Hardy-Littlewood-Sobolev inequalities on $\mathbb{R}^N$ and the Heisenberg group

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§1 Hardy-Littlewood-Sobolev inequalities on $\mathbb{R}^N$

▶ Hardy-Littlewood-Sobolev inequality on $\mathbb{R}^N$. Let $1 < r, s < \infty$ and $0 < \lambda < N$ such that $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2$, then

$$\left| \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\overline{f(x)} g(y)}{|x-y|^\lambda} dx dy \right| \leq C_{r,\lambda,N} \|f\|_r \|g\|_s, \tag{1}$$

- On $\mathbb{R}^1$: Hardy-Littlewood (1928, 30, 32); on $\mathbb{R}^N$: Sobolev (1938).
- Sharp version (with best constant and formulae for maximizers) when $r = s = 2N/(2N - \lambda)$: Lieb (1983), Carlen-Loss (1990), Frank-Lieb (2010).

  Special cases ($\lambda = N - 2$): Rosen (1971), Aubin (1976), Talenti (1976), Carlen-Carrillo-Loss (2010), etc.
- Existence of maximizers (optimizers or extremals) for all r, s: Lieb (1983), Lions (1985), etc.
- Uniqueness of maximizers: Y. Li, Chen-C. Li-Ou (2004, 2005, 06).
- *Open*: Sharp versions when $r \neq s$.

Theorem 1 (Carlen-Loss (1990, 1992), Lieb-Loss (1997)). *The maximizers for sharp version of (1) when $r = s = 2N/(2N-\lambda)$ assume the form after translation and dilation*

$$\frac{1}{(1+|x|^2)^{N/r}}.$$

Outline of proof of Theorem 1.

1. Define stereographic projection $\mathcal{S}$: $x \mapsto s$ from $\mathbb{R}^N \cup \{\infty\} \to \mathbb{S}^N \subseteq \mathbb{R}^{N+1}$

$$\mathcal{S}(x) = \left(\frac{2x_j}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}\right) \text{ and } \mathcal{S}^{-1}(s) = \left(\frac{s_j}{1+s_{n+1}}\right), \ j = 1, 2, ..., n,$$

$$\mathcal{J}_{\mathcal{S}}(x) = \left(\frac{2}{1+|x|^2}\right)^N = \left(\frac{1}{1+s_{n+1}}\right)^N = \mathcal{J}_{\mathcal{S}^{-1}}(s),$$

$$|s-t|^2 = \frac{2}{1+|x|^2}|x-y|^2\frac{2}{1+|y|^2}.$$

2.

$$F(s) = |\mathcal{J}_{\mathcal{S}^{-1}(s)}|^{1/r} f(\mathcal{S}^{-1}(s)) \text{ and } G(t) = |\mathcal{J}_{\mathcal{S}^{-1}(t)}|^{1/s} f(\mathcal{S}^{-1}(t)),$$

$$\|F\|_r = \|f\|_r \text{ and } \|G\|_s = \|g\|_s.$$

3. *Only if* $r = s = 2N/(2N-\lambda)$, then
$$\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\overline{f(x)}g(y)}{|x-y|^\lambda}dxdy = \int\int_{\mathbb{S}^N\times\mathbb{S}^N} \frac{\overline{F(s)}G(t)}{|s-t|^\lambda}dsdt.$$
4. One can assume $g = \overline{f}$ because the form is positive definite.
5. Maximizers for sharp version of (1) are radially symmetric on $\mathbb{R}^N$ (by symmetrization or moving plane/sphere methods), thus f is invariant under rotations $\mathcal{R} \in O(\mathbb{R}^N)$. As a result, F is invariant under rotations $\mathcal{R} \in O(\mathbb{R}^N)$ that keep the "north pole" n, that is, constant on ever level.
6. $$F \xrightarrow{\mathcal{R}\in O(N+1)} \widetilde{F} \xrightarrow{\mathcal{S}^{-1}} \widetilde{f} \xrightarrow{\mathcal{S}} \widetilde{F} \xrightarrow{\mathcal{R}^{-1}} F.$$
   $\widetilde{f}$ is also a maximizer thus radially symmetric. Then $F(n) = F(s)$ for s on any level, and F is constant on $\mathbb{S}^N$.
7. $$f(x) = |\mathcal{J}_{\mathcal{S}^{-1}(s)}|^{-1/r} F(\mathcal{S}^{-1}(s)) = c(1+|x|^2)^{-N/r}.$$

$\square$

Remark.

- Too beautiful to be generalized.
- May be used to get the formulae for maximizers under special cases in other inequalities. E.g. Radon-like transform inequalities, see Christ (2010, 11).

▶ Weighted Hardy-Littlewood-Sobolev inequality on $\mathbb{R}^N$:

$$\left| \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\overline{f(x)} g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta} dx dy \right| \leq C_{\alpha,\beta,r,\lambda,N} \|f\|_r \|g\|_s,$$

where $1 < r, s < \infty$, $0 < \lambda < N$ and $\alpha + \beta \geq 0$ such that $\lambda + \alpha + \beta \leq N$, $\alpha < N/r'$, $\beta < N/s'$ and $\frac{1}{r} + \frac{1}{s} + \frac{\lambda+\alpha+\beta}{N} = 2$.

