"A tribute to Eric Sawyer"

Carlos Pérez

Universidad de Sevilla

Conference in honor of Eric Sawyer

Fields Institute, Toronto

July, 2011

CV

cv

B.Sc.   McGill University, Mathematics 1973

CV

B.Sc.   McGill University, Mathematics 1973

Ph.D.   McGill University, Mathematics 1977

cv

B.Sc.   McGill University, Mathematics 1973

Ph.D.   McGill University, Mathematics 1977

Title of his thesis:

CV

B.Sc.   McGill University, Mathematics 1973

Ph.D.   McGill University, Mathematics 1977

Title of his thesis:

"Inner Functions on Polydiscs"

CV

B.Sc.   McGill University, Mathematics 1973

Ph.D.   McGill University, Mathematics 1977

Title of his thesis:

"Inner Functions on Polydiscs"

Advisor: **Kohur N. Gowrisankaran**

CV

B.Sc.   McGill University, Mathematics 1973

Ph.D.   McGill University, Mathematics 1977

Title of his thesis:

"Inner Functions on Polydiscs"

Advisor: **Kohur N. Gowrisankaran**

Posdoctoral position at the University of Wisconsin, Madison

<p align="center">**CV**</p>

B.Sc.   McGill University, Mathematics 1973

Ph.D.   McGill University, Mathematics 1977

Title of his thesis:

"Inner Functions on Polydiscs"

Advisor: **Kohur N. Gowrisankaran**

Posdoctoral position at the University of Wisconsin, Madison

McMaster University, 1980

CV

B.Sc.   McGill University, Mathematics 1973

Ph.D.   McGill University, Mathematics 1977

Title of his thesis:

"Inner Functions on Polydiscs"

Advisor: **Kohur N. Gowrisankaran**

Posdoctoral position at the University of Wisconsin, Madison

McMaster University, 1980

McKay Professor since 2000

CV

B.Sc.   McGill University, Mathematics 1973

Ph.D.   McGill University, Mathematics 1977

Title of his thesis:

"Inner Functions on Polydiscs"

Advisor: **Kohur N. Gowrisankaran**

Posdoctoral position at the University of Wisconsin, Madison

McMaster University, 1980

McKay Professor since 2000

Two PhD students: **Scott Rodney** and **Lu Qi Wang**

CV

B.Sc.   McGill University, Mathematics 1973

Ph.D.   McGill University, Mathematics 1977

Title of his thesis:

"**Inner Functions on Polydiscs**"

Advisor: **Kohur N. Gowrisankaran**

Posdoctoral position at the University of Wisconsin, Madison

McMaster University, 1980

McKay Professor since 2000

Two PhD students: **Scott Rodney** and **Lu Qi Wang**

Many posdoctoral researchers: **Steve Hofman, Cristian Rios, Alex Iosevich, Yong Sheng Han, Serban Costea.**

Co-authors

Andersen, Kenneth F.

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban
Csima, József

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban
Csima, József
Guan, Pengfei

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban
Csima, József
Guan, Pengfei
Han, Yong Sheng

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban
Csima, József
Guan, Pengfei
Han, Yong Sheng
Iosevich, Alexander

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban
Csima, József
Guan, Pengfei
Han, Yong Sheng
Iosevich, Alexander
Kerman, Ronald

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban
Csima, József
Guan, Pengfei
Han, Yong Sheng
Iosevich, Alexander
Kerman, Ronald
Lacey, Michael T.

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban
Csima, József
Guan, Pengfei
Han, Yong Sheng
Iosevich, Alexander
Kerman, Ronald
Lacey, Michael T.
Martin-Reyes, Francisco Javier

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban
Csima, József
Guan, Pengfei
Han, Yong Sheng
Iosevich, Alexander
Kerman, Ronald
Lacey, Michael T.
Martin-Reyes, Francisco Javier
Rios, Cristian

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban
Csima, József
Guan, Pengfei
Han, Yong Sheng
Iosevich, Alexander
Kerman, Ronald
Lacey, Michael T.
Martin-Reyes, Francisco Javier
Rios, Cristian
Rochberg, Richard

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban
Csima, József
Guan, Pengfei
Han, Yong Sheng
Iosevich, Alexander
Kerman, Ronald
Lacey, Michael T.
Martin-Reyes, Francisco Javier
Rios, Cristian
Rochberg, Richard
Seeger, Andreas

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban
Csima, József
Guan, Pengfei
Han, Yong Sheng
Iosevich, Alexander
Kerman, Ronald
Lacey, Michael T.
Martin-Reyes, Francisco Javier
Rios, Cristian
Rochberg, Richard
Seeger, Andreas
Uriarte-Tuero, Ignacio

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban
Csima, József
Guan, Pengfei
Han, Yong Sheng
Iosevich, Alexander
Kerman, Ronald
Lacey, Michael T.
Martin-Reyes, Francisco Javier
Rios, Cristian
Rochberg, Richard
Seeger, Andreas
Uriarte-Tuero, Ignacio
Wheeden, Richard L.

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban
Csima, József
Guan, Pengfei
Han, Yong Sheng
Iosevich, Alexander
Kerman, Ronald
Lacey, Michael T.
Martin-Reyes, Francisco Javier
Rios, Cristian
Rochberg, Richard
Seeger, Andreas
Uriarte-Tuero, Ignacio
Wheeden, Richard L.
Wick, Brett D.

Co-authors

Andersen, Kenneth F.
Arcozzi, Nicola
Baernstein, Albert
Chanillo, Sagun
Chen, Wengu
Costea, Serban
Csima, József
Guan, Pengfei
Han, Yong Sheng
Iosevich, Alexander
Kerman, Ronald
Lacey, Michael T.
Martin-Reyes, Francisco Javier
Rios, Cristian
Rochberg, Richard
Seeger, Andreas
Uriarte-Tuero, Ignacio
Wheeden, Richard L.
Wick, Brett D.
Zhao, Shi Ying

More information

Number of publications: 70

More information

Number of publications: 70

Areas of research:

More information

Number of publications: 70

Areas of research:

Convex and discrete geometry
Fourier analysis
Functional analysis
Functions of a complex variable
Measure and integration
Number theory
Operator theory
Partial differential equations
Potential theory
Real functions
Several complex variables
Analytic spaces

Schrödinger operators and the theory of weights

Schrödinger operators and the theory of weights

Fefferman-Phong problem

Schrödinger operators and the theory of weights
Fefferman-Phong problem

Let

$$-\Delta - V$$

be a Schrödinger operator with potential V.

Schrödinger operators and the theory of weights
Fefferman-Phong problem

Let

$$-\Delta - V$$

be a Schrödinger operator with potential V.

Consider the first eigenvalue defined by its quadratic, called the Rayleigh quotient

$$\lambda_1 = \inf_{g \in C_o^\infty} \frac{\langle Hg, g \rangle}{\langle g, g \rangle}$$

Schrödinger operators and the theory of weights
Fefferman-Phong problem

Let

$$-\Delta - V$$

be a Schrödinger operator with potential V.

Consider the first eigenvalue defined by its quadratic, called the Rayleigh quotient

$$\lambda_1 = \inf_{g \in C_o^\infty} \frac{\langle Hg, g \rangle}{\langle g, g \rangle}$$

By a formula due to of **R. Kerman and E. Sawyer**

Schrödinger operators and the theory of weights
Fefferman-Phong problem

Let

$$-\Delta - V$$

be a Schrödinger operator with potential V.

Consider the first eigenvalue defined by its quadratic, called the Rayleigh quotient

$$\lambda_1 = \inf_{g \in C_o^\infty} \frac{\langle Hg, g \rangle}{\langle g, g \rangle}$$

By a formula due to of **R. Kerman and E. Sawyer**

$$\boxed{-\lambda_1 = \inf\{\lambda \geq 0 : C_\lambda(V) \leq 1\},}$$

where $C_\lambda(V)$ is the smallest constant satisfying

Schrödinger operators and the theory of weights
Fefferman-Phong problem

Let

$$-\Delta - V$$

be a Schrödinger operator with potential V.

