A nonlinear Calderón-Zymund theory for quasilinear operators and its apllications

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Introduction: The Harmonic Transform

The harmonic transform:
Given a vector field $\mathbf{f} \in L^2(\Omega, \mathbb{R}^n)$, consider the Dirichlet problem

$$\begin{cases} \Delta u = \operatorname{div} \mathbf{f}, \\ u \in W_0^{1,\,2}(\Omega). \end{cases} \qquad (1)$$

We have the L^2 estimate

$$||\nabla u||_{L^2(\Omega)} \leq ||\mathbf{f}||_{L^2(\Omega)}.$$

The harmonic transform is defined by

$$\mathcal{H} : L^2(\Omega, \mathbb{R}^n) \to L^2(\Omega, \mathbb{R}^n)$$

$$\mathcal{H}(\mathbf{f}) = \nabla u.$$

Question: Is $\mathcal{H}$ also bounded on other Lebesgue spaces?

The Harmonic Transform

The case $\Omega = \mathbb{R}^n$: By means of Fourier transform we find

$$
\begin{aligned}
\mathcal{H}(\mathbf{f}) &= -[R_{ij}]\mathbf{f} \\
&= -c(n)\,\mathrm{p.v.}\int_{\mathbb{R}^n} \frac{<x-y, \mathbf{f}(y)>(x-y)}{|x-y|^{n+2}}dy.
\end{aligned}
$$

Here $[R_{ij}]$ is the matrix of second order Riesz transforms:

$$
\begin{aligned}
R_{ij}(\varphi) &= R_i(R_j(\varphi)) \\
&= c(n)\,\mathrm{p.v.}\int_{\mathbb{R}^n} \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^{n+2}}\varphi(y)dy.
\end{aligned}
$$

Calderón-Zygmund, 1952:

$$
||\mathcal{H}(\mathbf{f})||_{L^q(\mathbb{R}^n)} \leq C_q\,||\mathbf{f}||_{L^q(\mathbb{R}^n)}, \qquad \forall q > 1.
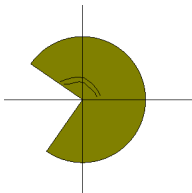$$

The Harmonic Transform on bounded domains

For bounded C^1-domains, Jerison-Kenig, 1995:

$$\mathcal{H} : L^q(\Omega) \to L^q(\Omega) \text{ for all } q > 1.$$

However, such gradient bounds generally fail to hold on Lipschitz domains. Here is an example. Let $\frac{\pi}{2} < \theta_0 < \pi$ and consider the (non-convex) domain:

$$\Omega_{\theta_0} = \{(r, \theta) : 0 < r < 1 \text{ and } -\theta_0 < \theta < \theta_0\}.$$

For $\lambda = \frac{\pi}{2\theta_0} < 1$, let $u(r, \theta) = r^\lambda \cos(\lambda\theta)$. Near the origin,
$$|\nabla u| = \lambda r^{\lambda-1} = \lambda r^{\frac{\pi}{2\theta_0}-1}.$$
Bad news: For any $q > 4$ we can find a θ_0 (near π) such that $|\nabla u| \notin L^q(\Omega_{\theta_0})$.
Good news: For any $q > 4$ we can find a θ_0 (near $\frac{\pi}{2}$) such that $|\nabla u| \in L^q(\Omega_{\theta_0})$.

The Harmonic Transform with coefficients

One can generalize (1) to the equation

$$
\begin{cases}
\operatorname{div} A(x)\nabla u &= \operatorname{div} \mathbf{f} \text{ in } \Omega, \\
u &= 0 \text{ on } \partial\Omega,
\end{cases}
$$

where the matrix $A = [A_{ij}]$ is uniformly elliptic.
If $A \in VMO$ or if $A \in BMO$ with small BMO seminorms then

$$
||\nabla u||_{L^q(\Omega)} \leq C \, ||\mathbf{f}||_{L^q(\Omega)}, \qquad \forall q > 1.
$$

Di Fazio, 1996 ($C^{1,1}$-domains).
Auscher-Qafsaoui, 2002 (C^1-domains).
Byun-Wang, CPAM 2004, AIM 2008 (Reifenberg flat domains).
N. G. Meyer (1963): A counter example for bad A.

