Boundedness of singular integrals and their commutators with BMO functions on Hardy spaces

Xuan Thinh Duong (Macquarie University, Australia)

Joint work with The Anh Bui (Macquarie)

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1. Introduction and main results

Standard Calderón-Zygmund theory: Let T be a bounded operator on $L^2(X)$ where X is a doubling space. Assume that the kernel of T satisfies the Hörmander condition, then T is of weak type $(1,1)$, bounded on $L^p(X)$, $1 < p < \infty$, bounded from the Hardy space $H^1(X)$ into $L^1(X)$ and bounded from $L^\infty(X)$ into $\mathrm{BMO}(X)$. One can also obtain boundedness of the commutator of T and a BMO function on $L^p(X)$.

In the last ten years or so, there are lots of research devoted to the study of boundedness of T when the kernel of T is rough and does not satisfy the Hörmander condition. A successful approach is to define certain Hardy spaces associated to T itself, then show that T maps this Hardy space into $L^1(X)$. Then one can interpolate between L^2 and this Hardy space to obtain boundedness of T on L^p or H^p for $1 < p < 2$. For $p > 2$, one can study boundedness of T from $L^\infty(X)$ into certain BMO space associated to T.

In this talk, we present certain results along this line for T and its commutator with a BMO function.

Let (X, d, μ) be a metric measure space endowed with a distance d and a doubling measure μ, i.e.

$$V(x, 2r) \leq CV(x, r) < \infty, \tag{1}$$

where $B(x, r) = \{y \in X : d(x, y) < r\}$ and $V(x, r) = \mu(B(x, r))$.
Note that the doubling property implies the following strong homogeneity property,

$$V(x, \lambda r) \leq c\lambda^n V(x, r) \tag{2}$$

for some $c, n > 0$ uniformly for all $\lambda \geq 1$ and $x \in X$. The smallest value of the parameter n is a measure of the dimension of the space. There also exist c and $N, 0 \leq N \leq n$, so that

$$V(y, r) \leq c \left(1 + \frac{d(x, y)}{r}\right)^N V(x, r) \tag{3}$$

uniformly for all $x, y \in X$ and $r > 0$.
We will write B for $B(x_B, r_B)$ and for $\lambda > 0$, λB means the λ-dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$. For each ball $B \subset X$ we set

$$S_0(B) = B \text{ and } S_j(B) = 2^j B \backslash 2^{j-1} B \text{ for } j \in \mathbb{N}.$$

Assume that there exists an operator L defined on $L^2(X)$. For our results, we will need some of the following assumptions:

(**H1**) L is a non-negative self-adjoint operator on $L^2(X)$;

(**H2**) The operator L generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ which satisfies the Davies-Gaffney condition. That is, there exist constants C, $c > 0$ such that for any open subsets $U_1,\ U_2 \subset X$,

$$|\langle e^{-tL} f_1, f_2\rangle| \leq C \exp\Big(-\frac{\mathrm{dist}(U_1, U_2)^2}{c\, t}\Big)\|f_1\|_{L^2(X)}\|f_2\|_{L^2(X)}, \quad \forall\, t > 0, \qquad (4)$$

for every $f_i \in L^2(X)$ with $\operatorname{supp} f_i \subset U_i$, $i = 1, 2$, where $\mathrm{dist}(U_1, U_2) := \inf_{x\in U_1, y\in U_2} d(x, y)$.

(**H3**) The kernel $p_t(x, y)$ of e^{-tL} satisfies the Gaussian upper bound, i.e. there exist constants C, $c > 0$ such that for almost every $x, y \in X$,

$$|p_t(x, y)| \leq \frac{C}{V(x, \sqrt{t})} \exp\left(-\frac{d^2(x, y)}{ct}\right), \forall t > 0. \qquad (5)$$

Remark

The Gaussian bound (**H3**) *implies the Davies-Gaffney condition* (**H2**).

We list a number of examples:

(i) The Laplace operator Δ *on the Euclidean space* $\mathbb{R}^n$ *satisfies* (**H1**) *and* (**H3**). *So do the second order non-negative self-adjoint divergence form operators with real bounded measurable coefficients on* $\mathbb{R}^n$. *Second order divergence form operators with complex bounded measurable coefficients on* $\mathbb{R}^n$ *would satisfy* (**H2**). *They satisfy* (**H3**) *for low dimensions* n *but might not satisfy* (**H3**) *for higher dimensions* n.