Consider the first eigenvalue defined by its quadratic, called the Rayleigh quotient

$$\lambda_1 = \inf_{g \in C_o^\infty} \frac{\langle Hg, g \rangle}{\langle g, g \rangle}$$

By a formula due to of **R. Kerman and E. Sawyer**

$$\boxed{-\lambda_1 = \inf\{\lambda \geq 0 : C_\lambda(V) \leq 1\},}$$

where $C_\lambda(V)$ is the smallest constant satisfying

$$\int_{\mathbb{R}^n} g(y)^2 V(y) dy \leq C_\lambda(V)^2 \int_{\mathbb{R}^n} \left(\nabla g(y)^2 + \lambda^2 g(y)^2 \right) dy$$

The trace inequality I

The trace inequality I

By taking the Fourier transform, is easy to see that this is equivalent to,

The trace inequality I

By taking the Fourier transform, is easy to see that this is equivalent to,

$$\int_{\mathbb{R}^n} |J_{1,\lambda}f|^2 \, V(y)dy \leq C_\lambda(V)^2 \int_{\mathbb{R}^n} |g(y)|^2 \, dy \qquad \lambda > 0$$

The trace inequality I

By taking the Fourier transform, is easy to see that this is equivalent to,

$$
\int_{\mathbb{R}^n} |J_{1,\lambda}f|^2 \, V(y)dy \leq C_\lambda(V)^2 \int_{\mathbb{R}^n} |g(y)|^2 \, dy \qquad \lambda > 0
$$

where $J_{\alpha,\lambda}, \; \alpha, \lambda > 0$ is the Bessel potential defined by

$$
J_{\alpha,\lambda}f = K_{\alpha,\lambda} \star f.
$$

The trace inequality I

By taking the Fourier transform, is easy to see that this is equivalent to,

$$\int_{\mathbb{R}^n} |J_{1,\lambda}f|^2\, V(y)dy \leq C_\lambda(V)^2 \int_{\mathbb{R}^n} |g(y)|^2\, dy \qquad \lambda > 0$$

where $J_{\alpha,\lambda}, \ \alpha, \lambda > 0$ is the Bessel potential defined by

$$J_{\alpha,\lambda}f = K_{\alpha,\lambda} \star f.$$

This is a what is called a **trace inequality.**

The trace inequality I

By taking the Fourier transform, is easy to see that this is equivalent to,

$$\int_{\mathbb{R}^n} |J_{1,\lambda}f|^2 \, V(y)dy \leq C_\lambda(V)^2 \int_{\mathbb{R}^n} |g(y)|^2 \, dy \qquad \lambda > 0$$

where $J_{\alpha,\lambda}$, $\alpha, \lambda > 0$ is the Bessel potential defined by

$$J_{\alpha,\lambda}f = K_{\alpha,\lambda} \star f.$$

This is a what is called a **trace inequality.**

In other words

$$C_\lambda(V) = \|J_{1,\lambda}\|_{L^2 \to L^2(V)}$$

The trace inequality I

By taking the Fourier transform, is easy to see that this is equivalent to,

$$\int_{\mathbb{R}^n} |J_{1,\lambda}f|^2 \, V(y) dy \leq C_\lambda(V)^2 \int_{\mathbb{R}^n} |g(y)|^2 \, dy \qquad \lambda > 0$$

where $J_{\alpha,\lambda}, \; \alpha, \lambda > 0$ is the Bessel potential defined by

$$J_{\alpha,\lambda}f = K_{\alpha,\lambda} \star f.$$

This is a what is called a **trace inequality.**

In other words
$$C_\lambda(V) = \|J_{1,\lambda}\|_{L^2 \to L^2(V)}$$

The case $\lambda = 0$ is specially interesting and we have

The trace inequality I

By taking the Fourier transform, is easy to see that this is equivalent to,

$$\int_{\mathbb{R}^n} |J_{1,\lambda}f|^2 \, V(y)dy \leq C_\lambda(V)^2 \int_{\mathbb{R}^n} |g(y)|^2 \, dy \qquad \lambda > 0$$

where $J_{\alpha,\lambda}$, $\alpha, \lambda > 0$ is the Bessel potential defined by

$$J_{\alpha,\lambda}f = K_{\alpha,\lambda} \star f.$$

This is a what is called a **trace inequality.**

In other words

$$C_\lambda(V) = \|J_{1,\lambda}\|_{L^2 \to L^2(V)}$$

The case $\lambda = 0$ is specially interesting and we have

$$C_0(V) = \|I_1\|_{L^2 \to L^2(V)}$$

where now I_α, $0 < \alpha < n$, is the Riesz potential operator defined by

The trace inequality I

By taking the Fourier transform, is easy to see that this is equivalent to,

$$\int_{\mathbb{R}^n} |J_{1,\lambda}f|^2\, V(y)dy \leq C_\lambda(V)^2 \int_{\mathbb{R}^n} |g(y)|^2\, dy \qquad \lambda > 0$$

where $J_{\alpha,\lambda}$, $\alpha,\lambda > 0$ is the Bessel potential defined by

$$J_{\alpha,\lambda}f = K_{\alpha,\lambda} \star f.$$

This is a what is called a **trace inequality.**

In other words $$C_\lambda(V) = \|J_{1,\lambda}\|_{L^2 \to L^2(V)}$$

The case $\lambda = 0$ is specially interesting and we have

$$C_0(V) = \|I_1\|_{L^2 \to L^2(V)}$$

where now I_α, $0 < \alpha < n$, is the Riesz potential operator defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

The trace inequality II

The trace inequality II

Then, is interesting to find conditions on the positive measure $d\mu$ such that

The trace inequality II

Then, is interesting to find conditions on the positive measure $d\mu$ such that

$$\int_{\mathbb{R}^n} T_\Phi f(x)^p \, d\mu(x) \leq c \int_{\mathbb{R}^n} f(x)^p \, dx$$

The trace inequality II

Then, is interesting to find conditions on the positive measure $d\mu$ such that

$$\int_{\mathbb{R}^n} T_\Phi f(x)^p \, d\mu(x) \leq c \int_{\mathbb{R}^n} f(x)^p \, dx$$

Kerman and Sawyer related these operators with their dual ones, namely

The trace inequality II

Then, is interesting to find conditions on the positive measure $d\mu$ such that

$$\int_{\mathbb{R}^n} T_\Phi f(x)^p \, d\mu(x) \le c \int_{\mathbb{R}^n} f(x)^p \, dx$$

Kerman and Sawyer related these operators with their dual ones, namely

$$T_\Phi^\mu(f)(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, d\mu(y)$$

The trace inequality II

Then, is interesting to find conditions on the positive measure $d\mu$ such that

$$
\int_{\mathbb{R}^n} T_\Phi f(x)^p \, d\mu(x) \leq c \int_{\mathbb{R}^n} f(x)^p \, dx
$$

Kerman and Sawyer related these operators with their dual ones, namely

$$
T_\Phi^\mu(f)(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y) \, d\mu(y)
$$

and its associated maximal type fractional operators

The trace inequality II

Then, is interesting to find conditions on the positive measure $d\mu$ such that

$$\int_{\mathbb{R}^n} T_\Phi f(x)^p \, d\mu(x) \leq c \int_{\mathbb{R}^n} f(x)^p \, dx$$

Kerman and Sawyer related these operators with their dual ones, namely

$$T_\Phi^\mu(f)(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, d\mu(y)$$

and its associated maximal type fractional operators

$$M_{\tilde{\Phi}}^\mu(f)(x) = \sup_{x \in Q} \frac{\tilde{\Phi}(\ell(Q))}{|Q|} \int_Q f(y) \, d\mu(y).$$

Kerman-Sawyer's theorem for the trace inequality

Kerman-Sawyer's theorem for the trace inequality

Φ is a decreasing function and μ a positive measure on $\mathbb{R}^n$.

