol}(\Gamma)$, a basis is given by the set of all sets of the form

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So for Pol(Γ), a basis is given by the set of all sets of the form

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- The topological clone of Γ determines the finitely related members of the pseudo-variety generated by the polymorphism algebra of Γ (for related results for endomorphism monoids, see (B.+Junker'10))
- When can we reconstruct the topological polymorphism clone of Γ from its abstract clone?

The Finite Dimension Property

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Definition (FDP)

An ω-categorical structure Δ has the FDP if there are $c, d \in \mathbb{N}$ such that for every model-complete core of a reduct Γ of Δ the following are equivalent:

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- The equivalence relation with infinitely many infinite classes (Wrona'11).

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Do all finitely bounded homogeneous structures have the FDP?

From Abstract Clones to Taylor terms

Assume that Γ is finite, and that $\mathrm{Pol}(\Gamma)$ is idempotent.

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Theorem (Taylor'77,Hobby+McKenzie'88,McKenzie+Maroti'08,Siggers'10).

The following are equivalent.

1. Γ does not pp-interpret $\big(\{0,1\}; \{(0,0,1),(0,1,0),(1,0,0)\}\big)$
2. Γ is preserved by a Taylor operation, i.e., an n-ary f s.t. for every $1 \leq i \leq n$ there are $x_1,\ldots,x_n,y_1,\ldots,y_n \in \{x,y\}$ satisfying
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(One) problem in the ω-categorical: cannot assume idempotency.

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Recall from Michael's talk:

CSP(Γ) is tractable iff Γ is preserved by one out of 17 operations: (long list)

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Recall from Michael's talk:

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Theorem 5 (B.+Pinsker'11).

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- Γ has a 4-ary (canonical) polymorphism f and $\alpha_1, \alpha_2 \in \mathrm{Aut}(G)$ such that for all $x, y \in V$

$$f(y,y,x,x) = \alpha_1 f(x,x,x,y) = \alpha_2 f(y,x,y,x) \; ,$$

  and CSP(Γ) is in P.

An Ideal World: Canonical Operations

Definition (Canonical Operations)

Let Γ be ω-categorical with domain D. An operation $f : D^k \to D$ is canonical if for all $m \geq 1$ and all $t_1, \ldots, t_k \in D^m$, the m-type of $f(t_1, \ldots, t_k)$ only depends on the m-types of the tuples $t_1, \ldots, t_k$.

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- the variety generated by the polymorphism algebra $\mathbf{A}$ of Γ contains a 2-element algebra all of whose operations are projections, or

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Proof. Let *p* be the number of *m*-types in Δ. $\mathbf{A}^m$ is homomorphic to an (idempotent) algebra $T(\mathbf{A})$ (the type algebra) with *p* elements, and there is an *f* such that

$$T(\mathbf{A}) \models f(x_1, \ldots, x_n) = f(x_{i_1}, \ldots, x_{i_n})$$

if and only if there exists an $\alpha \in \mathrm{Aut}(\Gamma)$ and a $g \in \mathrm{Aut}(\Gamma)$ such that

$$\mathbf{A} \models g(x_1, \ldots, x_n) = \alpha\, g(x_{i_1}, \ldots, x_{i_n}) \, .$$

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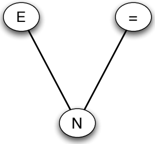

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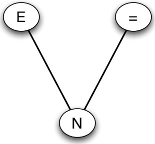

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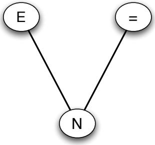

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But by compactness, $\mathrm{Aut}(G) \cup \{f\}$ generates g, α such that $g(x, y) = \alpha g(y, x)$.

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And/Or Precedence Constraints:

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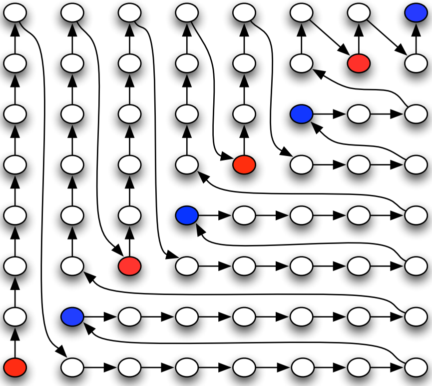

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4. Reduce $\mathrm{CSP}(\Gamma)$ to the CSP for the (finite) type algebra $T(\mathrm{Pol}(\Gamma'))$ for an expansion Γ' of Γ by finitely many constants.

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