Regularity of Solutions of Degenerate Quasilinear Equations

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Outline

Original motivation

Quasilinear equation

$$Lu = divA(x,u)\nabla u + b(x,u,\nabla u) = f$$

Monge-Ampère equation

$$detD^2 u = k(x,u,Du), \quad x \in \Omega$$

where k is smooth and nonnegative in $\Omega \times \mathbf{R} \times \mathbf{R}^n$, Ω is a convex domain in $\mathbf{R}^n$.

Partial Legendre transform

Change of variables

$$\begin{cases} s = x_1 \\ t_2 = u_{x_2}(x) \\ \dots \\ t_n = u_{x_n}(x) \end{cases}$$

Quasilinear system

$$Lv_p \equiv \left\{ \frac{\partial^2}{\partial s^2} + \frac{\partial}{\partial \mathbf{t}'} k \left(co \left[\frac{\partial \mathbf{v}}{\partial \mathbf{t}'} \right] \right)' \frac{\partial}{\partial \mathbf{t}} \right\} v_p = 0, \quad 2 \leq p \leq n$$

where $\mathbf{v} = (v_p)_{p=2}^n = (x_p(s, \mathbf{t}))_{p=2}^n$.

Regularity of solutions

$$Monge - Ampère\ equation \quad \leftrightarrow \quad Quasilinear\ equation$$

$$detD^2u = k \qquad div \begin{pmatrix} 1 & 0 \\ 0 & kM \end{pmatrix} \nabla v = 0$$

$$u \in C^{1+\alpha} \qquad v \in C^{\alpha}$$

Ellipticity

$$Lu = divA(x,u)\nabla u + b(x,u,\nabla u)$$

- Ellipticity

$$0 < \lambda(x,z)|\xi|^2 \leq \xi' A(x,z)\xi \leq \Lambda(x,z)|\xi|^2$$

- Subellipticity

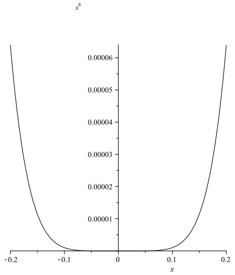
$$||u||_{C^\alpha} \leq C(||u||_{L^2}, ||Lu||_{L^\infty})$$

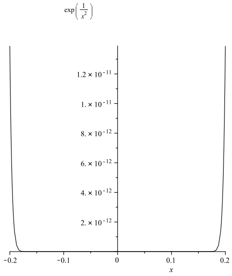
- Hypoellipticity

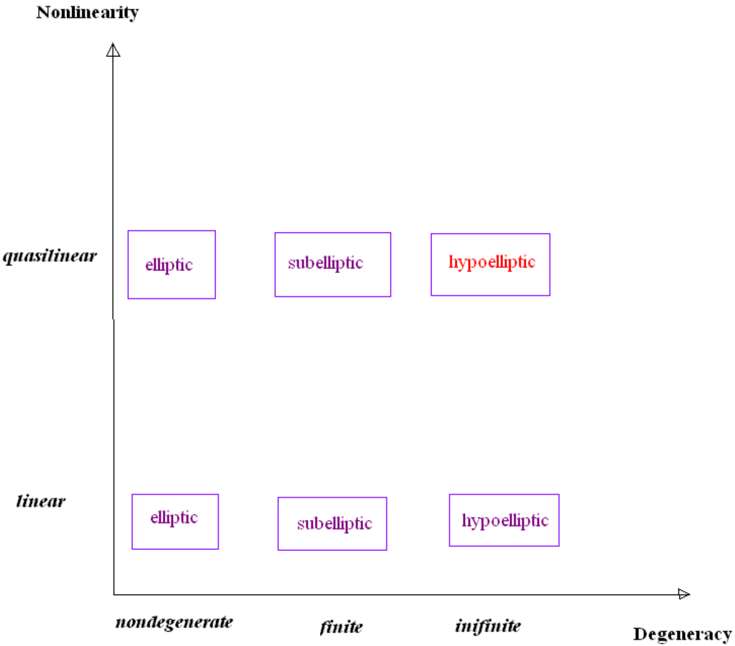
$$Lu \in C^\infty \;\Rightarrow\; u \in C^\infty$$

Two main difficulties

- Non-linearity $A(x,u)$, $\tilde{A}(x) := A(x,u(x))$
  $\tilde{A}(x)$ is as rough as $u(x)$ is
- Degeneracy $detA(x^0) = 0$
- Graphs of the functions x^6 and $exp(-1/x^2)$






Nonlinearity
quasilinear
elliptic
subelliptic
hypoelliptic
linear
elliptic
subelliptic
hypoelliptic
nondegenerate
finite
inifinite
Degeneracy

Finitely degenerate case

- Hörmander's theorem [Hörmander, 1967]
- Fefferman-Phong characterization of subellipticity [Fefferman, 1981]
- Extension to rough vector fields [Sawyer, 2006]

