A new geometric regularity condition for the end-point estimates of bilinear Calderón-Zygmund operators

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This is joint work in progress with:

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$$\left.\begin{array}{c} |\partial^\alpha k(x)| \leq C_\alpha |x|^{-(n+|\alpha|)} \\ \text{plus some cancellation} \end{array}\right\} \Rightarrow T : L^p \to L^p, \ 1 < p < \infty$$

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In fact, the L^p theory is obtained from the L^2 (or any fixed L^{p_0}) estimate via an $L^{1,\infty}$ estimate, interpolation, and duality as you all know.

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It is very useful to have representations of operators and conditions both on the time and frequency domains. For convolution operators the regularity conditions on the kernel or symbol are almost equivalent.
(But they can be weakened in some particular cases of *rough* operators which we will not discuss here.)

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$$T(f)(x) = \int K(x,y)\, f(y)\, dy$$

with

$$(R) \qquad |\partial^{\gamma}_{x,y} K(x,y)| \leq C_{\gamma} |x-y|^{-n-|\gamma|}$$

for $|\gamma| = 0, 1$, then

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The condition (R) alone is not enough though to imply L^2-boundedness. Some *cancellation* is again needed.

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$$T : L^2 \to L^2 \iff \sup_\xi (\|T(e^{ix\cdot\xi})\|_{BMO} + \|T^*(e^{ix\cdot\xi})\|_{BMO}) \leq C$$

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(The WBP property essentially says that T is “well-behaved” with respect to translations and dilations)

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But there other useful characterizations more in the spirit of *Eric Sawyer's testing conditions*

$$\iff \|T(\varphi_{z,R})\|_{L^2} + \|T^*(\varphi_{z,R})\|_{L^2} \leq CR^{n/2}$$

(Stein, 1993)

for all *normalized bumps* supported on the unit ball and such that $\|\partial^\alpha \varphi\|_{L^\infty} \leq 1$, for all $|\alpha| \leq N$ and where $\varphi_{z,R}(x) = \varphi(\frac{x-z}{R})$.

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or even more similar to the testing conditions,

$$\iff \|T_\epsilon(\chi_B)\|_{L^2} + \|T_\epsilon^*(\chi_B)\|_{L^2} \leq C|B|^{1/2}$$
(Nazarov-Treil-Volberg, 1998)

(T_ϵ are the usual truncated integrals)

Bilinear Calderón-Zygmund Theory

Most of the above have been extended to the bilinear or multilinear setting. There is by now a fairly developed multilinear Calderón-Zygmund theory:

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Coifman-Meyer introduced the multipliers

$$T(f,g)(x) = \int m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}\,d\xi d\eta$$

$$|\partial^\alpha m(\xi,\eta)| \leq C_\alpha(|\xi|+|\eta|)^{-|\alpha|},$$

and showed $T : L^p \times L^q \to L^r$ for $1/p + 1/q = 1/r$ and $r > 1$.

These operators have a singular integral representation

$$T(f,g)(x) = \int_{\mathbb{R}^{2n}} K(x-y, x-z) f(y) g(z)\, dydz$$

More generally we can consider

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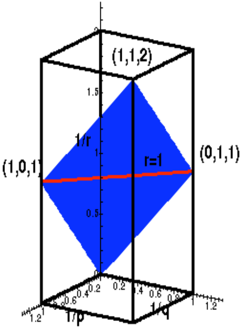
However, after the results of Lacey-Thiele (1997-1999) on the bilinear Hilbert transform, Kenig-Stein and Grafakos-T. extended the theory for $1/2 < r \leq 1$.

Points with coordinates $(1/p, 1/q, 1/r)$ and $1/p + 1/q = 1/r$



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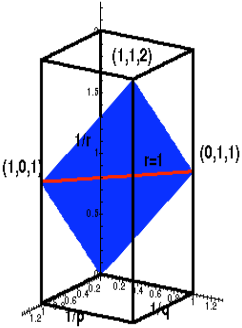

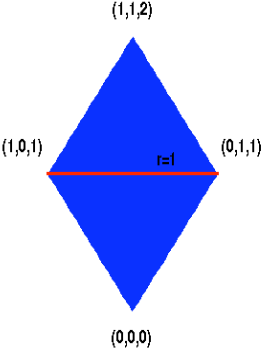

$$(R) \quad |\partial^\alpha K(y_0, y_1, y_2)| \lesssim \left(\sum |y_j - y_k|\right)^{-(2n+\alpha)}, \quad \alpha = 0, 1$$

$$T^{*0} = T, \;\; \langle T^{*1}(f,g), h\rangle = \langle T(h,g), f\rangle, \;\; \langle T^{*2}(f,g), h\rangle = \langle T(f,h), g\rangle$$

