Multiparameter Hardy spaces associated to compositions of operators and Littlewood-Paley theory

In honor of Eric Sawyer on his 60th Birthday

(joint work with Han, Lin, Ruan and Sawyer)

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Earlier works on pure product Hardy spaces: Gundy-Stein, R. Fefferman, Chang-R. Fefferman, Journe, Pipher, Lacey, ...,
our work is motivated by the work of Phong and Stein on weak type $(1,1)$ estimates on composition operators.

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- Let $e(\xi)$ be a function on $\mathbb{R}^m$ homogeneous of degree 0 in the isotropic sense and smooth away from the origin. Similarly, suppose that $h(\xi)$ is a function on $\mathbb{R}^m$ homogeneous of degree 0 in the non-isotropic sense related to the heat equation, and also smooth away from the origin. Then it is well-known that the Fourier multipliers T_1 defined by $\widehat{T_1(f)}(\xi) = e(\xi)\widehat{f}(\xi)$ and T_2 given by $\widehat{T_2(f)}(\xi) = h(\xi)\widehat{f}(\xi)$ are both bounded on L^p for $1 < p < \infty$, and satisfy various other regularity properties such as being of weak-type $(1, 1)$.

- It was well known that T_1 and T_2 are bounded on the classical isotropic and non-isotropic Hardy spaces, respectively. Rivieré asked the question: Is the composition $T_1 \circ T_2$ still of weak-type (1,1)? Phong and Stein answered this question and gave a necessary and sufficient condition for which $T_1 \circ T_2$ is of weak-type (1,1). The operators Phong and Stein studied are in fact compositions with different kind of homogeneities which arise naturally in the $\bar{\partial}$-Neumann problem.

- We write $\mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R}$ with $x = (x', x_m)$ where $x' \in \mathbb{R}^{m-1}$ and $x_m \in \mathbb{R}$. We consider two kinds of homogeneities

$$\delta : (x', x_m) \to (\delta x', \delta x_m), \delta > 0$$

  and

$$\delta : (x', x_m) \to (\delta x', \delta^2 x_m), \delta > 0.$$

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- For $x = (x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R}$ we denote $|x|_e = (|x'|^2 + |x_m|^2)^{\frac{1}{2}}$ and $|x|_h = (|x'|^2 + |x_m|)^{\frac{1}{2}}$. We also use notations $j \wedge k = \min\{j, k\}$ and $j \vee k = \max\{j, k\}$.

- Definition 1.1: A locally integrable function K_1 on $\mathbb{R}^m \backslash \{0\}$ is said to be a Calderón-Zygmund kernel associated with the isotropic homogeneity if

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} K_1(x)\right| \leq A|x|_e^{-m-|\alpha|} \qquad \text{for all } |\alpha| \geq 0,$$

$$\int_{r_1<|x|_e<r_2} K_1(x)\, dx = 0$$

for all $0 < r_1 < r_2 < \infty$.
We say that an operator T_1 is a Calderón-Zygmund singular integral operator associated with the isotropic homogeneity if $T_1(f)(x) = p.v.(K_1 * f)(x)$, where K_1 satisfies the above conditions.

- It is well-known that any Calderón-Zygmund singular integral operator associated with the isotropic homogeneity is bounded on $L^p(\mathbb{R}^m)$ for $1 < p < \infty$ and is also bounded on the classical Hardy space $H^p(\mathbb{R}^m)$ with $0 < p \leq 1$. Here the classical Hardy space $H^p(\mathbb{R}^m)$ is associated with the isotropic homogeneity. To see this, let $\psi^{(1)} \in S(\mathbb{R}^m)$ with

$$supp\ \widehat{\psi^{(1)}} \subseteq \{(\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \frac{1}{2} \leq |\xi|_e \leq 2\}, \quad (1.5)$$

and

$$\sum_{j \in \mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}\xi', 2^{-j}\xi_m)|^2 = 1 \ for\ all\ (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} \backslash \{(0,0)\}. \quad (1$$

- The Littlewood-Paley-Stein square function of $f \in S'(\mathbb{R}^m)$ then is defined by

$$g(f)(x) = \Big\{ \sum_{j \in \mathbb{Z}} |\psi_j^{(1)} * f(x)|^2 \Big\}^{\frac{1}{2}},$$

where $\psi_j^{(1)}(x', x_m) = 2^{jm}\psi^{(1)}(2^j x', 2^j x_m)$. Note that the isotropic homogeneity is involved in $g(f)$.

