

The L^p Dirichlet problem for second-order, non-divergence form operators: solvability and perturbation results

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The setting

Our work is concerned with second-order, linear, non-divergence form, uniformly elliptic operators \mathcal{L} on a bounded Lipschitz domain $D \subset \mathbb{R}^n$, with $n \geq 2$. That is, $\mathcal{L} = a^{ij} \partial_{ij}$, where $A(x) = (a^{ij}(x))_{i,j}$ is a symmetric matrix with ellipticity constant $0 < \lambda < \infty$ such that for all $x, \xi \in \mathbb{R}^n$,

$$\lambda |\xi|^2 \leq \xi^t A(x) \xi \leq \lambda^{-1} |\xi|^2.$$

The problem we consider is the Dirichlet problem

$$\begin{aligned} \mathcal{L}u &= 0 && \text{in } D, \\ u &= g && \text{on } \partial D, \end{aligned}$$

where $g \in L^p(\partial D, d\sigma)$ (σ is surface measure on ∂D and is the assumed measure on ∂D unless otherwise stated).

Non-divergence form difficulties

Unlike the divergence-form case, if A is simply bounded and measurable, we are not guaranteed a unique solution. More care needs to be taken, so, following Rios [Ri03], we define

Definition (\mathcal{CD})

Given an operator \mathcal{L} and a domain D , we say that \mathcal{CD} holds for \mathcal{L} on D if, for every continuous function g on ∂D , there exist a unique solution u and some $1 \leq q \leq \infty$ such that $u \in C(\overline{D}) \cap W_{loc}^{2,q}(D)$.

Chiarenza-Frasca-Longo [CFL] showed that if the coefficients a^{ij} are in VMO, then \mathcal{CD} holds for any $1 < q < \infty$. This can be extended to allow for coefficients that lie in BMO_ρ , in which case \mathcal{CD} holds for any $1 < q < q_0(\rho)$.

The fundamental result

We start with the following result of Dahlberg [Da77]

Theorem

Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then

- (i) ω (harmonic measure on ∂D) and σ are mutually absolutely continuous.
- (ii) There exists $\epsilon = \epsilon(D) > 0$ such that if $2 - \epsilon < p \leq \infty$ and $g \in L^p(\partial D)$, then $\Delta u = 0$, $u|_{\partial D} = g$ can be uniquely solved for a u satisfying $Nu(Q) = \sup_{x \in \Gamma(Q)} |u(x)|$ in $L^p(\partial D)$.
- (iii) $\omega \in RH_2(d\sigma)$,

where $RH_q(d\sigma)$ is the class of measures μ mutually absolutely continuous with σ so that $k = d\mu/d\sigma$ satisfies, $\forall Q \in \partial D, r > 0$

$$\left(\frac{1}{\sigma(\Delta(Q, r))} \int_{\Delta(Q, r)} k^q d\sigma \right)^{1/q} \leq C \left(\frac{1}{\sigma(\Delta(Q, r))} \int_{\Delta(Q, r)} k d\sigma \right).$$

The L^p Dirichlet problem

The goal of our work is to extend Dahlberg's result to more general operators. Given the importance of the non-tangential maximal function N , we define

Definition (\mathcal{D}_p)

The L^p Dirichlet problem is solvable for \mathcal{L} on D (or \mathcal{D}_p holds for \mathcal{L} on D) if

- (i) \mathcal{CD} holds for \mathcal{L} on D and
- (ii) there is a constant C (depending on, at most, \mathcal{L} , λ , n , D , and p) such that for all $g \in C(\partial D)$, the \mathcal{CD} solution u_g satisfies

$$\|Nu_g\|_{L^p(\partial D)} \leq C \|g\|_{L^p(\partial D)}.$$

Perturbation theorems

If we have two operators \mathcal{L}_0 and \mathcal{L}_1 , and we know that \mathcal{D}_p holds for \mathcal{L}_0 , certain conditions on the difference between the operators' coefficients allow us to conclude that \mathcal{D}_q holds for \mathcal{L}_1 , for some q . We frequently use the following notation:

$$\varepsilon(x) = \left(a_0^{ij}(x) - a_1^{ij}(x) \right)_{i,j}$$

$$\delta(x) = \text{dist}(x, \partial D)$$

$$\mathbf{a}(x) = \sup_{z \in B_{\delta(x)/2}(x)} |\varepsilon(z)|.$$

Perturbation theorems, cont.

Several theorems control $\varepsilon(x)$ by assuming that $\mathbf{a}^2(x)/\delta(x)$ is **the density of a Carleson measure**, that is, there is a C such that

$$h(r, Q) = \left(\frac{1}{\sigma(\Delta(Q, r))} \int_{T(\Delta(Q, r))} \frac{\mathbf{a}^2(x)}{\delta(x)} dx \right)^{1/2} \leq C.$$

The smallest such C is called the Carleson constant.

If, in addition,

$$\lim_{r \rightarrow 0} \sup_{|Q|=1} h(r, Q) = 0,$$

we say that the Carleson measure has *vanishing trace*.