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Grafakos-T. (2000)

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Also,

$$T : L^p \times L^q \to L^r \iff \sum_{j=0}^{2} \sup_{\xi_1,\xi_2} \| T^{*j}(e^{ix\cdot\xi_1}, e^{ix\cdot\xi_2})\|_{BMO} \le C$$

Certainly there are m-linear versions

$$(R) \quad |\partial^\alpha K(y_0, y_1, \ldots, y_m)| \leq C_\alpha \left(\sum_{k,l=0}^{m} |y_k - y_l| \right)^{-mn-|\alpha|}$$

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We call these operators m-CZO

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$$\mathcal{R}_{ij}(\mathbf{f})(x) = \text{p.v.} \int_{(\mathbb{R}^n)^m} \frac{x_i - (y_j)_i}{\left(\sum_{j=1}^m |x - y_j|^2\right)^{\frac{nm+1}{2}}} \mathbf{f}(\mathbf{y}) d\mathbf{y}$$

for $i = 1, \cdots, n$ and $j = 1, \ldots, M$, and where $\mathbf{f}(\mathbf{y}) = f_1(y_1) \ldots f(y_m)$ and $(y_j)_i$ denotes the i-th coordinate of y_j.

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For example on $\mathbb{R} \times \mathbb{R}$,

$$R_1(f, g)(x) = \text{p.v.} \int_{\mathbb{R}^2} \frac{x - y}{|(x - y, x - z)|^3} f(y) g(z)\, dydz$$

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There is also a very extensive literature about other multilinear operators which do not fall within the scope of Calderón-Zygmund theory and that we will not consider in this talk.

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$$(H) \quad \int_{|x-y|>2|z-y|} |K(x,y) - K(x,z)|\, dx \leq C \quad y, z \in \mathbb{R}^n,\ y \neq z$$

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for $\alpha > n/2$ and ψ a smooth bump supported on $|\xi| \sim 1$ as considered by Hörmander (1960).

Conditions of the form

$$\sup_j \|m(2^j\cdot)\psi\|_{L^r_s} \leq C$$

where also considered by Kurtz-Wheeden (1979) to obtain weighted estimates

$$\|T(f)\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$$

for w in some A_t classes, t depeding on r, s, p and n.

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However all this only applies to convolution operators.

In general, for non-convolution operators,

$$(H) \quad \int_{|x-y|>2|z-y|} |K(x,y)-K(x,z)|\,dx \leq C \quad y,z\in\mathbb{R}^n,\ y\neq z$$

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- There is no A_p theory of weights.

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It is barely enough to get the end-point estimate.

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We want a condition in the spirit of Hörmander’s integral condition.

To present the result we have obtained with C. Pérez we need to look one more time to the condition (H).

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$$\sum_{Q \in D} |Q| \sup_{y \in Q} \int_{\mathbb{R}^n \setminus Q^*} |K(x,y) - K(x,c_Q)|\, dx \leq c \left| \bigcup_{Q \in D} Q \right|$$

(Notation: $Q^* = 3Q$ and c_Q is the center of the cube Q.)

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$$\sum_{(P,Q)\in D_1\times D_2} |P||Q| \sup_{(y,z)\in P\times Q} \int_{\mathbb{R}^n\setminus(P^*\cup Q^*)} |K(x,y,z)-K(x,c_P,c_Q)|\, dx$$

$$\leq c\left(|\bigcup_{P\in D_1} P| + |\bigcup_{Q\in D_2} Q|\right)$$

The main result

Theorem (Pérez-T.)

Let T be a bilinear operator with kernel K satisfying the GBH condition as defined before and such that

$$T : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$$

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This result is due to L. Grafakos-J.Soria (2009).

On the other hand, from Minkowski's inequality, if $K \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$

$$T_K : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n) \qquad \text{if} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \quad r \geq 1$$

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So even in this is case of integrable kernels, it is of interest to have some additional condition that allows for $r < 1$.

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Proof of the main result

Recall the result we are trying to show.

