# Higher integrability of the Harmonic Measure and Uniform Rectifiability

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joint work with

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Spain

Conference in Harmonic Analysis
and Partial Differential Equations
in honour of Eric Sawyer

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Section 1

# Introduction

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**Theorem (F. & M. Riesz 1916)**

$\Omega \subset \mathbb{C}$ **simply connected domain** *with* **rectifiable boundary**

$$\textit{harmonic measure } \omega \ll \sigma = \mathcal{H}^1\big|_{\partial\Omega}$$

- [Lavrentiev 1936] Quantitative version
- [Bishop-Jones 1990]
  - $E \subset \partial\Omega$, $E$ rectifiable $\Longrightarrow$ $\omega \ll \sigma$ on $E$
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**Goal:** Obtain **higher dimensional**, **scale-invariant** (quantitative) version of F. & M. Riesz

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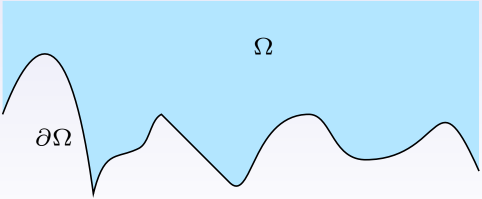
# Harmonic measure

- $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, connected and open $\rightsquigarrow$ $\sigma = \mathcal{H}^n\big|_{\partial\Omega}$
- Surface ball $\Delta(x,r) = B(x,r) \cap \partial\Omega$ with $x \in \partial\Omega$
- Harmonic measure $\{\omega^X\}_{X\in\Omega}$ family of probabilities on $\partial\Omega$

$$u(X) = \int_{\partial\Omega} f(x)\, d\omega^X(x) \quad \text{solves} \quad (D) \begin{cases} \mathcal{L}u = 0 \text{ in } \Omega \\ u\big|_{\partial\Omega} = f \in C_c(\partial\Omega) \end{cases}$$



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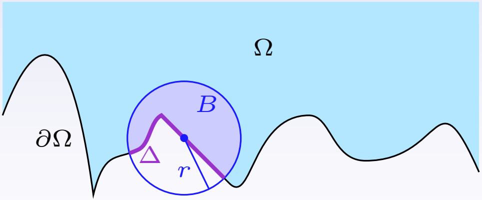

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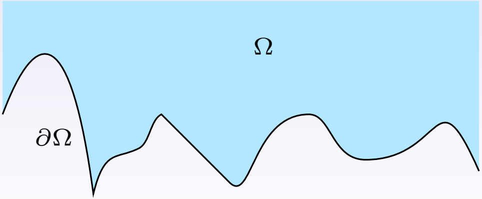

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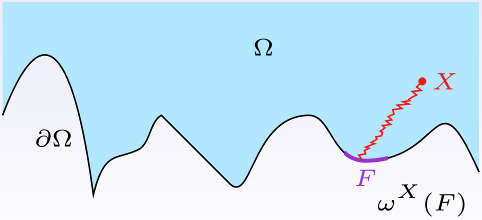

# Qualitative vs. Quantitative (scale-invariant)

- $\omega \ll \sigma$ $\rightsquigarrow$ $\omega \in A_\infty(\sigma)$

$$\sigma(F) = 0 \;\Rightarrow\; \omega(F) = 0 \qquad \frac{\omega(F)}{\omega(\Delta)} \lesssim \left(\frac{\sigma(F)}{\sigma(\Delta)}\right)^\theta, \quad F \subset \Delta$$

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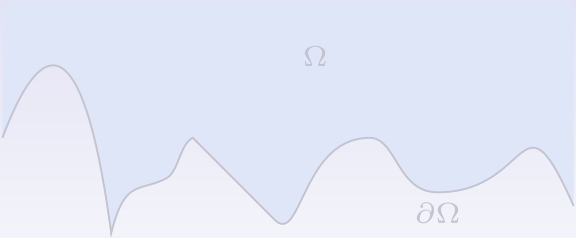


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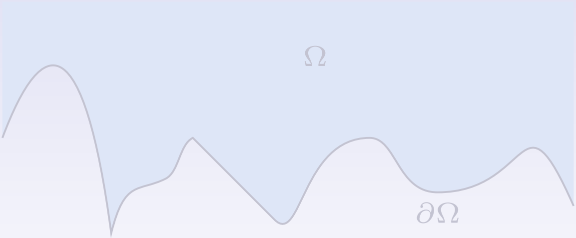


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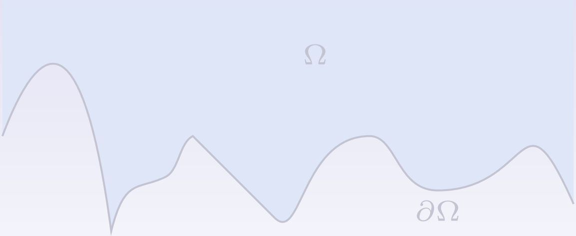


