

# Wolff potentials and regularity of solutions to integral systems on homogeneous spaces

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## Introduction

Wolff potentials on homogeneous spaces  
Lane-Emden type integral system

## Main results

HLS type inequality for Wolff potentials  
Integrability estimates  
Lipschitz continuity estimates

## Main tools

Dyadic cubes on homogeneous spaces  
Regularity lifting  
Modified version of regularity lifting

- ▶ Based on systematic use of Wolff potentials, Phuc and Verbitsky [PV] (2008) studied p-Laplacian equations

$$-\Delta_p u = -\operatorname{div}(\nabla u |\nabla u|^{p-2}) = u^q$$

and Hessian equations

$$F_k[-u] = u^q.$$

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we will concentrate on some analogous results on homogeneous spaces.

A quasi-metric  $d$  on a set  $\mathbb{X}$  is a mapping  $d : \mathbb{X} \times \mathbb{X} \rightarrow [0, \infty)$  satisfying

- (i)  $d(x, y) = 0$  if and only if  $x = y$ ;
- (ii)  $d(x, y) = d(y, x)$  for all  $x, y \in \mathbb{X}$ ;
- (iii)  $d(x, z) \leq k_1(d(x, y) + d(y, z))$  for all  $x, y, z \in \mathbb{X}$  and some constant  $k_1 \in [1, \infty)$  independent of  $x, y$  and  $z$ .

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Such quasi-metric  $d$  defines a topology on  $\mathbb{X}$ , for which the balls  $B_t(x) = \{y \in \mathbb{X} : d(x, y) < t\}$  form a base. Then homogeneous spaces introduced by Coifman and Weiss [CW] is defined as follows.

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### Definition 1 (Homogeneous spaces)

A space of homogeneous type  $(\mathbb{X}, d, \mu)$  is a set  $\mathbb{X}$  equipped with a quasi-metric  $d$  and a doubling measure  $\mu$ , that is,  $\mu$  is a locally finite nonnegative measure on Borel subsets of  $\mathbb{X}$  satisfying  $\mu(B_{2t}(x)) \leq k_2\mu(B_t(x))$  for all balls  $B_t(x) \subseteq \mathbb{X}$  and some constant  $k_2 \in (0, \infty)$  independent of  $x$  and  $t$ .

In [MS], Macias and Segovia have proved that one can replace the quasi-metric  $d$  by another quasi-metric  $d^* \approx d$  such that  $d^*$  yields the same topology on  $\mathbb{X}$  as  $d$  does

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$$\mu(B_r(x)) \sim r^N \quad (1.1)$$

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where  $B_r(x) = \{y \in \mathbb{X} : d^*(y, x) < r\}$  and  $d^*$  has the following regularity property

$$|d^*(x, y) - d^*(x', y)| \leq C_0 d^*(x, x')^\theta [d^*(x, y) + d^*(x', y)]^{1-\theta} \quad (1.2)$$

for some regularity exponent  $\theta : 0 < \theta < 1, 0 < r < \infty$  and all  $x, x', y \in \mathbb{X}$ .

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From now on, we drop the  $*$  in the quasi-metric  $d^*$  and simply assume  $d$  satisfies (1.1) and (1.2). We also call  $N$  as the **homogeneous dimension** of  $(\mathbb{X}, d, \mu)$ .

- ▶ The continuous truncated version of Wolff potentials on homogeneous spaces for  $\omega \in \mathbb{M}^+(\mathbb{X})$  is defined as

$$W_{\alpha,p}^r \omega(x) = \int_0^r \left[ \frac{\omega(B_t(x))}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t}.$$

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- ▶ Similarly, we define the continuous version

$$W_{\alpha,p} \omega = W_{\alpha,p}^\infty \omega$$

and the discrete version

$$W_{\alpha,p}^D \omega(x) = \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \left[ \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \chi_Q(x),$$

using the dyadic construction on homogeneous spaces by Christ [C] and Sawyer and Wheeden [SW].