The *p*-harmonic Transform:

Let $\mathbf{f} \in L^p(\Omega, \mathbb{R}^n)$. Consider the problem

$$\begin{cases} \Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div}|\mathbf{f}|^{p-2}\mathbf{f}, \\ \qquad\qquad u \in W_0^{1,p}(\Omega). \end{cases} \quad (2)$$

The fundamental estimate (take u as a test function and IBP):

$$\int_\Omega |\nabla u|^p dx \leq \int_\Omega |\mathbf{f}|^p dx$$

The *p*-harmonic transform is defined by

$$\mathcal{H}_p : L^p(\Omega, \mathbb{R}^n) \to L^p(\Omega, \mathbb{R}^n)$$

$$\mathcal{H}_p(\mathbf{f}) = \nabla u.$$

The p-harmonic Transform: Basic question

Question: Is $\mathcal{H}_p$ also bounded on other Lebesgue spaces?

Theorem (Iwaniec 1983, DiBenedetto-Manfredi 1993 (p-systems))

Let $\Omega = \mathbb{R}^n$ and $p < q < \infty$. One has

$$\mathcal{H}_p : L^q(\Omega, \mathbb{R}^n) \to L^q(\Omega, \mathbb{R}^n)$$

$$||\mathcal{H}_p(\mathbf{f})||_{L^q} \leq C||\mathbf{f}||_{L^q}$$

provided that $\mathcal{H}_p(\mathbf{f}) \in L^p$.

Theorem (Iwaniec-Sbordone 1994)

Let Ω be a bounded regular domain. There exists small $\epsilon > 0$ such that for all $p - \epsilon < q < p$ one has

$$\mathcal{H}_p : L^q(\Omega, \mathbb{R}^n) \to L^q(\Omega, \mathbb{R}^n)$$

$$||\mathcal{H}_p(\mathbf{f})||_{L^q} \leq C||\mathbf{f}||_{L^q}.$$

Conjecture (Iwaniec, 1983): $\mathcal{H}_p$ is also bounded on L^q for all $\max\{1, p-1\} < q < p$. We show that under certain positivity condition this conjecture holds true.

Theorem (P. 2011)

Suppose that $p > 2 - \frac{1}{n}$, $p - 1 < q < p$. Let u be a p-superharmonic function in $\mathbb{R}^n$ such that

$$\Delta_p u = \operatorname{div}|\mathbf{f}|^{p-2}\mathbf{f}.$$

Then one has the estimate

$$\int_{\mathbb{R}^n} |\nabla u|^q dx \leq C \int_{\mathbb{R}^n} |\mathbf{f}|^q dx,$$

provided that

$$\lim_{R \to \infty} \fint_{B(O,R)} |\nabla u| dy = 0.$$

Some ingridients in the proofs

Recent work of F. Duzaar and G. Mingione 2009, 2010: Suppose that u solves the nonlinear equation with measure data

$$-\Delta_p u = \mu \geq 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

Then we have for a.e. $x \in \mathbb{R}^n$:

$$|\nabla u(x)| \leq C \int_0^\infty \left[\frac{\mu(B_t(x))}{t^{n-1}} \right]^{\frac{1}{p-1}} \frac{dt}{t}, \quad \text{if } p \geq 2,$$

and

$$|\nabla u(x)| \leq C \left[\int_0^\infty \frac{\mu(B_t(x))}{t^{n-1}} \frac{dt}{t} \right]^{\frac{1}{p-1}}, \quad \text{if } 2 - \frac{1}{n} < p < 2,$$

provided that

$$\lim_{R \to \infty} \fint_{B(O,R)} |\nabla u| dy = 0.$$

P.-Torres 2008: If $\mathbf{f} \in L^q$ satisfies $\mathrm{div}|\mathbf{f}|^{p-2}\mathbf{f} = \mu \geq 0$, then it holds that

$$\left\| \left[\int_0^\infty \frac{\mu(B_t(\cdot))}{t^{n-1}} \frac{dt}{t} \right]^{\frac{1}{p-1}} \right\|_{L^q} \leq C \left\| \mathbf{f} \right\|_{L^q}.$$