Subunit balls

- Subunit curve
  Lipschitz curve $\gamma : [0, r] \to \Omega$ such that
  $$(\gamma'(t)\xi)^2 \le \xi' A(\gamma(t))\xi, \ \ a.e.\ t \in [0, r], \ \xi \in \mathbf{R}^n$$
- Subunit metric
  $$d(x, y) = \inf\{r > 0 : \ \gamma(0) = x, \ \gamma(r) = y, \ \gamma \ is \ subunit \ in \ \Omega\}$$
- Subunit ball
  $$B(x, r) = \{y \in \Omega : d(x, y) < r\}$$
- Doubling condition
  $$|B(x, r)| \le C \left(\frac{r}{t}\right)^D |B(y, t)|, \ \ B(x, r) \supset B(y, t)$$
- Containment condition
  $$E(x, r) \subseteq B(x, Cr^{\varepsilon})$$

Fefferman-Phong characterization of subellipticity

Operator L is subelliptic

$$||u||_{H^{\varepsilon}} \leq C(||u||_{L^2} + ||Lu||_{L^2})$$

if and only if
the following containment condition holds

$$E(x, R) \subseteq B(x, CR^{\varepsilon})$$

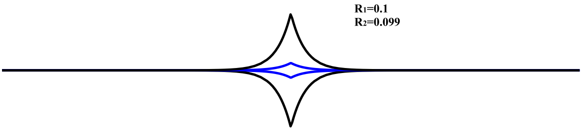
Extension of Fefferman-Phong result to rough vector fields

1. $Lu = f \in L^\infty$, L has bounded measurable coefficients
2. doubling condition holds
3. containment condition holds
4. $B(x, r) \subseteq E(x, cr)$
5. Sobolev and Poincaré inequalities hold
6. there is an "accumulating system of cutoff functions"

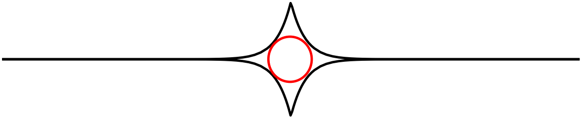
Then the operator L is subelliptic.

Subunit balls and non-doubling measures

Infinite degeneracy $\Rightarrow$ no doubling



Containment condition $E(x, \alpha(R)) \subseteq B(x, R)$, $\alpha(R) > 0$



What do we expect?

Assuming continuity (Rios, Sawyer, Wheeden 2011)

If $Lu \in C^{\infty}$ then every continuous weak soulution is smooth

Last step

Show continuity using “subunit metric” approach

Idea of proof I

- Weak solution

$$-\int (\nabla u)' A \nabla w = \int fw$$

  $w \in W_0^{1,2}(\Omega)$, nonnegative
- Weak Sobolev inequality

$$\left(\frac{1}{|B|} \int_B |w|^{2\sigma} \right)^{\frac{1}{2\sigma}} \leq Cr \left(\frac{1}{|B|} \int_B ||\nabla w||_A^2 \right)^{\frac{1}{2}} + C \left(\frac{1}{|B|} \int_B |w|^2 \right)^{\frac{1}{2}}$$

  for any $w \in W_0^{1,2}(B)$ and some $\sigma > 1$
- Moser iteration for $\overline{u}^\beta = (u+m)^\beta$, $\overline{u}$ — positive supersolution

Idea of proof II

- Harnack inequality

$$\operatorname*{ess\,sup}_{x\in B} \overline{u} \leq C \left[\frac{1}{|B|} \int_B \overline{u}^{\gamma} \right] \left[\frac{1}{|B|} \int_B \overline{u}^{-\gamma} \right] \operatorname*{ess\,inf}_{x\in B} \overline{u}$$

- $\log \overline{u} \in BMO \Rightarrow \overline{u}^{\gamma} \in A_2$
- “Hölder continuity”

$$|u(x) - u(y)| \leq C(||u||_{L^2}, ||f||_{L^\infty}) d(x,y)^{\alpha}$$

- Using containment condition

$$d(x,y) \leq C|x-y|^{\varepsilon}$$

Absense of doubling condition

Example: $|B_R| \sim e^{(-1/R^2)} \Rightarrow |B_{2R}| = e^{(3/4R^2)}|B_R|$

- Moser iteration, volumes of balls "accumulate"
- $BMO \not\leftrightarrow A_2$
  $RBMO$ space of Tolsa [Tolsa, 2001]
- John-Nirengberg inequality [Hytönen, 2010]

$$|\{x \in B_0 : |f(x) - f_{B_0}| > \alpha\}| \leq Ce^{\frac{c_2\alpha}{||f||_{RBMO_\rho}}}|\rho^{1+\varepsilon}B_0|$$

- Poincaré inequality
- Sobolev inequality
  Typically $\sigma = D/(D-2)$ where D is the doubling exponent

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Thank you for your attention!