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- The classical Hardy space $H^p(\mathbb{R}^m)$ then can be characterized by

$$H^p(\mathbb{R}^m) = \{f \in S'/P(\mathbb{R}^m) : g(f) \in L^p(\mathbb{R}^m)\},$$

where S'/P denotes the space of distributions modulo polynomials. If $f \in H^p(\mathbb{R}^m)$, the H^p norm of f is defined by $\|f\|_{H^p} = \|g(f)\|_{L^p}$.

- A Calderón-Zygmund singular integral operator associated with the non-isotropic homogeneity is bounded on $L^p, 1 < p < \infty$. It is not bounded on the classical Hardy space but bounded on the non-isotropic Hardy space. The non-isotropic Hardy space can also be characterized by the non-isotropic Littlewood-Paley-Stein square function. To be more precise, let $\psi^{(2)} \in S(\mathbb{R}^m)$ with

$$supp\ \ \widehat{\psi^{(2)}} \subseteq \{(\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \frac{1}{2} \leq |\xi|_h \leq 2\}, \quad (1.7)$$

$$\sum_{k \in \mathbb{Z}} |\widehat{\psi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m)|^2 = 1 \ \textit{for all}\ (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} \setminus \{(0,0)\}.$$

- We then define $g_h(f)$, the non-isotropic Littlewood-Paley-Stein square function of $f \in S'(\mathbb{R}^m)$, by

$$g_h(f)(x) = \left\{ \sum_{k\in\mathbb{Z}} |\psi_k^{(2)} * f(x)|^2 \right\}^{\frac{1}{2}},$$

where $\psi_k^{(2)}(x', x_m) = 2^{k(m+1)}\psi(2^k x', 2^{2k} x_m)$. The non-isotropic Hardy space $H_h^p(\mathbb{R}^m)$ then can be characterized by

$$H_h^p(\mathbb{R}^m) = \{f \in S'/P(\mathbb{R}^m) : g_h(f) \in L^p(\mathbb{R}^m)\}$$

and if $f \in H_h^p(\mathbb{R}^m)$, the H_h^p norm of f is defined by $\|f\|_{H_h^p} = \|g_h(f)\|_{L^p}$.

- Suppose that $\psi^{(1)}$ and $\psi^{(2)}$ are functions satisfying the above conditions. Let $\psi_{j,k}(x) = \psi_j^{(1)} * \psi_k^{(2)}(x)$. Define a new Littlewood-Paley-Stein square function by

$$g_{com}(f)(x) = \Big\{ \sum_{j,k \in \mathbb{Z}} |\psi_{j,k} * f(x)|^2 \Big\}^{\frac{1}{2}}.$$

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- As in the classical case, it is not difficult to check that for $1 < p < \infty$,

$$\|g_{com}(f)\|_{L^p} \approx \|f\|_{L^p}.$$

The estimates above suggest us to define the H^p norm of f in terms of the L^p norm of $g_{com}(f)$ when $0 < p \leq 1$. However, this continuous version of the Littlewood-Paley-Stein square function $g_{com}(f)$ is convenient to deal with the case for $1 < p < \infty$ but not for the case when $0 < p \leq 1$. The crucial idea is to replace the continuous version of $g_{com}(f)$ by the discrete version.