P.-Verbitsky 2008 (Wolff type inequalities): For all $q > p - 1 > 0$ one has the equivalence

$$\left\| \left[\int_0^\infty \frac{\mu(B_t(\cdot))}{t^{n-1}} \frac{dt}{t} \right]^{\frac{1}{p-1}} \right\|_{L^q} \simeq \left\| \int_0^\infty \left[\frac{\mu(B_t(\cdot))}{t^{n-1}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right\|_{L^q}.$$

The *p*-harmonic transform with general structures

More generally, one can consider the equation

$$\mathrm{div}\mathcal{A}(x, \nabla u) = \mathrm{div}|\mathbf{f}|^{p-2}\mathbf{f},$$

where

$$|\mathcal{A}(x, \xi)| \leq \beta|\xi|^{p-1},$$

$$\langle \mathcal{A}_\xi(x, \xi)\lambda, \lambda \rangle \geq \alpha|\xi|^{p-2}|\lambda|^2, \qquad |\mathcal{A}_\xi(x, \xi)| \leq \beta|\xi|^{p-2},$$

for all x, ξ, and λ in $\mathbb{R}^n$. For our purpose we also require that $\mathcal{A}$ satisfy a δ-BMO condition in the x-variable. For each ball $B \subset \mathbb{R}^n$ we let $\beta = \beta(B) : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\beta(B)(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|\mathcal{A}(x, \xi) - \overline{\mathcal{A}}_B(\xi)|}{|\xi|^{p-1}},$$

where

$$\overline{\mathcal{A}}_B(\xi) = \fint_B \mathcal{A}(x, \xi)dx = \frac{1}{|B|}\int_B \mathcal{A}(x, \xi)dx.$$

In the linear case where $\mathcal{A}(x, \xi) = A(x)\xi$, we see that

$$\beta(B)(x) \leq |A(x) - \overline{A}_B|$$

Definition

We say that $\mathcal{A}(x, \xi)$ satisfies a δ-BMO condition for some $\delta > 0$ (with exponent s) if

$$\|\mathcal{A}\|_s^{\#} := \sup_{B \subset \mathbb{R}^n} \left(\fint_B \beta(B)(x)^s dx \right)^{\frac{1}{s}} \leq \delta.$$

Some earlier works:

- Kinnunen-Zhou, 1999, 2001 (VMO coefficients, $C^{1,\alpha}$ domains)): lies deeply in the $C^{1,\alpha}$-regularity for the homogeneous p-harmonic equation $\Delta_p u = 0$.
  The boundedness of Fefferman-Stein sharp maximal function $M^{\#}$ was also employed.

- Byun-Wang-Zhou, 2007, Byun-Wang, 2008 (small BMO coefficients, Reifenberg flat domains)): relies on $W^{1,\infty}$ estimates for the homogeneous p-harmonic equation. Perturbation approach in Caffarelli-Peral, 1998.

P.-Mengesha 2011 and P.-Mengesha (in preparation):

Theorem

There exists a constant $\delta > 0$ such that if u is a solution to

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla u) & = & \operatorname{div} |\mathbf{f}|^{p-2}\mathbf{f} \text{ in } \Omega, \\ u & = & 0 \text{ on } \partial\Omega, \end{cases}$$

where A satisfies the (δ, R)-BMO condition and Ω is (δ, R)-Reifenberg flat then one has the estimate

$$||\nabla u||_{\mathcal{F}_q} \leq C \, ||\mathbf{f}||_{\mathcal{F}_q}, \qquad \forall q > p.$$

The $\mathcal{A}$-superharmonicity condition is not needed here.