Kerman-Sawyer's theorem for the trace inequality

Φ is a decreasing function and μ a positive measure on $\mathbb{R}^n$.

theorem

T.F.A.E.:

i) The trace inequality holds:

$$\left(\int_{\mathbb{R}^n} T_\Phi f(x)^p \, d\mu(x) \right)^{1/p} \leq c \left(\int_{\mathbb{R}^n} f(x)^p \, dx \right)^{1/p} \qquad f \geq 0$$

ii)

$$\left(\int_{\mathbb{R}^n} T_\Phi^\mu(\chi_Q)(x)^{p'} \, dx \right)^{1/p'} \leq c \, \mu(Q)^{1/p'} \qquad Q \in \mathcal{D}$$

iii)

$$\left(\int_Q M_{\tilde{\Phi}}^\mu(\chi_Q)(x)^{p'} \, dx \right)^{1/p'} \leq c \, \mu(Q)^{1/p'} \qquad Q \in \mathcal{D}$$

Some comments about of the proof

Some comments about of the proof

Use of duality:

Some comments about of the proof

Use of duality:

$$\left(\int_{\mathbb{R}^n} T_\Phi^\mu f(x)^{p'} \, dx \right)^{1/p'} \leq C_\mu \left(\int_{\mathbb{R}^n} f(x)^{p'} \, d\mu(x) \right)^{1/p'} \qquad f \geq 0$$

Some comments about of the proof

Use of duality:

$$\left(\int_{\mathbb{R}^n} T_\Phi^\mu f(x)^{p'} \, dx \right)^{1/p'} \leq C_\mu \left(\int_{\mathbb{R}^n} f(x)^{p'} \, d\mu(x) \right)^{1/p'} \qquad f \geq 0$$

Control by the maximal function:

Some comments about of the proof

Use of duality:

$$\left(\int_{\mathbb{R}^n} T_\Phi^\mu f(x)^{p'}\, dx\right)^{1/p'} \leq C_\mu \left(\int_{\mathbb{R}^n} f(x)^{p'}\, d\mu(x)\right)^{1/p'} \qquad f \geq 0$$

Control by the maximal function:

$$\left\|T_\Phi^\mu(f)\right\|_{L^{p'}} \approx \left\|M_{\tilde{\Phi}}^\mu(f)\right\|_{L^{p'}}$$

Some comments about of the proof

Use of duality:

$$\left(\int_{\mathbb{R}^n} T_{\Phi}^{\mu} f(x)^{p'} \, dx\right)^{1/p'} \leq C_{\mu} \left(\int_{\mathbb{R}^n} f(x)^{p'} \, d\mu(x)\right)^{1/p'} \qquad f \geq 0$$

Control by the maximal function:

$$\left\|T_{\Phi}^{\mu}(f)\right\|_{L^{p'}} \approx \left\|M_{\Phi}^{\mu}(f)\right\|_{L^{p'}}$$

Proof by good-λ as in the work of Muckenhoupt-Wheeden

Some comments about of the proof

Use of duality:

$$\left(\int_{\mathbb{R}^n} T_{\Phi}^{\mu} f(x)^{p'} \, dx \right)^{1/p'} \le C_{\mu} \left(\int_{\mathbb{R}^n} f(x)^{p'} \, d\mu(x) \right)^{1/p'} \qquad f \ge 0$$

Control by the maximal function:

$$\left\| T_{\Phi}^{\mu}(f) \right\|_{L^{p'}} \approx \left\| M_{\Phi}^{\mu}(f) \right\|_{L^{p'}}$$

Proof by good-λ as in the work of Muckenhoupt-Wheeden

Replace cubes by dyadic cubes:

Some comments about of the proof

Use of duality:

$$\left(\int_{\mathbb{R}^n} T_\Phi^\mu f(x)^{p'} \, dx \right)^{1/p'} \leq C_\mu \left(\int_{\mathbb{R}^n} f(x)^{p'} \, d\mu(x) \right)^{1/p'} \qquad f \geq 0$$

Control by the maximal function:

$$\left\| T_\Phi^\mu(f) \right\|_{L^{p'}} \approx \left\| M_{\tilde{\Phi}}^\mu(f) \right\|_{L^{p'}}$$

Proof by good-λ as in the work of Muckenhoupt-Wheeden

Replace cubes by dyadic cubes:

$$\approx \left\| M_{\tilde{\Phi}}^{\mu,d}(f) \right\|_{L^{p'}}$$

Some comments about of the proof

Use of duality:

$$\left(\int_{\mathbb{R}^n} T_\Phi^\mu f(x)^{p'}\, dx\right)^{1/p'} \leq C_\mu \left(\int_{\mathbb{R}^n} f(x)^{p'}\, d\mu(x)\right)^{1/p'} \qquad f \geq 0$$

Control by the maximal function:

$$\left\|T_\Phi^\mu(f)\right\|_{L^{p'}} \approx \left\|M_{\tilde{\Phi}}^\mu(f)\right\|_{L^{p'}}$$

Proof by good-λ as in the work of Muckenhoupt-Wheeden

Replace cubes by dyadic cubes:

$$\approx \left\|M_{\tilde{\Phi}}^{\mu,d}(f)\right\|_{L^{p'}}$$

Finally step: prove a **testing type condition "a la Sawyer"**

Solution to the Fefferman-Phong problem

Solution to the Fefferman-Phong problem

After proving

Solution to the Fefferman-Phong problem

After proving

$$C_\lambda(V) \approx \sup_Q \frac{1}{v(Q)} \int_Q I_2(v\chi_Q)\, v dx$$

Solution to the Fefferman-Phong problem

After proving

$$C_\lambda(V) \approx \sup_Q \frac{1}{v(Q)} \int_Q I_2(v\chi_Q)\, v dx$$

they proved,

Solution to the Fefferman-Phong problem

After proving

$$C_\lambda(V) \approx \sup_Q \frac{1}{v(Q)} \int_Q I_2(v\chi_Q) \, v dx$$

they proved,

Theorem   (Kerman-Sawyer)

There are dimensional constans c, C such that such that the least eigenvalue λ_1 of satisfies

$$E_{small} \leq -\lambda_1 \leq E_{large}$$

where

$$E_{small} = \sup_Q \left\{ |Q|^{-2/n} : \frac{1}{v(Q)} \int_Q I_2(v\chi_Q) \, v dx \geq C \right\}$$

and

$$E_{large} = \sup_Q \left\{ |Q|^{-2/n} : \frac{1}{v(Q)} \int_Q I_2(v\chi_Q) \, v dx \geq c \right\}$$

The power bump condition

The power bump condition

In the work of of Fefferman and Phong the functionals

$$\sup_Q \ell(Q)^2 \left(\frac{1}{|Q|} \int_Q v^r \, dx \right)^{1/r}$$

was first considered instead of

The power bump condition

In the work of of Fefferman and Phong the functionals

$$\sup_{Q} \ell(Q)^2 \left(\frac{1}{|Q|} \int_Q v^r \, dx \right)^{1/r}$$

was first considered instead of

$$\sup_{Q} \frac{1}{v(Q)} \int_Q I_2(v\chi_Q) \, v dx \approx C_\lambda(V)$$

The power bump condition

In the work of of Fefferman and Phong the functionals

$$\sup_Q \ell(Q)^2 \left(\frac{1}{|Q|} \int_Q v^r \, dx \right)^{1/r}$$

was first considered instead of

$$\sup_Q \frac{1}{v(Q)} \int_Q I_2(v\chi_Q) \, v dx \approx C_\lambda(V)$$

Related work by Chang-Wilson-Wolff

The power bump condition

In the work of of Fefferman and Phong the functionals

$$\sup_{Q} \ell(Q)^2 \left(\frac{1}{|Q|} \int_Q v^r \, dx \right)^{1/r}$$

was first considered instead of

$$\sup_{Q} \frac{1}{v(Q)} \int_Q I_2(v\chi_Q) \, v dx \approx C_\lambda(V)$$

Related work by Chang-Wilson-Wolff

References:

The power bump condition

In the work of of Fefferman and Phong the functionals

$$\sup_{Q} \ell(Q)^2 \left(\frac{1}{|Q|} \int_Q v^r \, dx \right)^{1/r}$$

was first considered instead of

$$\sup_{Q} \frac{1}{v(Q)} \int_Q I_2(v\chi_Q) \, v dx \approx C_\lambda(V)$$

Related work by Chang-Wilson-Wolff

References:

R. Kerman and E. Sawyer:

"Weighted norm inequalities for potentials with applications to Schrodinger operators, Fourier transforms and Carleson measures", Bulletin of the A.M.S (1985)

The power bump condition

In the work of of Fefferman and Phong the functionals

$$\sup_{Q} \ell(Q)^2 \left(\frac{1}{|Q|} \int_Q v^r \, dx \right)^{1/r}$$

was first considered instead of

$$\sup_{Q} \frac{1}{v(Q)} \int_Q I_2(v\chi_Q) \, v dx \approx C_\lambda(V)$$

Related work by Chang-Wilson-Wolff

References:

R. Kerman and E. Sawyer:

"Weighted norm inequalities for potentials with applications to Schrodinger operators, Fourier transforms and Carleson measures", Bulletin of the A.M.S (1985)

"The trace inequality and eigenvalue estimates for Schrodinger operators", Annales de l'Institut Fourier, 36 (1986).