Consider a Lane-Emden type integral system, that is,

$$\begin{cases} u = W_{\alpha,p}(v^{q_2}), \\ v = W_{\alpha,p}(u^{q_1}), \end{cases} \quad (1.3)$$

under the (critical) condition

$$\frac{p-1}{q_1+p-1} + \frac{p-1}{q_2+p-1} = \frac{N-\alpha p}{N}, \quad (1.4)$$

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## Remark

When  $v = u$  and  $q_1 = q_2 = q$ , (1.3) is reduced to

$$u = W_{\alpha,p}(u^q),$$

*which is the Lane-Emden type integral equation. Given special pairs of  $\alpha$  and  $p$ , the equation deduces  $p$ -Laplacian equations and Hessian equations.*

One of our main theorems is as follows.

### Theorem 2.1

Let  $\alpha > 0$ ,  $1 < p < \infty$ ,  $q > p - 1$ ,  $\omega \in \mathbb{M}^+(\mathbb{X})$  and  $0 < r \leq \infty$ , then

$$\|W_{\alpha,p}^r \omega\|_{L^q(d\mu)}^q = \int_{\mathbb{X}} \left\{ \int_0^r \left[ \frac{\omega(B_t(x))}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right\}^q d\mu \quad (2.1)$$

$$\simeq \|I_{\alpha p}^r \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} = \int_{\mathbb{X}} \left[ \int_0^r \frac{\omega(B_t(x))}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \frac{dt}{t} \right]^{\frac{q}{p-1}} d\mu. \quad (2.2)$$

By a HLS inequality proved by Sawyer and Wheeden [SW] (also see Sawyer, Wheeden and Zhao[SWZ]) for Riesz potentials on homogeneous spaces, it is not difficult to derive the following HLS type inequality for Wolff potentials.

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### Theorem 2.2 (HLS type inequality for Wolff potentials)

Let  $\alpha > 0$ ,  $1 < p < \infty$ ,  $q > p - 1$  and  $\alpha p < N$ . If  $f \in L^s(d\mu)$  for  $s > 1$ , then

$$\|W_{\alpha,p}(f)\|_{L^q(d\mu)} \leq C \|f\|_{L^s(d\mu)}^{\frac{1}{p-1}},$$

where  $\frac{p-1}{q} = \frac{1}{s} - \frac{\alpha p}{N}$ .

This inequality can be applied to study the Lane-Emden type integral system (1.3).

Our main regularity theorems state

### Theorem 2.3 (Integrability estimates)

Let  $\alpha > 0$ ,  $1 < p \leq 2$ ,  $\alpha p < N$  and  $q_1, q_2 > 1$ , assume that  $(u, v)$  is a pair of positive solutions of (1.3) and (1.4) satisfying  $(u, v) \in L^{q_1+p-1}(d\mu) \times L^{q_2+p-1}(d\mu)$ , then

$(u, v) \in L^{s_1}(d\mu) \times L^{s_2}(d\mu)$  for all  $s_1$  and  $s_2$  such that  $\frac{1}{s_1}$  belongs to

$$\left(0, \frac{p}{q_1 + p - 1}\right) \cap \left(\frac{1}{q_1 + p - 1} - \frac{1}{q_2 + p - 1}, \frac{p-1}{q_2 + p - 1} + \frac{1}{q_1 + p - 1}\right)$$

and  $\frac{1}{s_2}$  belongs to

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## Theorem 2.4 ( $L^\infty$ estimates)

*Under the same conditions in Theorem 2.3,  $u$  and  $v$  are both uniformly bounded on  $\mathbb{X}$ .*

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*Under the same conditions in Theorem 2.3, furthermore assume that  $k_1 = 1$ , then  $u$  and  $v$  are both Lipschitz continuous on  $\mathbb{X}$ , that is,  $u, v \in C^{0,1}(d\mu)$ .*

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## Remark

*Theorem 2.2, 2.3, 2.4 and 2.5 also hold for Euclidean spaces  $R^n$  and Heisenberg group  $H^n$ .*

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## Lemma 2 (Dyadic cubes on homogeneous spaces)

*For every integer  $k \in \mathbb{Z}_+$ , there exists a collection of open subsets  $\{Q_\tau^k \subseteq \mathbb{X} : \tau \in I_k\}$ , where  $I_k$  denotes some index set depending on  $k$ , and  $c_1, c_2 > 0$  such that*