- To define the discrete version of $g_{com}(f)$, the key tool is discrete Calderón's identity. To be more precise, we first recall classical continuous Calderón's identity on $L^2(\mathbb{R}^m)$. Let $\psi^{(1)}$ be a function satisfying the conditions of (1.5) and (1.6). By taking the Fourier transform, we have the following classical continuous Calderón's identity:

$$f(x) = \sum_{j\in} \psi_j^{(1)} * \psi_j^{(1)} * f(x),$$

where the series converges in $L^2(\mathbb{R}^m)$ and in $S_\infty(\mathbb{R}^m) := \{f \in S(\mathbb{R}^m) : \int_{\mathbb{R}^m} f(x)x^\alpha dx = 0 \text{ for any} |\alpha| \geq 0\}$.

- Note that the Fourier transforms of both $\psi_j^{(1)}$ and $\psi_j^{(1)} * f$ are compactly supported. Using a similar idea as in the Shannon sampling theorem, one can decompose $\psi_j^{(1)} * \psi_j^{(1)} * f(x)$ by

$$\sum_{\ell \in \mathbb{Z}^m} \psi_j^{(1)}(x - 2^{-j}\ell)(\psi_j^{(1)} * f)(2^{-j}\ell).$$

  Then classical discrete Calderón's identity is given by

$$f(x) = \sum_{j \in} \sum_{\ell \in \mathbb{Z}^m} \psi_j^{(1)}(x - 2^{-j}\ell)(\psi_j^{(1)} * f)(2^{-j}\ell), \quad (1.10)$$

  where the series converges in $L^2(\mathbb{R}^m)$ and $S_\infty(\mathbb{R}^m)$.

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where the series converges in $L^2(\mathbb{R}^m)$ and $S_\infty(\mathbb{R}^m)$.

- Now by considering $\psi_{j,k} = \psi_j^{(1)} * \psi_k^{(2)}$ and taking the Fourier transform, we obtain the following continuous Calderón's identity:

$$f(x) = \sum_{j,k \in \mathbb{Z}} \psi_{j,k} * \psi_{j,k} * f(x),$$

where the series converges in $L^2(\mathbb{R}^m)$ and in $S_\infty(\mathbb{R}^m)$. Furthermore, we will prove the following discrete Calderón's identity.

- Theorem 1.3: Suppose that $\psi^{(1)}$ and $\psi^{(2)}$ are functions satisfying conditions in (1.5) - (1.6) and (1.7) - (1.8), respectively. Let $\psi_{j,k}(x) = \psi_j^{(1)} * \psi_k^{(2)}(x)$. Then

$$f(x', x_m) = \sum_{j,k\in\mathbb{Z}} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} 2^{-(m-1)(j\wedge k)}\, 2^{-(j\wedge 2k)}$$

$$(\psi_{j,k} * f)(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m) \times \psi_{j,k}(x' - 2^{-(j\wedge k)}\ell', x_m - 2^{-(j\wedge 2k)}\ell_m),$$

where the series converges in $L^2(\mathbb{R}^m), S_\infty(\mathbb{R}^m)$ and $S'/P(\mathbb{R}^m)$.

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- Definition 1.4: For $f \in S'/P(\mathbb{R}^m)$, $G_\psi^d(f)$, the discrete Littlewood-Paley-Stein square function of f, is defined by

$$G_\psi^d(f)(x', x_m) =$$

$$\Big\{ \sum_{j,k\in\mathbb{Z}} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |(\psi_{j,k} * f)(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)|^2 \chi_I(x')\chi_J(x_m) \Big\}^{\frac{1}{2}}$$

where I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with the side length $\ell(I) = 2^{-(j\wedge k)}$ and $\ell(J) = 2^{-(j\wedge 2k)}$, and the left lower corners of I and the left end points of J are $2^{-(j\wedge k)}\ell'$ and $2^{-(j\wedge 2k)}\ell_m$, respectively.

- Now we can formally define the Hardy spaces associated with two different homogeneities by the following
  Definition 1.5: Let $0 < p \leq 1$.
  $H^p_{com}(\mathbb{R}^m) = \{f \in S'/P(\mathbb{R}^m) : G^d_\psi(f) \in L^p(\mathbb{R}^m)\}$. If $f \in H^p_{com}(\mathbb{R}^m)$ the norm of f is defined by $\|f\|_{H^p_{com}(\mathbb{R}^m)} = \|G^d_\psi(f)\|_{L^p(\mathbb{R}^m)}$.