Theorem

Let T be a biilinear operator with kernel K satisfying the GBH condition as defined before and such that

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Proof:
We want to show

$$|\{x \in \mathbb{R}^n : |T(f_1, f_2)(x)| > \lambda^2\}|$$
$$\leq C \left(\int_{\mathbb{R}^n} \frac{|f_1(x)|}{\lambda} \, dx \right)^{1/2} \left(\int_{\mathbb{R}^n} \frac{|f_2(x)|}{\lambda} \, dx \right)^{1/2}$$

and we may assume

$$\|f_1\|_1 = \|f_2\|_1 = 1.$$

Fixed $\lambda > 0$, and consider a Calderón-Zygmund decomposition at level λ for each f_j, $j = 1, 2$

We obtain a collection of dyadic non-overlapping cubes $Q_{j,k} = Q(c_{j,k}, r_{j,k})$, that satisfies

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$$\lambda < \frac{1}{|Q_{j,k}|} \int_{Q_{j,k}} |f_j(x)|\, dx \leq 2^n \lambda$$

We set $\Omega_j = \cup_k Q_{j,k}$, so $|\Omega_j| \leq \frac{C}{\lambda}$, and as usual

$$|f_j(x)| \leq \lambda \qquad a.e.\, x \in \mathbb{R}^n \setminus \Omega_j.$$

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We write $f_j = g_j + b_j$, where g_j is defined by

$$g_j(x) = \begin{cases} f_j(x), & x \in \mathbb{R}^n \setminus \Omega_j \\ f_{Q_{j,k}}, & x \in Q_{j,k}, \end{cases}$$

and for any $s \geq 1$

$$\|g_1\|_s \leq C \lambda^{1/s'} \|f_1\|_1^{\frac{1}{s}}$$

Also b_j is written as

$$b_j(x) = \sum_k b_{j,k}(x) = \sum_k (f_j(x) - f_{Q_{j,k}})\chi_{Q_{j,k}}(x).$$

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We split the distribution set in several parts using the Calderón-Zygmund decomposition of the functions f_1 and f_2 as follows,

$$|\{x \in \mathbb{R}^n : |T(f_1, f_2)(x)| > \lambda^2\}|$$

$$\begin{aligned}
&\leq && |\{x \in \mathbb{R}^n : |T(g_1, g_2)(x)| > \lambda^2/4\}| \\
&+ && |\{x \in \mathbb{R}^n \setminus \Omega^* : |T(g_1, b_2)(x)| > \lambda^2/4\}| \\
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&+ && |\Omega^*| \\
&= && |E_1| + |E_2| + |E_3| + |E_4| + |\Omega^*|
\end{aligned}$$

Let's see the estimate for E_4 to see how the GBH conditions appear (the other terms are similar or easier).

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In fact, the whole argument with the new condition is much simpler than the one in other proofs of the end-point estimate bilinear Calderón-Zygmund operators already in the literature.

First we split the operator

$$T(b_1, b_2) = \sum_{l,k} T(b_{1,l}, b_{2,k}).$$

Thus,

$$|E_4| \leq \frac{C}{\lambda^2} \sum_{l,k} \int_{\mathbb{R}^n \setminus \Omega^*} |T(b_{1,l}, b_{2,k})(x)| \, dx \leq$$

$$\frac{C}{\lambda^2} \sum_{l,k} \int_{\mathbb{R}^n \setminus (Q_{1,l}^* \cup Q_{2,k}^*)} \left| \int_{Q_{1,l}} \int_{Q_{2,k}} K(x,y,z) b_{1,l}(y) b_{2,k}(z) \, dz dy \right| dx$$

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We fix one of these $Q_{1,l}$ and $Q_{2,k}$ and use the cancellation of the b_j to obtain

$$\int_{\mathbb{R}^n\setminus(Q_{1,l}^*\cup Q_{2,k}^*)}\left|\int_{Q_{1,l}}\int_{Q_{2,k}}K(x,y,z)\,b_{2,k}(z)b_{1,l}(y)dz\,dy\right|\,dx$$

$$=\int_{(Q_{1,l}^*\cup Q_{2,k}^*)^C}\left|\int_{Q_{1,l}}\int_{Q_{2,k}}(K(x,y,z)-K(x,c_{Q_{1,l}},c_{Q_{2,k}}))\,b_{2,k}(z)b_{1,l}(y)dz\,dy\right|dx$$