Schrödinger operators with change of sign

Schrödinger operators with change of sign

Schrödinger operators: $H = -\Delta - V + W, \qquad V, W \geq 0$

Schrödinger operators with change of sign

Schrödinger operators: $H = -\Delta - V + W, \qquad V, W \geq 0$

where C is the smallest constant satisfying

$$\int_{\mathbb{R}^n} |f(y)|^2 \, v(y) dy \leq C \int_{\mathbb{R}^n} \left(|\nabla f(y)|^2 + (w + \lambda)|f(y)|^2 \right) dy$$

Schrödinger operators with change of sign

Schrödinger operators: $\qquad H = -\Delta - V + W, \qquad V, W \geq 0$

where C is the smallest constant satisfying

$$\int_{\mathbb{R}^n} |f(y)|^2 \, v(y) dy \leq C \int_{\mathbb{R}^n} \left(|\nabla f(y)|^2 + (w + \lambda)|f(y)|^2 \right) \, dy$$

In this connection, Eric considered the following functional:

Schrödinger operators with change of sign

Schrödinger operators: $\qquad H = -\Delta - V + W, \qquad V, W \geq 0$

where C is the smallest constant satisfying

$$\int_{\mathbb{R}^n} |f(y)|^2 \, v(y) dy \leq C \int_{\mathbb{R}^n} \left(|\nabla f(y)|^2 + (w + \lambda)|f(y)|^2 \right) \, dy$$

In this connection, Eric considered the following functional:

$$E(Q, v, w) = \min\{Z(Q, v), \frac{v(Q)}{w(Q)}\}$$

where

Schrödinger operators with change of sign

Schrödinger operators: $\qquad H = -\Delta - V + W, \qquad V, W \geq 0$

where C is the smallest constant satisfying

$$\int_{\mathbb{R}^n} |f(y)|^2 \, v(y) dy \leq C \int_{\mathbb{R}^n} \left(|\nabla f(y)|^2 + (w + \lambda)|f(y)|^2 \right) dy$$

In this connection, Eric considered the following functional:

$$E(Q, v, w) = \min\{Z(Q, v), \frac{v(Q)}{w(Q)}\}$$

where

$$Z(Q, v) = \frac{1}{v(Q)} \int_Q I_2(v\chi_Q) \, v dx$$

The main result

Theorem (Sawyer)

If

$$\sup_Q E(Q, v, w) < \infty$$

and v and w satisfy some extra condition then

$$\int_{\mathbb{R}^n} |f(y)|^2 \, v(y) dy \leq C \int_{\mathbb{R}^n} \left(|\nabla f(y)|^2 + (w + \lambda)|f(y)|^2 \right) dy$$

The main result

Theorem (Sawyer)

If

$$\sup_{Q} E(Q, v, w) < \infty$$

and v and w satisfy some extra condition then

$$\int_{\mathbb{R}^n} |f(y)|^2 \, v(y) dy \leq C \int_{\mathbb{R}^n} \left(|\nabla f(y)|^2 + (w + \lambda)|f(y)|^2 \right) dy$$

The proof is really complicated!!!

The main result

Theorem (Sawyer)

If

$$\sup_{Q} E(Q, v, w) < \infty$$

and v and w satisfy some extra condition then

$$\int_{\mathbb{R}^n} |f(y)|^2 \, v(y) dy \leq C \int_{\mathbb{R}^n} \left(|\nabla f(y)|^2 + (w + \lambda)|f(y)|^2 \right) dy$$

The proof is really complicated!!!

Eric Sawyer: A weighted inequality and eigenvalue estimates for Schrodinger operators", Indiana Journal of Mathematics, 35, (1986)

The model example: Muckenhoupt A_p class of weights

The model example: Muckenhoupt A_p class of weights

The **Hardy–Littlewood** maximal function:

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy.$$

The model example: Muckenhoupt A_p class of weights

The **Hardy–Littlewood** maximal function:

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy.$$

Theorem

Let $1 < p < \infty$, then

$$M : L^p(w) \longrightarrow L^p(w)$$

if and only if w satisfies the A_p condition:

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w\, dx \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'}\, dx \right)^{p-1} < \infty$$

Two weight theory for the maximal function

Two weight theory for the maximal function
Model example of Eric "testing philosphy"

Two weight theory for the maximal function

Model example of Eric "testing philosphy"

$$\int_{\mathbb{R}^n} (Mf)^p \, u dx \leq C \int_{\mathbb{R}^n} |f|^p \, v dx$$

Two weight theory for the maximal function
Model example of Eric "testing philosphy"

$$\int_{\mathbb{R}^n} (Mf)^p \, u dx \leq C \int_{\mathbb{R}^n} |f|^p \, v dx$$

The natural A_p condition

Two weight theory for the maximal function

Model example of Eric "testing philosphy"

$$\int_{\mathbb{R}^n} (Mf)^p \, u dx \le C \int_{\mathbb{R}^n} |f|^p \, v dx$$

The natural A_p condition

$$(A_p) \qquad \sup_Q \left(\frac{1}{|Q|} \int_Q u \, dx \right) \left(\frac{1}{|Q|} \int_Q v^{1-p'} \, dx \right)^{p-1} < \infty$$

Two weight theory for the maximal function
Model example of Eric "testing philosphy"

$$\int_{\mathbb{R}^n} (Mf)^p \, u dx \leq C \int_{\mathbb{R}^n} |f|^p \, v dx$$

The natural A_p condition

$$(A_p) \qquad \sup_Q \left(\frac{1}{|Q|} \int_Q u \, dx \right) \left(\frac{1}{|Q|} \int_Q v^{1-p'} \, dx \right)^{p-1} < \infty$$

is **NECESSARY** but **NOT SUFFICIENT**

Two weight theory for the maximal function
Model example of Eric "testing philosphy"

$$\int_{\mathbb{R}^n} (Mf)^p \, u dx \le C \int_{\mathbb{R}^n} |f|^p \, v dx$$

The natural A_p condition

$$(A_p) \qquad \sup_Q \left(\frac{1}{|Q|} \int_Q u \, dx \right) \left(\frac{1}{|Q|} \int_Q v^{1-p'} \, dx \right)^{p-1} < \infty$$

is **NECESSARY** but **NOT SUFFICIENT**

(known from the 70's, Muckenhoupt and Wheeden.)

Two weight theory for the maximal function
Model example of Eric "testing philosphy"

$$\int_{\mathbb{R}^n} (Mf)^p \, u dx \le C \int_{\mathbb{R}^n} |f|^p \, v dx$$

The natural A_p condition

$$(A_p) \qquad \sup_Q \left(\frac{1}{|Q|} \int_Q u \, dx \right) \left(\frac{1}{|Q|} \int_Q v^{1-p'} \, dx \right)^{p-1} < \infty$$

is **NECESSARY** but **NOT SUFFICIENT**

(known from the 70's, Muckenhoupt and Wheeden.)
However:

Two weight theory for the maximal function
Model example of Eric "testing philosphy"

$$\int_{\mathbb{R}^n} (Mf)^p \, u \, dx \leq C \int_{\mathbb{R}^n} |f|^p \, v \, dx$$

The natural A_p condition

$$(A_p) \qquad \sup_Q \left(\frac{1}{|Q|} \int_Q u \, dx \right) \left(\frac{1}{|Q|} \int_Q v^{1-p'} \, dx \right)^{p-1} < \infty$$

is **NECESSARY** but **NOT SUFFICIENT**

(known from the 70's, Muckenhoupt and Wheeden.)
However:

$$(u,v) \in A_p \iff M : L^p(v) \longrightarrow L^{p,\infty}(u)$$

Eric's theorem and Eric's condition

Eric's theorem and Eric's condition

Theorem [Sawyer] (1981)

Let $1 < p < \infty$ and let (u, v) be a couple of weights, then

$$\int_{\mathbb{R}^n} Mf(x)^p \, u dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v dx$$

if and only if (u, v) satisfies the S_p condition: there is a constant c s.t.

$$(S_p) \qquad \int_Q M(\chi_Q \sigma)^p \, u dx \leq c \, \sigma(Q)$$

where $\sigma = v^{1-p'}$

Eric's theorem and Eric's condition

Theorem [Sawyer] (1981)

Let $1 < p < \infty$ and let (u, v) be a couple of weights, then

$$\fint_{\mathbb{R}^n} Mf(x)^p \, u dx \leq C \fint_{\mathbb{R}^n} |f(x)|^p v dx$$

if and only if (u, v) satisfies the S_p condition: there is a constant c s.t.

$$(S_p) \qquad \fint_Q M(\chi_Q \sigma)^p \, u dx \leq c \, \sigma(Q)$$

where $\sigma = v^{1-p'}$

He tested this "philosophy" with many other operators.