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- To see that these Hardy spaces are well defined, we need to show that $H^p_{com}(\mathbb{R}^m)$ is independent of the choice of the functions $\psi^{(1)}$ and $\psi^{(2)}$. This will directly follow from the following

- Theorem 1.6: If $\varphi_{j,k}$ satisfies the same conditions as $\psi_{j,k}$, then for $0 < p \leq 1$ and $f \in S'/P(\mathbb{R}^m)$,

$$\|G_\psi^d(f)\|_{L^p(\mathbb{R}^m)} \approx \|G_\varphi^d(f)\|_{L^p(\mathbb{R}^m)}.$$

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- We now state the main results of this paper.
  Theorem 1.7: Let T_1 and T_2 be Calderón-Zygmund singular integral operators with the isotropic and non-isotropic homogeneity, respectively. Then for $0 < p \leq 1$, the composition operator $T = T_1 \circ T_2$ is bounded on $H^p_{com}(\mathbb{R}^m)$.

- Theorem 1.8: Let $0 < p \leq 1$. If $f \in L^2(\mathbb{R}^m) \cap H^p_{com}(\mathbb{R}^m)$, then there is a constant $C = C(p)$ such that

$$\|f\|_{L^p(\mathbb{R}^m)} \leq C\|f\|_{H^p_{com}(\mathbb{R}^m)},$$

where the constant C is independent of the L^2 norm of f.

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- We remark that the proof of the above theorem does not use atomic decomposition and hence Journé's covering lemma is not required. As a consequence, we obtain

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- Theorem 1.9: Let $0 < p \le 1$. Suppose that T is a composition of T_1 and T_2 as given in Theorem 1.7. Then T extends to a bounded operator from $H^p_{com}(\mathbb{R}^m)$ to $L^p(\mathbb{R}^m)$.

- Theorem 1.10: (Calderón-Zygmund decomposition for H^p_{com}) Let $0 < p_2 \leq 1, p_2 < p < p_1 < \infty$ and let $\alpha > 0$ be given and $f \in H^p_{com}$. Then we may write $f = g + b$ where $g \in H^{p_1}_{com}$ and $b \in H^{p_2}_{com}$ such that $\|g\|^{p_1}_{H^{p_1}_{com}} \leq C\alpha^{p_1-p}\|f\|^p_{H^p_{com}}$ and $\|b\|^{p_2}_{H^{p_2}_{com}} \leq C\alpha^{p_2-p}\|f\|^p_{H^p_{com}}$, where C is an absolute constant.

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- Theorem 1.11: (Interpolation theorem on H^p_{com}) Let $0 < p_2 < p_1 < \infty$ and T be a linear operator which is bounded from $H^{p_2}_{com}$ to L^{p_2} and bounded from $H^{p_1}_{com}$ to L^{p_1}, then T is bounded from H^p_{com} to L^p for all $p_2 < p < p_1$. Similarly, if T is bounded on $H^{p_2}_{com}$ and $H^{p_1}_{com}$, then T is bounded on H^p_{com} for all $p_2 < p < p_1$.

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- Using ideas from Frazier-Jawerth-Weiss in one parameter case. Taking the Fourier transform, we obtain the following continuous Calderón's identity:

$$f(x) = \sum_{j,k\in\mathbb{Z}} \psi_{j,k} * \psi_{j,k} * f(x), \quad (2.1)$$

where the convergence of series in $L^2(\mathbb{R}^m)$, $S_\infty(\mathbb{R}^m)$ and $S'/P(\mathbb{R}^m)$ follows from the results in the classical case.

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- To get a discrete version of Calderon's identity, we need to decompose $\psi_{j,k} * \psi_{j,k} * f$ in (2.1). Set $g = \psi_{j,k} * f$ and $h = \psi_{j,k}$. The Fourier transforms of g and h are given by $\widehat{g}(\xi', \xi_m) = \widehat{\psi^{(1)}}(2^{-j}\xi', 2^{-j}\xi_m)\widehat{\psi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m)\widehat{f}(\xi', \xi_m)$ and $\widehat{h}(\xi', \xi_m) = \widehat{\psi^{(1)}}(2^{-j}\xi', 2^{-j}\xi_m)\widehat{\psi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m)$.