(ii) Schrödinger operators or magnetic Schrödinger operators with non-negative potentials satisfy (**H1**) *and* (**H3**).

(iii) Laplace-Beltrami operators on all complete Riemannian manifolds satisfy (**H1**) *and* (**H2**) *but do not satisfy* (**H3**) *in general.*

(iv) Laplace type operators acting on vector bundles satisfy (**H1**) *and* (**H2**).

Our aim is to study boundedness of certain singular integral operators with non-smooth kernels and boundedness of their commutators via estimates on related function spaces.
Denote by $H_L^p(X), 0 < p \leq 1$, the Hardy spaces associated to the operator L. Assume that T is a bounded operator on $L^2(X)$. There are a number of known sufficient conditions on T or its associated kernel $k(x, y)$ so that L^2-boundedness of T can be extended to other spaces such as Lebesgue space L^p, $p \neq 2$, Hardy spaces, and BMO spaces. Another natural question is to study boundedness of the commutator of a BMO function b and T which is given by

$$[b, T]f := bTf - Tbf$$

for all functions f with compact supports.
We aim to establish a sufficient condition on an L^2 bounded operator T so that it implies the following:
(i) T is bounded from the Hardy spaces $H_L^p(X)$ to $L^p(X)$, $0 < p \leq 1$; and
(ii) the commutator $[b, T]$ is bounded from $H_L^1(X)$ to $L^{1,\infty}(X)$ under the extra assumption that T is of weak type $(1, 1)$.

The main result is as follows.

Theorem

Let L be an operator which satisfies (**H1**) *and* (**H2**). *Let $0 < p \leq 1$. Let a be a $(p, 2, m)$-atom in the Hardy space H_L^p associate to the operator L. Assume that T is a bounded operator on $L^2(X)$ so that Ta satisfies the estimate*

$$\left(\int_{S_j(B)} |Ta|^2 dx \right)^{\frac{1}{2}} \leq C 2^{-2jm} V(B)^{\frac{1}{2} - \frac{1}{p}} \tag{6}$$

for any $(p, 2, m)$-atom a supported in the ball B.
Then, we have:

(i) *T is bounded from $H_L^p(X)$ to $L^p(X)$; and*

(ii) *in addition, if T is of weak type $(1, 1)$ then the commutator $[T, b]$, where b is a BMO function, maps continuously from H_L^1 to $L^{1,\infty}$.*

Remark

(a) There is no explicit regularity condition on the kernel of T, so in general T is not a standard Calderón-Zygmund singular integral operator.
(b) As applications, we will obtain boundedness of various singular integral operators and their commutators which do not belong to the class of Calderón-Zygmund operators.

(c) By interpolation T is bounded from $H_L^p(X)$ to $L^p(X)$ for $0 < p \leq 2$.

2. Hardy spaces associated to operators

The theory of Hardy spaces associated to generators of semigroups was developed in the last fifteen years by P. Auscher, X. Duong, A. McIntosh, L. Yan, S. Hofmann, S. Mayboroda and others. Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates was developed recently by S. Hofmann, G. Lu, M. Mitrea, D. Mitrea and L. Yan for $p = 1$, and by J. Li and X. Duong for $0 < p < 1$.

Let L be an operator which satisfies (**H1**) and (**H2**). Consider the area integrals associated to L

$$S_{h,K}f(x) = \Big(\int_0^\infty \int_{d(x,y)<t} |(t^2L)^K e^{-t^2L} f(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \Big)^{1/2}, \quad x \in X \tag{7}$$

where K is a positive integer and $f \in L^2(X)$. We shall write S_h in place of $S_{h,1}$. For each integer $K \geq 1$ and $1 \leq p < \infty$, we now define

$$D_{K,p} = \Big\{ f \in L^2(X) : \ S_{h,K}f \in L^p(X) \Big\}, \quad 1 \leq p < \infty.$$