Eric's theorem and Eric's condition

Theorem [Sawyer] (1981)

Let $1 < p < \infty$ and let (u, v) be a couple of weights, then

$$\fint_{\mathbb{R}^n} Mf(x)^p\, u dx \leq C \fint_{\mathbb{R}^n} |f(x)|^p v dx$$

if and only if (u, v) satisfies the S_p condition: there is a constant c s.t.

$$(S_p) \qquad \fint_Q M(\chi_Q \sigma)^p\, u dx \leq c\, \sigma(Q)$$

where $\sigma = v^{1-p'}$

He tested this "philosophy" with many other operators.

For instance: **One sided maximal Hardy–Littlewood maximal function**

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(t)|\, dt$$

First E. Sawyer's theorem for fractional integrals

First E. Sawyer's theorem for fractional integrals

Recall the definition of the classical fractional integral: if $\quad 0 < \alpha < n$

$$I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy.$$

First E. Sawyer's theorem for fractional integrals

Recall the definition of the classical fractional integral: if $\quad 0 < \alpha < n$

$$I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy.$$

Theorem [Sawyer] (1985)

$$\left\| I_\alpha(f) \right\|_{L^{p,\infty}(u)} \leq c \, \| f \|_{L^p(v)}$$

if and only if (u, v) satisfies the following condition:

$$\int_Q I_\alpha(u\chi_Q)^{p'} \sigma \, dx \leq c \, u(Q)$$

where $\sigma = v^{1-p'}$

Second E. Sawyer's theorem for fractional integrals

Second E. Sawyer's theorem for fractional integrals

Theorem   [Sawyer] (1988)

$$\int_{\mathbb{R}^n} |I_\alpha(f)|^p \, u\,dx \leq C \int_{\mathbb{R}^n} |f|^p \, v\,dx$$

if and only if (u,v) satisfies the S_p condition:

$$\int_Q I_\alpha(\sigma\chi_Q)^p \, u\,dx \leq c\,\sigma(Q) \qquad \& \qquad \int_Q I_\alpha(u\chi_Q)^{p'} \sigma\,dx \leq c\,u(Q)$$

where $\sigma = v^{1-p'}$

Second E. Sawyer's theorem for fractional integrals

Theorem [Sawyer] (1988)

$$\int_{\mathbb{R}^n} |I_\alpha(f)|^p \, u \, dx \leq C \int_{\mathbb{R}^n} |f|^p \, v \, dx$$

if and only if (u, v) satisfies the S_p condition:

$$\int_Q I_\alpha(\sigma \chi_Q)^p \, u \, dx \leq c \, \sigma(Q) \qquad \& \qquad \int_Q I_\alpha(u\chi_Q)^{p'} \, \sigma \, dx \leq c \, u(Q)$$

where $\sigma = v^{1-p'}$

"A characterization of two weight norm inequalities for fractional and Poisson integrals", Trans. A.M.S. (1988)

Sawyer-Wheeden two weight bump condition

Sawyer-Wheeden two weight bump condition

The two weight Fefferman-Phong condition in 1992 by Sawyer and Wheeden showed that the

Theorem [Sawyer-Wheeden] (1992)

Suppose that (u, v) be a couple of weights such that for some $r > 1$

$$K = \sup_Q \ell(Q)^\alpha \left(\frac{1}{|Q|} \int_Q u^r \, dx \right)^{1/r} \left(\frac{1}{|Q|} \int_Q v^{-r} \, dx \right)^{1/r} < \infty$$

Then

$$\int_{\mathbb{R}^n} |I_\alpha f(x)|^2 \, u(x) dx \leq C \int_{\mathbb{R}^n} |f|^2 v(x) dx.$$

Sawyer-Wheeden two weight bump condition

The two weight Fefferman-Phong condition in 1992 by Sawyer and Wheeden showed that the

Theorem [Sawyer-Wheeden] (1992)

Suppose that (u, v) be a couple of weights such that for some $r > 1$

$$K = \sup_Q \ell(Q)^\alpha \left(\frac{1}{|Q|} \int_Q u^r \, dx\right)^{1/r} \left(\frac{1}{|Q|} \int_Q v^{-r} \, dx\right)^{1/r} < \infty$$

Then

$$\int_{\mathbb{R}^n} |I_\alpha f(x)|^2 \, u(x) dx \leq C \int_{\mathbb{R}^n} |f|^2 v(x) dx.$$

E. Sawyer and R.L. Wheeden, "Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces, American Journal of Mathematics, (1992)

Tips of the proof

Tips of the proof

$$I_\alpha f(x) \approx \sum_{Q \in \mathcal{D}} \frac{|Q|^{\alpha/n}}{|Q|} \int_Q f(y)\,dy\,\chi_Q(x),$$

Tips of the proof

$$I_\alpha f(x) \approx \sum_{Q \in \mathcal{D}} \frac{|Q|^{\alpha/n}}{|Q|} \int_Q f(y) \, dy \, \chi_Q(x),$$

and then after "averaging"

Tips of the proof

$$I_\alpha f(x) \approx \sum_{Q \in \mathcal{D}} \frac{|Q|^{\alpha/n}}{|Q|} \int_Q f(y)\, dy\, \chi_Q(x),$$

and then after "averaging"

$$I_\alpha f(x) \approx \sum_{k,j} \frac{|Q_{j,k}|^{\alpha/n}}{|Q_{j,k}|} \int_{Q_{j,k}} f(y)\, dy\, \chi_{Q_{j,k}}(x),$$

Tips of the proof

$$I_\alpha f(x) \approx \sum_{Q \in \mathcal{D}} \frac{|Q|^{\alpha/n}}{|Q|} \int_Q f(y)\, dy\, \chi_Q(x),$$

and then after "averaging"

$$I_\alpha f(x) \approx \sum_{k,j} \frac{|Q_{j,k}|^{\alpha/n}}{|Q_{j,k}|} \int_{Q_{j,k}} f(y)\, dy\, \chi_{Q_{j,k}}(x),$$

where the family $\{Q_{j,k}\}$ is formed by special dyadic Calderón-Zyygmund cubes.

Some comments

- This and other similar results were extended to spaces of homogeneous type.

Some comments

- This and other similar results were extended to spaces of homogeneous type.

- The case when $\alpha = 0$ corresponds to the Hilbert transform is due to Neugebauer.

Some comments

• This and other similar results were extended to spaces of homogeneous type.

• The case when $\alpha = 0$ corresponds to the Hilbert transform is due to Neugebauer.

• Improves: Chang-Wilson-Wolff   and   Fefferman-Phong

Some comments

- This and other similar results were extended to spaces of homogeneous type.

- The case when $\alpha = 0$ corresponds to the Hilbert transform is due to Neuge-bauer.

- Improves: Chang-Wilson-Wolff and Fefferman-Phong

- The failure of the Besicovitch covering lemma for the Heisenberg group was obtained (also independently by Koranyi and Reimann).

Some comments

- This and other similar results were extended to spaces of homogeneous type.

- The case when $\alpha = 0$ corresponds to the Hilbert transform is due to Neugebauer.

- Improves: Chang-Wilson-Wolff and Fefferman-Phong

- The failure of the Besicovitch covering lemma for the Heisenberg group was obtained (also independently by Koranyi and Reimann).

- A construction of a dyadic grid for spaces of homogeneous type was given (also independently by Christ, see as well David)

Some comments

- This and other similar results were extended to spaces of homogeneous type.

- The case when $\alpha = 0$ corresponds to the Hilbert transform is due to Neugebauer.