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Dahlberg's divergence-form perturbation theorem

Dahlberg's perturbation theorem (in [Da86]) states the following in the **divergence form case**:

Theorem

Assume that $\mathbf{a}^2(x)/\delta(x)$ is the density of a Carleson measure with vanishing trace. Then, if $\omega_0 \in RH_p(d\sigma)$ for some p , $\omega_1 \in RH_p(d\sigma)$ as well, where ω_i is the elliptic measure for the operator \mathcal{L}_i .

Building on earlier work, this implies that if \mathcal{D}_p holds for \mathcal{L}_0 , \mathcal{D}_p holds for \mathcal{L}_1 as well.

FKP's divergence-form perturbation theorem

In [FKP], R. Fefferman, Kenig and Pipher show that, again in the divergence case,

Theorem

If \mathbf{a}^2/δ is the density of a Carleson measure (with no restriction on its constant), then $\omega_1 \in A_\infty(d\sigma)$ if $\omega_0 \in A_\infty(d\sigma)$.

Recall that $A_\infty(d\sigma) = \cup_{q>1} RH_q(d\sigma)$. Therefore, this result implies that if \mathcal{D}_p holds for \mathcal{L}_0 , then \mathcal{D}_q holds for \mathcal{L}_1 for some q .

In addition, they prove that, without further restriction on the Carleson norm, this theorem is sharp.

Rios' non-divergence form perturbation theorem

In [Ri03], Rios proves that essentially the same results hold in the non-divergence case. The extra hypotheses required on the coefficients ensure uniqueness of solutions, as described above. In short, his theorem is

Theorem

Assume that \mathcal{L}_0 satisfies \mathcal{D}_p for some p . Then there is a ρ_0 such that if $a_k^{ij} \in BMO_{\rho_0}$, $k = 0, 1$, and \mathbf{a}^2/δ is the density of a Carleson measure, then \mathcal{L}_1 verifies \mathcal{D}_q for some q .

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Non-divergence form perturbation theorem

Bringing the theory full circle, we (in [DW]) have the following

Theorem (Dindoš-W)

For two operators \mathcal{L}_0 and \mathcal{L}_1 , let $\varepsilon_0 < \infty$ be the Carleson constant of the measure $\mathbf{a}^2(x)/\delta(x) dx$. Assume that \mathcal{D}_p holds for \mathcal{L}_0 with constant $C_p > 0$.

Then there are constants $\rho_0 > 0$ (independent of p) and $M = M(p, D, \lambda, C_p, \rho) > 0$ such that if $a_0^{ij} \in BMO_\rho$, with $\rho < \rho_0$, and if $\varepsilon_0 < M$, then \mathcal{D}_p holds for \mathcal{L}_1 .

Non-divergence form solvability theorem

Using the perturbation theorem, we are also able to establish a new solvability theorem, following in the footsteps of [KP] and [DPP]

Theorem (Dindoš-W)

Let $1 < p < \infty$, let $0 < \lambda < \infty$ be a fixed ellipticity constant, and let D be a Lipschitz domain with Lipschitz constant L . Let $\mathcal{L} = a^{ij}\partial_{ij}$ be an operator with ellipticity constant λ . If

$$\sup \left\{ \frac{|a^{ij}(x) - \text{avg}(a^{ij}(z))|^2}{\delta(x)} : x \in B_{\delta(z)/2}(z) \right\}$$

is the density of a Carleson measure in D with Carleson constant M , then there is a constant $C(p, \lambda)$ such that if $L < C(p, \lambda)$ and $M < C(p, \lambda)$, then \mathcal{D}_p holds for \mathcal{L} . [Here $\text{avg}(a^{ij}(z))$ is the average of a^{ij} over the ball $B_{\delta(z)/2}(z)$.]

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Preliminary steps

Recall that we assume the Carleson constant $\varepsilon_0 < M$.

By making M small enough, we can

- ▶ ensure that \mathcal{L}_1 is in \mathcal{CD} (since $\|A_0 - A_1\|_{L^\infty(D)} \lesssim \varepsilon_0$) and
- ▶ guarantee that the ellipticity constant of \mathcal{L}_1 stays bounded away from zero.

Thus, we can speak of solutions u_0 and u_1 to the corresponding Dirichlet problems with the same boundary data g .

Let $F = u_0 - u_1$.

The idea of the proof

The goal is to prove that there is a C such that

$$\|Nu_1\|_{L^p(\partial D)} \leq C \|g\|_{L^p(\partial D)},$$

and we show this by proving that, in fact,

$$\|NF\|_{L^p(\partial D)} \leq C \|g\|_{L^p(\partial D)}.$$

This is enough, since \mathcal{D}_p holds for \mathcal{L}_0 and $u_1 = u_0 - F$.