- (i)  $\mu(\{X \setminus \cup Q_\tau^k\}) = 0$ ;
- (ii) If  $\ell \geq k$ , then for all  $\tau' \in I_\ell$  and  $\tau \in I_k$  either  $Q_{\tau'}^\ell \subseteq Q_\tau^k$  or  $Q_{\tau'}^\ell \cap Q_\tau^k = \emptyset$ ;
- (iii) If  $\ell < k$ , for each  $\tau \in I_k$ , there is a unique  $\tau' \in I_\ell$  such that  $Q_\tau^k \subseteq Q_{\tau'}^\ell$ ,  $\text{diam}(Q_\tau^k) \leq c_1 2^{-k}$ , and each  $Q_{\tau'}^\ell$  contains some ball  $B(z_{\tau'}^k, c_2 2^{-k})$ .

- We say that a cube  $Q \subseteq \mathbb{X}$  is a dyadic cube if  $Q = Q_\tau^k$  for some  $k \in \mathbb{Z}_+$ ,  $\tau \in I_k$  and  $\text{diam}(Q) \sim 2^{-k}$ .

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- ▶ For  $\alpha > 0$ ,  $1 < p < \infty$  and  $\omega \in M^+(\mathbb{X})$ , we define the discrete Wolff potentials on homogeneous space  $\mathbb{X}$  by

$$W_{\alpha,p}^D \omega(x) = \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \left[ \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \chi_Q(x).$$

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- ▶ When  $\alpha = \lambda/2$  and  $p = 2$ , the discrete Riesz follows as

$$I_\lambda^D \omega(x) = \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\lambda}{N}}} \chi_Q(x).$$

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- ▶  $\|W_{\alpha,p}^D \omega\|_{L^q(d\mu)}^q \simeq \|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}}.$

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- ▶ Suppose  $V$  is a topological vector space with two extended norms,

$$\|\cdot\|_X, \|\cdot\|_Y : V \rightarrow [0, \infty],$$

let  $X := \{v \in V : \|v\|_X < \infty\}$  and  
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- ▶ The operator  $T : X \rightarrow Y$  is said to be **contracting** if

$$\|Tf - Th\|_Y \leq \eta \|f - h\|_X,$$

$\forall f, h \in X$  and some  $0 < \eta < 1$ .

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- ▶ And  $T$  is said to be **shrinking** if

$$\|Tf\|_Y \leq \theta \|f\|_X,$$

$\forall f \in X$  and some  $0 < \theta < 1$ .

### Theorem 3.1 (Regularity lifting by contracting operators)

*[MCL](2011) Let  $T$  be a contracting operator from  $X$  to itself and from  $Y$  to itself, and assume that  $X, Y$  are both complete. If  $f \in X$ , and there exists  $g \in Z := X \cap Y$  such that  $f = Tf + g$  in  $X$ , then  $f \in Z$ .*

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### Remark

We apply Theorem 3.1 to prove Theorem 2.3 by letting  $X = L^{q_1+p-1}(d\mu) \times L^{q_2+p-1}(d\mu)$  and  $Y = L^{s_1}(d\mu) \times L^{s_2}(d\mu)$ .

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- ▶ Two normed subspaces  $X$  and  $Y$  are called an “XY-pair”, if whenever the sequence  $\{u_n\} \subseteq X$  with  $u_n \rightarrow u$  in  $X$  and  $\|u_n\|_Y \leq C$  will imply  $u \in Y$ .

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## Remark

*There are some “XY-pairs” of important spaces, and the pair we use here is  $L^\infty$  and  $C^{0,1}$ .*

## Theorem 3.2 (Regularity lifting by combinations of contracting and shrinking operators)

[MCL] (2011) *Let  $X$  and  $Y$  be an “XY-pair”, and assume that  $X, Y$  are both complete. Let  $A$  and  $B$  be closed subsets of  $X$  and  $Y$  respectively, and  $T$  be an operator, which is contracting from  $A$  to  $X$  and shrinking from  $B$  to  $Y$ . Define  $Sw = Tw + g$  for some  $g \in A \cap B$ , and assume that  $S : A \cap B \rightarrow A \cap B$ . Then there exists a unique solution  $u$  of the equation  $w = Tw + g$  in  $A$ , and  $u \in Y$ .*

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$$u \in A \subseteq X \xrightarrow{\text{Regularity lifting}} u \in Y$$

Thank you!



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