- Improves: Chang-Wilson-Wolff and Fefferman-Phong

- The failure of the Besicovitch covering lemma for the Heisenberg group was obtained (also independently by Koranyi and Reimann).

- A construction of a dyadic grid for spaces of homogeneous type was given (also independently by Christ, see as well David)

- later with Cruz-Uribe and Martell, obtained nice results for singular integrals in this vein.

Mixed weak type inequalities and Eric's conjecture

Mixed weak type inequalities and Eric's conjecture

This result of E. Sawyer can be seen as a very sophisticated version of the classical weak type $(1, 1)$ estimate for the Hardy-Littlewood maximal function.

Mixed weak type inequalities and Eric's conjecture

This result of E. Sawyer can be seen as a very sophisticated version of the classical weak type $(1,1)$ estimate for the Hardy-Littlewood maximal function.

Theorem (E. Sawyer)   Let $u, v \in A_1(\mathbb{R})$, then there is a constant c such that,

$$\left\| \frac{M(f)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}} |f(x)| \, u \, dx.$$

Mixed weak type inequalities and Eric's conjecture

This result of E. Sawyer can be seen as a very sophisticated version of the classical weak type $(1,1)$ estimate for the Hardy-Littlewood maximal function.

Theorem (E. Sawyer)   Let $u, v \in A_1(\mathbb{R})$, then there is a constant c such that,

$$\left\| \frac{M(f)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}} |f(x)| \, u \, dx.$$

E. Sawyer, "A weighted weak type inequality for the maximal function", Proceedings of the A.M.S., 93(1985).

Mixed weak type inequalities and Eric's conjecture

This result of E. Sawyer can be seen as a very sophisticated version of the classical weak type $(1,1)$ estimate for the Hardy-Littlewood maximal function.

Theorem (E. Sawyer)   Let $u, v \in A_1(\mathbb{R})$, then there is a constant c such that,

$$\left\| \frac{M(f)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}} |f(x)| \, u \, dx.$$

E. Sawyer, "A weighted weak type inequality for the maximal function", Proceedings of the A.M.S., 93(1985).

• Of course, the case $v = u = 1$ is the classical unweighted result.

Mixed weak type inequalities and Eric's conjecture

This result of E. Sawyer can be seen as a very sophisticated version of the classical weak type $(1, 1)$ estimate for the Hardy-Littlewood maximal function.

Theorem (E. Sawyer)   Let $u, v \in A_1(\mathbb{R})$, then there is a constant c such that,

$$\left\| \frac{M(f)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}} |f(x)| \, u dx.$$

E. Sawyer, "A weighted weak type inequality for the maximal function", Proceedings of the A.M.S., 93(1985).

• Of course, the case $v = u = 1$ is the classical unweighted result.

• The case $v = 1$ is the classical Fefferman-Stein's inequality and the case $u = 1$ is also immediate:

Mixed weak type inequalities and Eric's conjecture

This result of E. Sawyer can be seen as a very sophisticated version of the classical weak type $(1, 1)$ estimate for the Hardy-Littlewood maximal function.

Theorem (E. Sawyer)   Let $u, v \in A_1(\mathbb{R})$, then there is a constant c such that,

$$\left\| \frac{M(f)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}} |f(x)| \, u dx.$$

E. Sawyer, "A weighted weak type inequality for the maximal function", Proceedings of the A.M.S., 93(1985).

● Of course, the case $v = u = 1$ is the classical unweighted result.

● The case $v = 1$ is the classical Fefferman-Stein's inequality and the case $u = 1$ is also immediate:

$$\frac{M(f)}{v} \leq C \, M_v(\frac{f}{v})$$

Mixed weak type inequalities and Eric's conjecture

This result of E. Sawyer can be seen as a very sophisticated version of the classical weak type $(1, 1)$ estimate for the Hardy-Littlewood maximal function.

Theorem (E. Sawyer)   Let $u, v \in A_1(\mathbb{R})$, then there is a constant c such that,

$$\left\| \frac{M(f)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}} |f(x)| \, u dx.$$

E. Sawyer, "A weighted weak type inequality for the maximal function", Proceedings of the A.M.S., 93(1985).

• Of course, the case $v = u = 1$ is the classical unweighted result.

• The case $v = 1$ is the classical Fefferman-Stein's inequality and the case $u = 1$ is also immediate:

$$\frac{M(f)}{v} \leq C \, M_v(\frac{f}{v})$$

• The problem in general is that **the product uv can be very singular...**

Mixed weak type inequalities and Eric's conjecture

This result of E. Sawyer can be seen as a very sophisticated version of the classical weak type $(1,1)$ estimate for the Hardy-Littlewood maximal function.

Theorem (E. Sawyer)   Let $u, v \in A_1(\mathbb{R})$, then there is a constant c such that,

$$\left\|\frac{M(f)}{v}\right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}} |f(x)| \, u dx.$$

E. Sawyer, "A weighted weak type inequality for the maximal function", Proceedings of the A.M.S., 93(1985).

• Of course, the case $v = u = 1$ is the classical unweighted result.

• The case $v = 1$ is the classical Fefferman-Stein's inequality and the case $u = 1$ is also immediate:

$$\frac{M(f)}{v} \leq C \, M_v(\frac{f}{v})$$

• The problem in general is that **the product uv can be very singular...**

• To assume that $u \in A_1$ is natural because of the case $v = 1$.

Why Eric look for such a sophisticated a result?

Why Eric look for such a sophisticated a result?

To give a different proof of Muckenhoupt's A_p theorem assuming that the weights are already factored:

$$w = uv^{1-p} \qquad u, v \in A_1.$$

Why Eric look for such a sophisticated a result?

To give a different proof of Muckenhoupt's A_p theorem assuming that the weights are already factored:

$$w = uv^{1-p} \qquad u, v \in A_1.$$

Indeed, let

Why Eric look for such a sophisticated a result?

To give a different proof of Muckenhoupt's A_p theorem assuming that the weights are already factored:

$$w = uv^{1-p} \qquad u, v \in A_1.$$

Indeed, let

$$S_v(f) = \frac{M(f\,v)}{v}$$

Why Eric look for such a sophisticated a result?

To give a different proof of Muckenhoupt's A_p theorem assuming that the weights are already factored:

$$w = uv^{1-p} \qquad u, v \in A_1.$$

Indeed, let

$$S_v(f) = \frac{M(f\,v)}{v}$$

Then, Eric's result says that S_v is of weak type $(1, 1)$ w. r. to the measure uv:

$$S_v : L^1(uv) \to L^{1,\infty}(uv)$$

Why Eric look for such a sophisticated a result?

To give a different proof of Muckenhoupt's A_p theorem assuming that the weights are already factored:

$$w = uv^{1-p} \qquad u, v \in A_1.$$

Indeed, let

$$S_v(f) = \frac{M(f\,v)}{v}$$

Then, Eric's result says that S_v is of weak type $(1,1)$ w. r. to the measure uv:

$$S_v : L^1(uv) \to L^{1,\infty}(uv)$$

But observe that S_v is also bounded on L^∞, $(v \in A_1)$ and hence on $L^\infty(uv)$.

Why Eric look for such a sophisticated a result?

To give a different proof of Muckenhoupt's A_p theorem assuming that the weights are already factored:

$$w = uv^{1-p} \qquad u, v \in A_1.$$

Indeed, let

$$S_v(f) = \frac{M(f\,v)}{v}$$

Then, Eric's result says that S_v is of weak type $(1,1)$ w. r. to the measure uv:

$$S_v : L^1(uv) \to L^{1,\infty}(uv)$$

But observe that S_v is also bounded on L^∞, $(v \in A_1)$ and hence on $L^\infty(uv)$.

Then by the Marcinkiewicz interpolation theorem:

Why Eric look for such a sophisticated a result?

To give a different proof of Muckenhoupt's A_p theorem assuming that the weights are already factored:

$$w = uv^{1-p} \qquad u, v \in A_1.$$

Indeed, let

$$S_v(f) = \frac{M(f\,v)}{v}$$

Then, Eric's result says that S_v is of weak type $(1, 1)$ w. r. to the measure uv:

$$S_v : L^1(uv) \to L^{1,\infty}(uv)$$

But observe that S_v is also bounded on L^∞, $(v \in A_1)$ and hence on $L^\infty(uv)$.