$$=\int_{Q_{1,l}}\int_{Q_{2,k}}\int_{(Q_{1,l}^*\cup Q_{2,k}^*)^c}\left|K(x,y,z)-K(x,c_{Q_{1,l}},c_{Q_{2,k}})\right|\,dx\,|b_{2,k}(z)||b_{1,l}(y)|dzdy$$

$$\lesssim\lambda^2\,|Q_{1,l}||Q_{2,k}|\sup_{(y,z)\in P\times Q}\int_{(Q_{1,l}^*\cup Q_{2,k}^*)^c}\left|K(x,y,z)-K(x,c_{Q_{1,l}},c_{Q_{2,k}})\right|\,dx$$

and therefore

$$|E_4| \lesssim$$

$$\sum_{l,k} |Q_{1,l}||Q_{2,k}| \sup_{(y,z)\in P\times Q} \int_{\mathbb{R}^n\setminus(Q_{1,l}^*\cup Q_{2,k}^*)} \left|K(x,y,z) - K(x,c_{Q_{1,l}},c_{Q_{2,k}})\right| \, dx$$

$$\lesssim (|\cup_l Q_{1,l}| + |\cup_k Q_{2,k}|) = (|\Omega_1| + |\Omega_2|) \lesssim \frac{1}{\lambda}$$

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as we wanted to prove.

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or similarly the m-linear one

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There are however other naturally appearing weak-type end-point estimates in the multilinear case.
They take the form

$$|\{x \in \mathbb{R}^n : |T(f_1, \cdots, f_m)(x)| > \lambda^m\}| \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} \Phi\left(\frac{|f_j(x)|}{\lambda} \right) dx \right)^{1/m}$$

Examples:

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Multilinear commutators of m-CZOs and BMO functions:

Lerner, Ombrosi, Pérez, T. , Trujillo-González

$$T_{\Sigma\,\mathbf{b}}(\mathbf{f})(x) = \int \sum_{j=1}^{m} (b_j(x) - b_j(y_j))\, K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m)\, d\mathbf{y}$$

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$$T_{\Pi\,\mathbf{b}}(\mathbf{f})(x) = \int \prod_{j=1}^{m} (b_j(x) - b_j(y_j))\, K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m)\, d\mathbf{y}$$

Examples:

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Multi(sub)linear strong maximal function:

Grafakos, Liu, Pérez, T.

$$\mathcal{M}_{\mathcal{R}}(f_1, \ldots, f_m)(x) = \sup_{R \ni x} \prod_{i=1}^{m} \left(\frac{1}{|R|} \int_R |f_i(y)|\, dy \right)$$

Let

$$\Phi_n(t) = t(1 + (\log^+ t)^{n-1}) \approx t(\log(e+t))^{n-1}$$

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We have the end-point estimates

$$\left|\left\{x \in \mathbb{R}^n : |T_{\Sigma\,\mathbf{b}}(\mathbf{f})(x)| > t^m\right\}\right| \lesssim \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} \Phi_2\left(\frac{|f_j(x)|}{t}\right) dx \right)^{1/m}$$

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They can also be use to interpolate!

torres@math.ku.edu The University of Kansas

Theorem (Grafakos, Liu, Pérez, T.)

Suppose a bisublinear operator T maps $L^{s_1} \times L^{s_2} \to L^{s,\infty}$ for all $1 < s_1, s_2, s < \infty$ with $1/s_1 + 1/s_2 = 1/s$ and also satisfies the endpoint distributional estimate

$$\left|\left\{|T(f_1, f_2)| > \lambda\right\}\right| \leq C\left(\int \Phi\left(\frac{f_1}{\sqrt{\lambda}}\right) dx\right)^{\frac{1}{2}}\left(\int \Phi\left(\frac{f_2}{\sqrt{\lambda}}\right) dx\right)^{\frac{1}{2}},$$

where Φ is a nonnegative function that satisfies $\Phi(0) = 0$ and

$$\int_0^1 \lambda^\alpha \Phi\left(\frac{1}{\lambda}\right) d\lambda < \infty$$

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Then $T : L^{p_1} \times L^{p_2} \to L^p$ for all $1/p_1 + 1/p_2 = 1/p$ with $1 < p_1, p_2 < \infty$ and $1/2 < p < \infty$.

Many thanks for your attention

Happy Birthday Eric!!!