We will prove this using Rios' modified non-tangential maximal function \tilde{N} [Ri03] and the following pointwise estimate, for $\alpha' < \alpha$:

$$N_{\alpha'} u_1(Q) \lesssim \tilde{N}_\alpha u_1(Q) + \tilde{N}_\alpha (\delta |\nabla u_1|)(Q).$$

Key lemma

This lemma is analogous to one in [FKP], and is proven in a similar fashion.

Lemma

There is a constant $C = C(\lambda, n)$ such that, under the hypotheses of the perturbation theorem,

$$\tilde{N}F(Q) + \tilde{N}(\delta|\nabla F|)(Q) \leq C\varepsilon_0 M_{\omega_0}(A_\alpha u_1)(Q).$$

Here, A_α is Rios' modified second area function (again, see [Ri03]). M_{ω_0} is the Hardy-Littlewood maximal function associated to the elliptic measure ω_0 for \mathcal{L}_0 .

Using the key lemma

Recall that assuming \mathcal{D}_p holds for \mathcal{L}_0 in D gives us that $\omega_0 \in RH_{p'}(d\sigma)$, which is equivalent to $\sigma \in A_p(d\omega_0)$.

$$\begin{aligned} \int_{\partial B_1} \tilde{N}(F)^p d\sigma &\leq \int_{\partial B_1} \left(\tilde{N}(F)^p + \tilde{N}(\delta|\nabla F|)^p \right) d\sigma \\ &\leq C\epsilon_0 \int_{\partial B_1} (M_{\omega_0}(A_{\tilde{\alpha}}u_1))^p d\sigma \\ &\leq C\epsilon_0 \int_{\partial B_1} (M_{\omega_0}(A_{\tilde{\alpha}}u_1))^p \frac{d\sigma}{d\omega_0} d\omega_0 \\ &\leq C'\epsilon_0 \int_{\partial B_1} A_{\tilde{\alpha}}(u_1)^p \frac{d\sigma}{d\omega_0} d\omega_0, \end{aligned}$$

using the fundamental property of A_p weights.

Stepping forward

Further work reduces the estimate to

$$\int_{\partial B_1} \left(\tilde{N}(F)^p + \tilde{N}(\delta |\nabla F|)^p \right) d\sigma \leq \tilde{C} \varepsilon_0 \int_{\partial B_1} (S_\beta(u_0)^p + S_\beta(F)^p) d\sigma,$$

where S_β is the modified area function (or square function), again from [Ri03].

The fact that \mathcal{D}_p holds for \mathcal{L}_0 gives us that

$$\int_{\partial B_1} S_\beta(u_0)^p d\sigma \lesssim \int_{\partial B_1} g^p d\sigma,$$

so the task left to us is to estimate $\int_{\partial B_1} S_\beta(F)^p d\sigma$.

The good-lambda inequality

Our second key lemma is an unwieldy good-lambda inequality that gives rise to the following estimate:

Corollary

For any $1 < p < \infty$:

$$\begin{aligned} \int_{\partial D} S(F)^p d\sigma &\leq C(p) \int_{\partial D} \left(\tilde{N}(F)^p + \tilde{N}(\delta|\nabla F|)^p \right) d\sigma \\ &\quad + \int_{\partial D} S(u_0)^p d\sigma, \end{aligned}$$

where the area function S is defined over cones of smaller aperture than the modified non-tangential maximal function \tilde{N} .

Putting it all together

The corollary reduces the estimate to

$$\int_{\partial B_1} \left(\tilde{N}(F)^p + \tilde{N}(\delta|\nabla F|)^p \right) d\sigma \leq C\varepsilon_0 \int_{\partial B_1} \left(\tilde{N}(F)^p + \tilde{N}(\delta|\nabla F|)^p + g^p \right) d\sigma,$$

Making ε_0 small enough so that $C\varepsilon_0 \leq 1/2$, we get

$$\int_{\partial B_1} \left(\tilde{N}(F)^p + \tilde{N}(\delta|\nabla F|)^p \right) d\sigma \leq C \int_{\partial B_1} g^p d\sigma,$$

which is all we need to prove the theorem. □

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The good-lambda inequality

Lemma

Let $\alpha > 0$. Then there exists $0 < \beta < \alpha$ depending only on the dimension, the number α and the Lipschitz constant of the domain D such that the following holds:

Suppose that $S_\beta(F)(P) \leq \lambda$ for some P in a surface ball $\Delta = \Delta(P_0, r) \subset \partial D$. Then there exists $c > 0$, $\delta > 0$ depending only on the Lipschitz character of the domain D and the ellipticity constant of the operator \mathcal{L}_0 such that for any $\gamma > 0$

$$\sigma(\{Q \in \Delta \ ; \ S_\beta(F) > 2\lambda, \tilde{N}_\alpha(F) \leq \gamma\lambda, \tilde{N}_\alpha(\delta|\nabla F|) \leq \gamma\lambda, \\ \tilde{N}_\alpha(F)A_\alpha(u_1) \leq (\gamma\lambda)^2\}) \leq c\gamma^\delta \sigma(\Delta). \quad (1)$$