Then by the Marcinkiewicz interpolation theorem:

$$M : L^p(uv^{1-p}) \to L^p(uv^{1-p})$$

Two conjectures by Eric

Two conjectures by Eric

- First conjecture: to replace M by the Hilbert transform H, namely

Two conjectures by Eric

- First conjecture: to replace M by the Hilbert transform H, namely

$$\left\| \frac{H(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}} |f(x)| uv \, dx$$

Two conjectures by Eric

- First conjecture: to replace M by the Hilbert transform H, namely

$$\left\|\frac{H(fv)}{v}\right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}} |f(x)| uv \, dx$$

- Second conjecture: to replace M in one dimension by M in $\mathbb{R}^n$, namely

Two conjectures by Eric

- First conjecture: to replace M by the Hilbert transform H, namely

$$\left\|\frac{H(fv)}{v}\right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}} |f(x)| uv \, dx$$

- Second conjecture: to replace M in one dimension by M in $\mathbb{R}^n$, namely

$$\left\|\frac{M(fv)}{v}\right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)| uv \, dx$$

Results

Results

Joint work with J. M. Martell and D. Cruz-Uribe

Results

Joint work with J. M. Martell and D. Cruz-Uribe

T is any **Calderón-Zygmund** .

Results

Joint work with J. M. Martell and D. Cruz-Uribe

T is any **Calderón-Zygmund** .

Theorem   If $u \in A_1(\mathbb{R}^n)$, and $v \in A_1(\mathbb{R}^n)$, then there is a constant c such that

$$\left\| \frac{M(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)| uv \, dx$$

and

$$\left\| \frac{T(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)| uv \, dx$$

Results

Joint work with J. M. Martell and D. Cruz-Uribe

T is any **Calderón-Zygmund** .

Theorem   If $u \in A_1(\mathbb{R}^n)$, and $v \in A_1(\mathbb{R}^n)$, then there is a constant c such that

$$\left\| \frac{M(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)|uv\,dx$$

and

$$\left\| \frac{T(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)|uv\,dx$$

and also

Results

Joint work with J. M. Martell and D. Cruz-Uribe

T is any **Calderón-Zygmund** .

Theorem   If $u \in A_1(\mathbb{R}^n)$, and $v \in A_1(\mathbb{R}^n)$, then there is a constant c such that

$$\left\| \frac{M(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)| uv \, dx$$

and

$$\left\| \frac{T(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)| uv \, dx$$

and also

Theorem If $u \in A_1(\mathbb{R}^n)$, and $v \in A_\infty(u)$, then the same conclusion holds.

Results

Joint work with J. M. Martell and D. Cruz-Uribe

T is any **Calderón-Zygmund** .

Theorem   If $u \in A_1(\mathbb{R}^n)$, and $v \in A_1(\mathbb{R}^n)$, then there is a constant c such that

$$\left\|\frac{M(fv)}{v}\right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)|uv\,dx$$

and

$$\left\|\frac{T(fv)}{v}\right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)|uv\,dx$$

and also

Theorem If $u \in A_1(\mathbb{R}^n)$, and $v \in A_\infty(u)$, then the same conclusion holds.

Example: $u = 1$ and $v \in A_\infty$

Results

Joint work with J. M. Martell and D. Cruz-Uribe

T is any **Calderón-Zygmund** .

Theorem If $u \in A_1(\mathbb{R}^n)$, and $v \in A_1(\mathbb{R}^n)$, then there is a constant c such that

$$\left\| \frac{M(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)| uv \, dx$$

and

$$\left\| \frac{T(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)| uv \, dx$$

and also

Theorem If $u \in A_1(\mathbb{R}^n)$, and $v \in A_\infty(u)$, then the same conclusion holds.

Example: $u = 1$ and $v \in A_\infty$

Some special cases were considered in the 70's by **Muckenhoupt-Wheeden** for the Hilbert Transform.

Key Lemma

Key Lemma

Our approach is completely different

Key Lemma

Our approach is completely different

We found a sort of "**mechanism**" (a very general extrapolation type theorem) to reduce everything to prove the corresponding estimate for the **dyadic maximal operator**.

Key Lemma

Our approach is completely different

We found a sort of "**mechanism**" (a very general extrapolation type theorem) to reduce everything to prove the corresponding estimate for the **dyadic maximal operator**.

Lemma Let $u \in A_1(\mathbb{R}^n)$ and $v \in A_\infty(\mathbb{R}^n)$, then

$$\left\|\frac{T(fv)}{v}\right\|_{L^{1,\infty}(uv)} \leq c \left\|\frac{M^d(fv)}{v}\right\|_{L^{1,\infty}(uv)}$$

Key Lemma

Our approach is completely different

We found a sort of "**mechanism**" (a very general extrapolation type theorem) to reduce everything to prove the corresponding estimate for the **dyadic maximal operator**.

Lemma Let $u \in A_1(\mathbb{R}^n)$ and $v \in A_\infty(\mathbb{R}^n)$, then

$$\left\|\frac{T(fv)}{v}\right\|_{L^{1,\infty}(uv)} \leq c \left\|\frac{M^d(fv)}{v}\right\|_{L^{1,\infty}(uv)}$$

In fact this results holds for **non** A_∞ weights on v.

The final key inequality and Sawyer's conjecture

The final key inequality and Sawyer's conjecture

Then the proof of the Theorem is reduced to prove a Sawyer´s type estimate for the dyadic maximal operator but in $\mathbb{R}^n$:

The final key inequality and Sawyer's conjecture

Then the proof of the Theorem is reduced to prove a Sawyer´s type estimate for the dyadic maximal operator but in $\mathbb{R}^n$:

Theorem   If $u \in A_1(\mathbb{R}^n)$, and $v \in A_1(\mathbb{R}^n)$, then there is a constant c such that

$$\left\| \frac{M^d(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)| \, uv \, dx$$

The final key inequality and Sawyer's conjecture

Then the proof of the Theorem is reduced to prove a Sawyer´s type estimate for the dyadic maximal operator but in $\mathbb{R}^n$:

Theorem   If $u \in A_1(\mathbb{R}^n)$, and $v \in A_1(\mathbb{R}^n)$, then there is a constant c such that

$$\left\| \frac{M^d(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)| uv \, dx$$

We were very lucky because the proof is basically the same as in the Eric's paper, we have reduced everything to the dyadic and is a bit simpler but

The final key inequality and Sawyer's conjecture

Then the proof of the Theorem is reduced to prove a Sawyer´s type estimate for the dyadic maximal operator but in $\mathbb{R}^n$:

Theorem   If $u \in A_1(\mathbb{R}^n)$, and $v \in A_1(\mathbb{R}^n)$, then there is a constant c such that

$$\left\| \frac{M^d(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)| uv \, dx$$

We were very lucky because the proof is basically the same as in the Eric's paper, we have reduced everything to the dyadic and is a bit simpler but

We **DON'T UNDERSTAND IT**

The final key inequality and Sawyer's conjecture

Then the proof of the Theorem is reduced to prove a Sawyer´s type estimate for the dyadic maximal operator but in $\mathbb{R}^n$:

Theorem   If $u \in A_1(\mathbb{R}^n)$, and $v \in A_1(\mathbb{R}^n)$, then there is a constant c such that

$$\left\| \frac{M^d(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)| uv \, dx$$

We were very lucky because the proof is basically the same as in the Eric's paper, we have reduced everything to the dyadic and is a bit simpler but

We **DON'T UNDERSTAND IT**

We are not happy neither with the proof nor with the result.

The final key inequality and Sawyer's conjecture

Then the proof of the Theorem is reduced to prove a Sawyer´s type estimate for the dyadic maximal operator but in $\mathbb{R}^n$:

Theorem   If $u \in A_1(\mathbb{R}^n)$, and $v \in A_1(\mathbb{R}^n)$, then there is a constant c such that

$$\left\|\frac{M^d(fv)}{v}\right\|_{L^{1,\infty}(uv)} \leq c \int_{\mathbb{R}^n} |f(x)| uv \, dx$$

We were very lucky because the proof is basically the same as in the Eric's paper, we have reduced everything to the dyadic and is a bit simpler but

We **DON'T UNDERSTAND IT**

We are not happy neither with the proof nor with the result.