The case $q > p$ on bounded domains

Here $\mathcal{F}_q$ can be any of the following spaces: $p < q < \infty$

- $L^q(\Omega)$; $L^q_w(\Omega)$ where $w \in A_{q/p}$. This gives a nonlinear version of weighted norm inequalities for singular integrals: Hunt-Muckenhoupt-Wheeden, 1973 (1D), Coifman-Fefferman, 1974.

- Weighted Lorentz spaces $L^{q;t}_w(\Omega)$ where $w \in A_{q/p}$. Here

$$\|g\|_{L^{q;\,t}_w(\Omega)} := \left[q \int_0^\infty (\alpha^q w(\{x \in \Omega : |g(x)| > \alpha\}))^{\frac{t}{q}} \frac{d\alpha}{\alpha} \right]^{\frac{1}{t}}.$$

- Morrey spaces: $\mathcal{M}^{q;\,\theta}(\Omega)$, $0 < \theta \leq n$ with

$$\|g\|_{\mathcal{M}^{q;\,\theta}(\Omega)} := \sup_{\substack{0 < r \leq \mathrm{diam}(\Omega) \\ z \in \Omega}} r^{\frac{\theta - n}{q}} \|g\|_{L^q(B_r(z) \cap \Omega)}.$$

- Lorentz-Morrey spaces $\mathcal{LM}^{q,\,t;\,\theta}(\Omega)$, $0 < t \leq \infty$, $0 < \theta \leq n$.

$$\|g\|_{\mathcal{LM}^{q,\,t;\,\theta}(\Omega)} := \sup_{\substack{0 < r \leq \mathrm{diam}(\Omega) \\ z \in \Omega}} r^{\frac{\theta - n}{q}} \|g\|_{L^{q,\,t}(B_r(z) \cap \Omega)} < +\infty.$$

When $\theta = n$: $\mathcal{LM}^{q,\,t;\,\theta}(\Omega) = L^{q,\,t}(\Omega)$.
When $q = t$: $\mathcal{LM}^{q,\,t;\,\theta}(\Omega) = \mathcal{M}^{q;\,\theta}(\Omega)$.

- $\mathcal{F}_q$ can also be a "capacitary space"' $\mathcal{C}_q(\Omega)$ defined via

$$\|g\|_{\mathcal{C}_q(\Omega)} := \sup_{K \subset \Omega} \left\{ \frac{\int_K |g|^q dy}{\mathrm{Cap}_{1,s}(K)} \right\}^{\frac{1}{q}},$$

where $\mathrm{Cap}_{1,s}(\cdot)$ is the capacity associated to the Sobolev space $W^{1,s}(\mathbb{R}^n)$.

This estimate has an application to Riccati type equations with super-natural growth in the gradient.

$$\begin{cases} -\text{div}\mathcal{A}(x, \nabla u) = |\nabla u|^q + \mu \text{ in } \Omega, \\ \qquad\qquad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $q > p > 1$.

$$\Updownarrow$$

$$\mu(K) \le C \operatorname{Cap}_{1, \frac{q}{q-p+1}}(K) \tag{3}$$

P. 2009: $A \in \text{VMO}$, $\partial\Omega \in C^1$.

Some consequences

Solvability in Lebesgue spaces:

$$\omega \in L^{\frac{n(q-p+1)}{q}, \infty}(\Omega) \Rightarrow (3).$$

For example,

$$-\Delta_p u = |\nabla u|^q + \frac{c}{|x|^s}, \qquad O \in \Omega,$$

has a solution if and only if $s \leq \frac{q}{q-p+1}$.

Solvability in Morrey Spaces: Suppose $d\omega = fdx$. Let $\epsilon > 0$.

$$\omega \in \mathcal{L}^{1+\epsilon, \frac{(1+\epsilon)q}{q-p+1}} \Rightarrow \omega(E) \leq C \operatorname{Cap}_{1, \frac{q}{q-p+1}}(E).$$

This is also called Fefferman-Phong condition.

THANK YOU FOR YOUR ATTENTION!