- Note that the Fourier transforms of g and h are both compactly supported. More precisely, *supp* $\widehat{g}$, *supp* $\widehat{h}$ is

$$\subseteq \{(\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : |\xi'| \leq 2^{j \wedge k}\pi, |\xi_m| \leq 2^{j \wedge 2k}\pi\}.$$

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- Thus, we first expand $\widehat{g}$ in a Fourier series on the rectangle $R_{j,k} = \{\xi' \in \mathbb{R}^{m-1}, \xi_m \in \mathbb{R} : |\xi'| \leq 2^{j \wedge k}\pi, |\xi_m| \leq 2^{j \wedge 2k}\pi\}$:

$$\widehat{g}(\xi', \xi_m) = \sum_{(\ell', \ell_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}} 2^{-(m-1)(j \wedge k)}\, 2^{-(j \wedge 2k)} (2\pi)^{-m}$$

$$\times \int_{R_{j,k}} \widehat{g}(\eta', \eta_m) e^{i(2^{-(j \wedge k)}\ell' \cdot \eta' + 2^{-(j \wedge 2k)}\ell_m \eta_m)} d\eta' d\eta_m$$

$$\times e^{-i(2^{-(j \wedge k)}\ell' \cdot \xi' + 2^{-(j \wedge 2k)}\ell_m \xi_m)}$$

and then replace $R_{j,k}$ by $\mathbb{R}^m$ since $\widehat{g}$ is supported in $R_{j,k}$.

- Finally, we obtain

$$\widehat{g}(\xi', \xi_m) = \sum_{(\ell', \ell_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}} 2^{-(m-1)(j \wedge k)}\ 2^{-(j \wedge 2k)}$$

$$\times g(2^{-(j \wedge k)} \ell', 2^{-(j \wedge 2k)} \ell_m)\ e^{-i(2^{-(j \wedge k)} \ell' \cdot \xi' + 2^{-(j \wedge 2k)} \ell_m \xi_m)}.$$

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- Multiplying $\widehat{h}(\xi', \xi_m)$ from both sides yields

$$\widehat{g}(\xi', \xi_m) \widehat{h}(\xi', \xi_m)$$

$$= \sum_{(\ell', \ell_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}} 2^{-(m-1)(j \wedge k)}\ 2^{-(j \wedge 2k)}\ g(2^{-(j \wedge k)} \ell', 2^{-(j \wedge 2k)} \ell_m)$$

$$\times \widehat{h}(\xi', \xi_m)\ e^{-i(2^{-(j \wedge k)} \ell' \cdot \xi' + 2^{-(j \wedge 2k)} \ell_m \xi_m)}.$$

- Note that $\widehat{h}(\xi', \xi_m)\ e^{-i(2^{-(j\wedge k)}\ell'\cdot\xi' + 2^{-(j\wedge 2k)}\ell_m\xi_m)} = \widehat{h}(\cdot - 2^{-(j\wedge k)}\ell', \cdot - 2^{-(j\wedge 2k)}\ell_m))(\xi', \xi_m)$. Therefore, applying the identity $g * h = (\widehat{g}\ \widehat{h})^{\vee}$ implies that

$$(g * h)(x', x_m)$$

$$= \sum_{(\ell', \ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} 2^{-(m-1)(j\wedge k)}\ 2^{-(j\wedge 2k)} g(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)$$

$$\times h(x' - 2^{-(j\wedge k)}\ell', x_m - 2^{-(j\wedge 2k)}\ell_m). \quad (2.2)$$