Sawyer's conjecture: The result should hold in the following case

$$u \in A_1(\mathbb{R}^n) \qquad \& \qquad v \in A_\infty(\mathbb{R}^n)$$

which corresponds to the most singular case.

Proof: the two basic Ingredients

Proof: the two basic Ingredients

- The **Coifman-Fefferman's** estimate: if $0 < p < \infty$ and $w \in A_\infty$

Proof: the two basic Ingredients

- The **Coifman-Fefferman's** estimate: if $0 < p < \infty$ and $w \in A_\infty$

Let T be any Calderón-Zygmund Operator and let $0 < p < \infty$ and $w \in A_\infty$. Then

$$\|Tf\|_{L^p(w)} \leq c \|Mf\|_{L^p(w)}$$

Proof: the two basic Ingredients

- The **Coifman-Fefferman's** estimate: if $0 < p < \infty$ and $w \in A_\infty$

Let T be any Calderón-Zygmund Operator and let $0 < p < \infty$ and $w \in A_\infty$. Then

$$\|Tf\|_{L^p(w)} \le c \, \|Mf\|_{L^p(w)}$$

- and the use of the so called Rubio de Francia's algorithm for a non standard operator, namely Eric's operator:

Proof: the two basic Ingredients

• The **Coifman-Fefferman's** estimate: if $0 < p < \infty$ and $w \in A_\infty$

Let T be any Calderón-Zygmund Operator and let $0 < p < \infty$ and $w \in A_\infty$. Then

$$\|Tf\|_{L^p(w)} \leq c \|Mf\|_{L^p(w)}$$

• and the use of the so called Rubio de Francia's algorithm for a non standard operator, namely Eric's operator:

$$Rh(x) = \sum_{k=0}^{\infty} \frac{S_v^k h(x)}{2^k \|S_v\|^k}.$$

Proof: the two basic Ingredients

- The **Coifman-Fefferman's** estimate: if $0 < p < \infty$ and $w \in A_\infty$

Let T be any Calderón-Zygmund Operator and let $0 < p < \infty$ and $w \in A_\infty$. Then

$$\|Tf\|_{L^p(w)} \leq c \, \|Mf\|_{L^p(w)}$$

- and the use of the so called Rubio de Francia's algorithm for a non standard operator, namely Eric's operator:

$$Rh(x) = \sum_{k=0}^{\infty} \frac{S_v^k h(x)}{2^k \, \|S_v\|^k}.$$

where

$$S_v f(x) = \frac{M(fv)(x)}{v(x)}$$

Singular Integrals and the C_p class of weights

Singular Integrals and the C_p class of weights

Motivation: to understand the following version of the theorem of Coifman-Fefferman.

Singular Integrals and the C_p class of weights

Motivation: to understand the following version of the theorem of Coifman-Fefferman.

Let H be the Hilbert transform let $1 < p < \infty$ and $w \in A_\infty$.
Then

$$\int_{\mathbb{R}} |Hf|^p \, w \, dx \leq C \int_{\mathbb{R}} (Mf)^p \, w \, dx$$

Singular Integrals and the C_p class of weights

Motivation: to understand the following version of the theorem of Coifman-Fefferman.

Let H be the Hilbert transform let $1 < p < \infty$ and $w \in A_\infty$.
Then

$$\int_{\mathbb{R}} |Hf|^p \, w \, dx \leq C \int_{\mathbb{R}} (Mf)^p \, w \, dx$$

This question was raised by Muckenhoupt in the 70's. He showed that if the weight satisfies this estimate then:

Singular Integrals and the C_p class of weights

Motivation: to understand the following version of the theorem of Coifman-Fefferman.

Let H be the Hilbert transform let $1 < p < \infty$ and $w \in A_\infty$.
Then

$$\int_{\mathbb{R}} |Hf|^p \, w \, dx \leq C \int_{\mathbb{R}} (Mf)^p \, w \, dx$$

This question was raised by Muckenhoupt in the 70's. He showed that if the weight satisfies this estimate then:

$$\omega(E) \leq C \left(\frac{|E|}{|Q|} \right)^{\epsilon} \int_{\mathbb{R}} (M(\chi_Q))^p \, w \, dx$$

for any cube Q and for any set $E \subset Q$.

The C_p condition: necessity

The C_p condition: necessity

R_j will denote the Riesz transforms.

The C_p condition: necessity

R_j will denote the Riesz transforms.

Theorem (Eric)

Let $1 < p < \infty$. If the weight w satisfies

$$\int_{\mathbb{R}^n} |R_j f|^p \, w \, dx \leq C \int_{\mathbb{R}^n} (Mf)^p \, w \, dx, \qquad 1 \leq j \leq n, \quad f \in L_c^\infty$$

Then w satifies the C_p condition:

$$\omega(E) \leq C \left(\frac{|E|}{|Q|} \right)^\epsilon \int_{\mathbb{R}^n} (M(\chi_Q))^p \, w \, dx$$

for any cube Q and for any set $E \subset Q$.

The C_p condition: sufficiency

The C_p condition: sufficiency

Theorem (Eric)

Let $1 < p < \infty$ and let $\epsilon > 0$. Let T be a Singular Integral, then if weight w satisfies the $C_{p+\delta}$ condition:

$$\omega(E) \leq C\left(\frac{|E|}{|Q|}\right)^{\epsilon} \int_{\mathbb{R}^n} (M(\chi_Q))^{p+\delta}\, w\, dx$$

for any cube Q and for any set $E \subset Q$.
Then

$$\int_{\mathbb{R}^n} |Tf|^p\, w\, dx \leq C \int_{\mathbb{R}^n} (Mf)^p\, w\, dx.$$

The Monge-Ampere Equation

The Monge-Ampere Equation

$$u_{xx}u_{yy} - u_{xy}^2 = k(x,y)$$

near the origin with $0 \leq k(x,y) \in C^\infty(\mathbb{R}^2)$.

The Monge-Ampere Equation

$$u_{xx}u_{yy} - u_{xy}^2 = k(x,y)$$

near the origin with $0 \leq k(x,y) \in C^\infty(\mathbb{R}^2)$.

Then under minimal assumption on k and a minimal initial assumption on u then $u \in C^\infty$ near the origin.

The Monge-Ampere Equation

$$u_{xx}u_{yy} - u_{xy}^2 = k(x,y)$$

near the origin with $0 \leq k(x,y) \in C^\infty(\mathbb{R}^2)$.

Then under minimal assumption on k and a minimal initial assumption on u then $u \in C^\infty$ near the origin.

This theorem is due to P. Guan.

The Monge-Ampere Equation

$$u_{xx}u_{yy} - u_{xy}^2 = k(x, y)$$

near the origin with $0 \le k(x, y) \in C^\infty(\mathbb{R}^2)$.

Then under minimal assumption on k and a minimal initial assumption on u then $u \in C^\infty$ near the origin.

This theorem is due to P. Guan.

Improved in joint work with Eric (TAMS)

The Monge-Ampere Equation

$$u_{xx}u_{yy} - u_{xy}^2 = k(x,y)$$

near the origin with $0 \leq k(x,y) \in C^\infty(\mathbb{R}^2)$.

Then under minimal assumption on k and a minimal initial assumption on u then $u \in C^\infty$ near the origin.

This theorem is due to P. Guan.

Improved in joint work with Eric (TAMS)

Theorem [Rios, Sawyer and Wheeden] (2008)

Let $n \geq 3$ and suppose $k \approx |x|^{2m}$ can be written as a sum of squares of smooth functions in $\Omega \subset \mathbb{R}^n$. If u is a C^2 convex solution u to the subelliptic Monge-Ampére equation

$$det D^2 u(x) = k(x,u,Du) \qquad x \in \Omega,$$

then u is smooth if the elementary $(n-1)^{st}$ symmetric curvature k_{n-1} of u is positive (the case $m \geq 2$ uses an additional nondegeneracy condition on the sum of squares).

Advances in Math 2008

Comments

Comments

As a conseqence they obtain the following geometric result: a C^2 convex function u whose graph has smooth Gaussian curvature $k \approx |x|^2$ is itself smooth if and only if the subGaussian curvature k_{n-1} of u is positive in Ω.

future

future

I am sure that Eric is going to produce more mathematics for us!!!

future

I am sure that Eric is going to produce more mathematics for us!!!

THANK YOU VERY MUCH