- On $\mathbb{R}^N$: Stein-Weiss (1958).
- Existence of maximizers when $\alpha, \beta \geq 0$: Lieb (1983).
- Uniqueness of maximizers: Chen-C. Li (2007).
- Singularity analysis and asymptotic behavior of maximizers: Jin-C. Li (2006), C. Li-Lim (2007), etc.

- Special cases: $r = s$ and $\alpha = \beta = \frac{N}{r+1} - \frac{N-2}{N}$, maximizers assume

$$c\left[\frac{t}{t^2 + |z|^{\frac{(N-2)(r-1)}{2}}}\right]^{\frac{2}{r-1}}$$

for some constants c and t: Chen-C. Li (2007).

- *Open*: Existence of maximizers when $\alpha > 0$ and $\beta < 0$ (or *vise versa*), formulae of maximizers and all others.

§2 Hardy-Littlewood-Sobolev inequalities on $\mathbb{H}^n$

► Hardy-Littlewood-Sobolev inequality on $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$:

$$\left| \int \int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\overline{f(u)} g(v)}{|u^{-1}v|^{\lambda}} dudv \right| \leq C_{r,\lambda,n} \|f\|_r \|g\|_s. \tag{2}$$

- On $\mathbb{H}^n$: Folland-Stein (1973, 74).
- Sharp version when $\lambda = Q - 2$ and $r = s = 2Q/(2Q - \lambda) = 2Q/(Q + 2)$: Jerison-Lee (1988).
- Sharp version for all λ and when $r = s = 2Q/(2Q - \lambda)$: Frank-Lieb (2010).

The maximizers assume the form after dilation and translation

$$\frac{1}{\left[(1+|z|^2)^2+t^2\right]^{\frac{Q}{2r}}}.$$

- An upper bound for best constants when $r \neq s$: H. (2011).

$$\frac{Q|B_1|^{\frac{\lambda}{Q}}}{rs(Q-\lambda)}\left[\left(\frac{\lambda/Q}{1-1/r}\right)^{\frac{\lambda}{Q}}+\left(\frac{\lambda/Q}{1-1/s}\right)^{\frac{\lambda}{Q}}\right],$$

in which B_1 is the unit Heisenberg ball, and

$$|B_1| = \frac{2\pi^{\frac{Q-2}{2}}\Gamma(1/2)\Gamma((Q+2)/4)}{(Q-2)\Gamma((Q-2)/2)\Gamma((Q+4)/4)}.$$

- *Open*: All others.

Strategy of proving existence of maximizers.

1. Define

$$I_\lambda(f)(u) = \int_{\mathbb{H}^n} \frac{f(v)}{|u^{-1}v|} dv,$$

and there exists a maximizing sequence $\{f_j\}$ with $\|f_j\|_r = 1$ for $I_\lambda : L^r \to L^{s'}$.

2. Loss of compactness in $\{f_j\}$ is caused by dilation and translation invariance of I_λ.

3.

$$\boxed{\text{Symmetrization}} \xrightarrow{\textit{excluding}} \text{Existence} \xleftarrow{\textit{including}} \text{Concentration compactness.}$$

4.

$$\left\{ \frac{1}{\delta_j^{Q/r}} f_j \left(\frac{u_j^{-1} u}{\delta_j} \right) \right\}$$

is compact for some u_j and δ_j.

$\square$

▶ Weighted HLS inequality on $\mathbb{H}^n$:

Theorem 2 ($|u|$ weighted HLS inequality)**.** *For* $1 < r, s < \infty$, $0 < \lambda < Q = 2n + 2$ *and* $\alpha + \beta \geq 0$ *such that* $\lambda + \alpha + \beta \leq Q$, $\alpha < Q/r'$, $\beta < Q/s'$ *and* $\frac{1}{r} + \frac{1}{s} + \frac{\lambda+\alpha+\beta}{Q} = 2$,

$$\left| \int \int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\overline{f(u)} g(v)}{|u|^\alpha |u^{-1}v|^\lambda |v|^\beta} du dv \right| \leq C_{\alpha,\beta,r,\lambda,n} \|f\|_r \|g\|_s. \tag{3}$$

- On $\mathbb{H}^n$: H.-Lu-Zhu (2011).
- Singularity analysis and asymptotic behavior of maximizers: H.-Lu-Zhu (2011)
- *Open*: All others.

Theorem 3 ($|z|$ weighted HLS inequality)**.** *For* $1 < r, s < \infty$, $0 < \lambda < Q = 2n + 2$ *and* $0 \leq \alpha + \beta \leq n\lambda$ *such that* $\lambda + \alpha + \beta \leq Q$, $\alpha < 2n/r'$, $\beta < 2n/s'$ *and* $\frac{1}{r} + \frac{1}{s} + \frac{\lambda+\alpha+\beta}{Q} = 2$,

$$\left| \int \int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\overline{f(u)} g(v)}{|z|^\alpha |u^{-1}v|^\lambda |z'|^\beta} du dv \right| \leq C_{\alpha,\beta,r,\lambda,n} \|f\|_r \|g\|_s. \tag{4}$$

Here, $u = (z, t)$ *and* $v = (z', t')$*.*

- On $\mathbb{H}^n$: H.-Lu-Zhu (2011).
- Nonexistence of maximizers when $\alpha = \beta = (Q - \lambda)/2$ and $r = s = 2$: Beckner (1997).
- Singularity analysis and asymptotic behavior of maximizers: H.-Lu-Zhu (2011)
- *Open*: All others.

Thank you!