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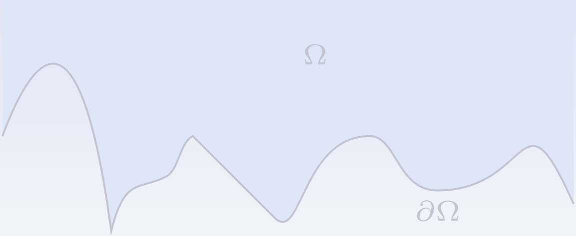


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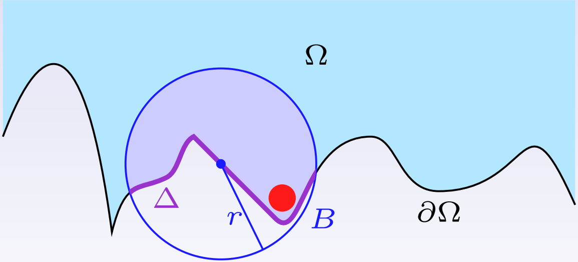


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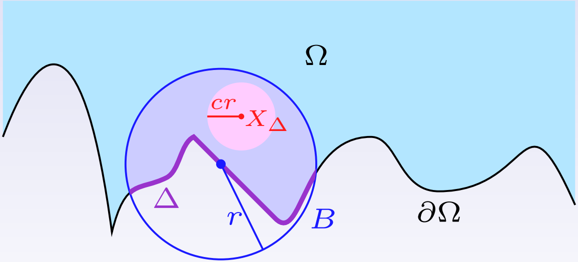


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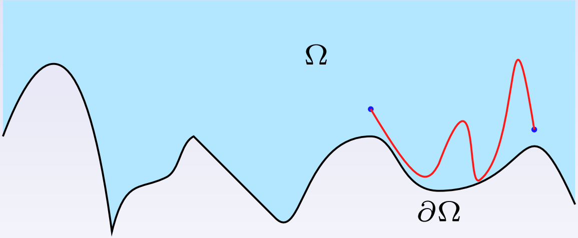


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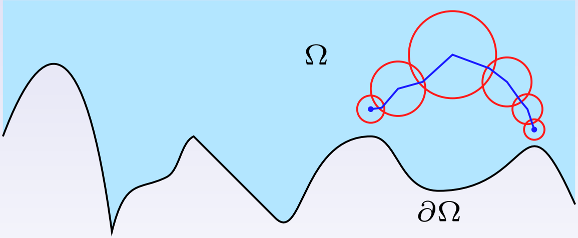


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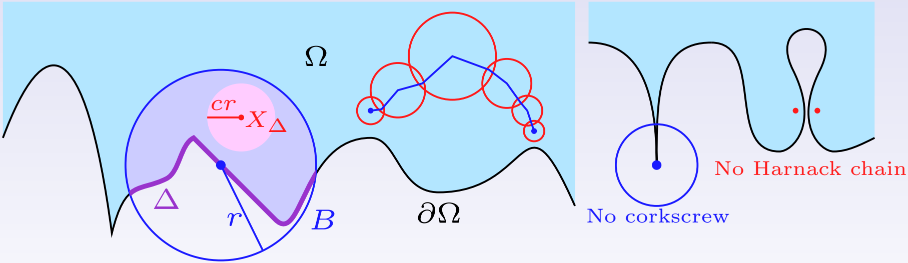


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**Definition (Jerison-Kenig 1982)**

$\Omega \subset \mathbb{R}^{n+1}$ is NTA if

- $\Omega$ satisfies the Corkscrew condition
- $\Omega_{\text{ext}} = \mathbb{R}^{n+1} \setminus \overline{\Omega}$ satisfies the Corkscrew condition
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**Theorem (David-Jerison 1990; Semmes 1989)**

- $\Omega \subset \mathbb{R}^{n+1}$ *is NTA*
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*Then* $\omega \in A_\infty(\sigma)$.

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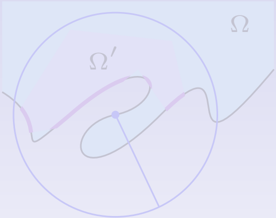
# NTA domains: Key steps in [David-Jerison 1990]

1. BPBLSD: Big Pieces of Boundaries of Lipschitz Sub-Domains

For every $B(x,r)$, $x \in \partial\Omega$,

- $\exists\, \Omega' \subset \Omega$ Lipschitz
- "Ample contact"

$$\sigma(\partial\Omega' \cap \partial\Omega \cap B(x,r)) \gtrsim r^n$$




2. Maximum principle + [Dahlberg 77]: $0 < \eta \ll 1$ ("Big pieces")

$$F \subset \Delta, \quad \sigma(F) \geq (1-\eta)\,\sigma(\Delta) \quad \Longrightarrow \quad \omega^{X_\Delta}(F) \geq c_0 > 0 \qquad (\star)$$