- Note that $\widehat{h}(\xi', \xi_m)\ e^{-i(2^{-(j\wedge k)}\ell'\cdot\xi' + 2^{-(j\wedge 2k)}\ell_m\xi_m)} = \widehat{h(\cdot - 2^{-(j\wedge k)}\ell', \cdot - 2^{-(j\wedge 2k)}\ell_m)}(\xi', \xi_m)$. Therefore, applying the identity $g * h = (\widehat{g}\ \widehat{h})^{\vee}$ implies that

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$$= \sum_{(\ell', \ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} 2^{-(m-1)(j\wedge k)}\ 2^{-(j\wedge 2k)} g(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)$$

$$\times h(x' - 2^{-(j\wedge k)}\ell', x_m - 2^{-(j\wedge 2k)}\ell_m). \quad (2.2)$$

- Substituting g by $\psi_{j,k} * f$ and h by $\psi_{j,k}$ into Calderón's identity in (2.1) gives the discrete Calderón's identity in (1.12) and the convergence of the series in the $L^2(\mathbb{R}^m)$.

- Lemma 3.1 (Almost orthogonality estimates)
Suppose that $\psi_{j,k}$ and $\varphi_{j',k'}$ satisfy the same conditions in (1.5)-(1.8). Then for any given integers L and M, there exists a constant $C = C(L, M) > 0$ such that

$$|\psi_{j,k} * \varphi_{j',k'}(x', x_m)|$$

$$\leq C2^{-|j-j'|L}2^{-|k-k'|L}\frac{2^{(j\wedge j'\wedge k\wedge k')(m-1)}}{(1+2^{j\wedge j'\wedge k\wedge k'}|x'|)^{(M+m-1)}}\frac{2^{j\wedge j'\wedge 2(k\wedge k')}}{(1+2^{j\wedge j'\wedge 2(k\wedge k')}|x_m|)$$

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- Lemma 3.2: Let I, I' be dyadic cubes in $\mathbb{R}^{m-1}$ and J, J' be dyadic intervals in $\mathbb{R}$ with the side lengths $\ell(I) = 2^{-(j\wedge k)}$, $\ell(I') = 2^{-(j'\wedge k')}$ and $\ell(J) = 2^{-(j\wedge 2k)}$, $\ell(J') = 2^{-(j'\wedge 2k')}$, and the left lower corners of I, I' and the left end points of J, J' are $2^{-(j\wedge k)}\ell'$, $2^{-(j'\wedge k')}\ell''$, $2^{-(j\wedge 2k)}\ell_m$ and $2^{-(j'\wedge 2k')}\ell'_m$, respectively. Then for any $u', v' \in I$, $u_m, v_m \in J$, and any $\frac{m-1}{M+m-1} < \delta \leq 1$,

$$\sum_{(\ell'',\ell'_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \frac{2^{(m-1)(j\wedge j'\wedge k\wedge k')}\ 2^{j\wedge j'\wedge 2k\wedge 2k'}\ 2^{-(m-1)(j'\wedge k')}\ 2^{-(j'\wedge 2k')}}{(1+2^{j\wedge j'\wedge k\wedge k'}|u'-2^{-(j'\wedge k')}\ell''|)^{(M+m-1)}}$$

$$\times \frac{|(\varphi_{j',k'}*f)(2^{-(j'\wedge k')}\ell'', 2^{-(j'\wedge 2k')}\ell'_m)|}{(1+2^{j\wedge j'\wedge 2k\wedge 2k'}|u_m-2^{-(j'\wedge 2k')}\ell'_m|)^{(M+1)}}$$

$$\leq C_1\Big\{M_s\Big[\Big(\sum_{(\ell'',\ell'_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |(\varphi_{j',k'}*f)(2^{-(j'\wedge k')}\ell'', 2^{-(j'\wedge 2k')}\ell'_m)|^2\chi_{I'}\chi_{J'}$$

where $C_1 = C2^{(m-1)(\frac{1}{\delta}-1)(j'\wedge k'-j\wedge k)_+}\ 2^{(\frac{1}{\delta}-1)(j'\wedge 2k'-j\wedge 2k)_+}$, here $(a-b)_+ = \max\{a-b, 0\}$, and M_s is the strong maximal function.