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$(\star)$ self-improves to $\omega \in A_\infty(\sigma)$

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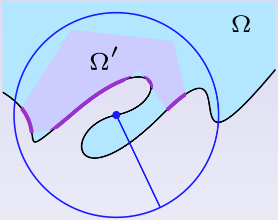


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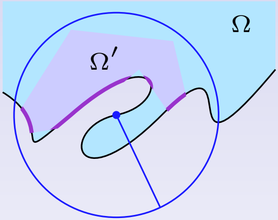


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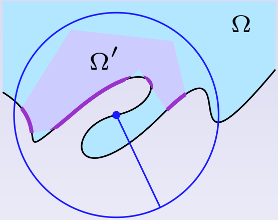


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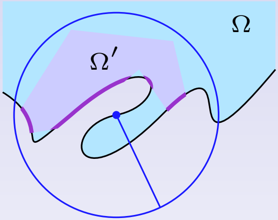


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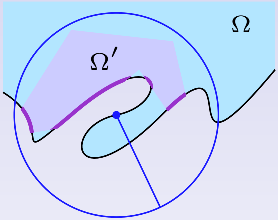


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## Related results

- [Bennewitz-Lewis 2004] Remove ext. corkscrew and Harnack chain
  - $\partial\Omega$ ADR
  - $\Omega$ interior corkscrew
  - BPBLSD
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  - $(\star)$ self-improves to "weak-$A_\infty$" ("weak Reverse Hölder") (even without comparison principle)

$$\omega(F) \lesssim \left(\frac{\sigma(F)}{\sigma(\Delta)}\right)^\theta \omega(2\,\Delta)$$

  - Sharp by counterexample

- [Badger 2011]
  - $\Omega$ NTA
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## Related results

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# Uniform rectifiability

## Definition

$E \subset \mathbb{R}^{n+1}$ closed ADR is UR if

$$\int_{\mathbb{R}^{n+1}\setminus E} |\nabla^2 \mathcal{S} f(X)|^2 \operatorname{dist}(X,E)\, dX \le C \int_E |f(y)|^2 \, d\mathcal{H}^n(y)$$

where $\mathcal{S}f$ single layer potential

$$\mathcal{S}f(X) := c_n \int_E \frac{f(y)}{|X-y|^{n-1}}\, d\mathcal{H}^n(y), \qquad X \notin E$$

- [David-Semmes 1991]
  $E$ is UR $\iff$ $E$ is ADR + all "nice" SIO are bounded on $L^2(E)$
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## Qualitative vs. Quantitative (scale-invariant)

- Rectifiability $\rightsquigarrow$ Uniform Rectifiablity

  Existence approx. tangent planes P. Jones's $\beta$-functionals

$$\beta_2(x,t) = \inf_P \left( \frac{1}{t^n} \int_{B(x,t)\cap E} \left( \frac{\operatorname{dist}(y,P)}{t} \right)^2 d\mathcal{H}^n(y) \right)^{1/2}, \quad x \in E,\ t > 0$$

- [David-Semmes 1991] $E$ ADR is UR if and only if

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Section 2

# Main results

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### Theorem

- $\partial\Omega$ *ADR*
- $\Omega$ *interior Corkscrew and Harnack chain*

[Hofmann, M.] $\partial\Omega$ $UR \Longrightarrow \omega \in A_\infty^{\text{weak}}$ *(weak-Reverse Hölder)*

$$\int_\Delta (k^{X_\Delta})^q \, d\sigma \lesssim \sigma(\Delta)^{1-q} \qquad (RH_q^{\text{weak}})$$

[Hofmann, Uriarte-Tuero, M.] $(RH_q^{\text{weak}}) \Longrightarrow \partial\Omega$ $UR$

- No exterior assumptions. No assumption BPBLSD.
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## "Large constant" analogue of Kenig-Toro 1997-2003

- [Kenig-Toro] • $\partial\Omega$ ADR • Reifenberg flatness

  $\Omega$ "vanishing chord-arc" $\iff$ $\log k \in VMO$

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  "vanishing chord-arc" $\iff$ 2-sided NTA + "Vanishing UR"

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Section 3

# Strategy of the Proof

# Strategy of the Proof: $w \in A_\infty^{\text{weak}}(\sigma) \implies \text{UR}$

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2. UR for approximating domains $\Omega_N$ (uniformly in $N$)
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Not always true

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$$\int_{\mathbb{R}^{n+1}} |\nabla^2 \mathcal{S} f(Y)|^2 \, \delta(Y) \, dY \lesssim \|f\|_{L^2(\partial\Omega)}$$

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**Theorem (Grau de la Herran-Mourgoglou)**

- $\Omega \subset \mathbb{R}^{n+1}$ *connected and open*
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- $b_Q = \sigma(Q)\,\eta_Q\,k^{X_Q}$ (normalized and localized Poisson kernel)

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