Definition

Let L be a non-negative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney condition
(i) For each $1 \leq p \leq 2$, the Hardy space $H^p_{L,S_h}(X)$ associated to L is the completion of the space $D_{1,p}$ in the norm

$$\|f\|_{H^p_{L,S_h}(X)} = \|S_h f\|_{L^p(X)}.$$

(ii) For each $2 < p < \infty$, the Hardy space $H^p_L(X)$ associated to L is the completion of the space $D_{K_0,p}$ in the norm

$$\|f\|_{H^p_{L,S_h}(X)} = \|S_{h,K_0} f\|_{L^p(X)}, \quad K_0 = \left[\frac{n}{2}\right] + 1.$$

It can be verified that $H^1_L(X) \subseteq L^1(X)$, $H^2_{L,S_h}(X) = L^2(X)$ and the dual space of $H^p_{L,S_h}(X)$ is $H^{p'}_{L,S_h}(X)$ where $1/p + 1/p' = 1$
Hence in general, we have $H^p_L(X) \subset L^p(X)$ for $p \in [1,2)$ and by duality $L^p(X) \subset H^p_L(X)$ for $p \in (2,\infty)$.
However, if L satisfies (H1) and (H3), then it was known that $H^p_L(X)$ and $L^p(X)$ coincide for all $p \in (1,\infty)$.

Let us describe the notion of a $(p, 2, M)$-*atom*, $0 < p \leq 1$, associated to operators on spaces (X, d, μ). In what follows, assume that

$$M \in \mathbb{N} \quad \text{and} \quad M > \frac{n(2-p)}{4p}, \tag{8}$$

where the parameter n, thought of as a measure of the dimension of the space X, is the smallest value for the doubling property holds. Let us denote by $\mathcal{D}(T)$ the domain of an operator T.

Definition

A function $a(x) \in L^2(X)$ is called a $(p, 2, M)$-atom associated to an operator L if there exist a function $b \in \mathcal{D}(L^M)$ and a ball B of X such that

(i) $a = L^M b$;

(ii) $\mathrm{supp} L^k b \subset B, \;\; k = 0, 1, \ldots, M$;

(iii) $||(r_B^2 L)^k b||_{L^2(X)} \leq r_B^{2M} V(B)^{\frac{1}{2} - \frac{1}{p}}, \;\; k = 0, 1, \ldots, M.$

In the case $\mu(X) < \infty$ the constant function having value $[\mu(X)]^{-\frac{1}{p}}$ is also considered to be an atom.

Definition

Given $0 < p \leq 1$ and $M > \frac{n(2-p)}{4p}$, the atomic Hardy space $H^p_{L,at,M}(X)$ is defined as follows. We say that $f = \sum \lambda_j a_j$ is an atomic $(p, 2, M)$-representation if $\{\lambda_j\}_{j=0}^{\infty} \in l^p$, each a_j is a $(p, 2, M)$-atom, and the sum converges in $L^2(X)$. Set

$$\mathbb{H}^p_{L,at,M}(X) = \{f : f \text{ has an atomic } (p, 2, M)\text{-representation}\},$$

with the norm given by

$$||f||_{\mathbb{H}^p_{L,at,M}(X)} = \inf\{(\sum |\lambda_j|^p)^{1/2} : f = \sum \lambda_j a_j \text{ is an atomic } (p, 2, M)\text{-representation}\}$$

The space $H^p_{L,at,M}(X)$ is then defined as the completion of $\mathbb{H}^p_{L,at,M}(X)$ with respect to the quasi-metric d defined by $d(h, g) = ||h - g||_{\mathbb{H}^p_{L,at,M}(X)}$ for all $h, g \in \mathbb{H}^p_{L,at,M}(X)$.

In this case the mapping $h \longrightarrow ||h||_{H^p_{L,at,M}(X)}, 0 < p < 1$ is not a norm and $d(h, g)) = ||h - g||_{H^p_{L,at,M}(X)}$ is a quasi-metric. For $p = 1$, the mapping $h \longrightarrow ||h||_{H^1_{L,at,M}(X)}$ is a norm. A straightforward argument shows that $H^p_{L,at,M}(X)$ is complete. In particular, $H^1_{L,at,M}(X)$ is a Banach space and $H^1_{L,at,M}(X) \hookrightarrow L^1$. In general, for $p \in (0, 1]$, by Hölder inequality we obtain $H^p_L(X) \subset L^p(X)$.

A basic result concerning these spaces is the following proposition.

Proposition

If an operator L satisfies conditions $(H1)$ and $(H2)$, then for every $0 < p \leq 1$ and for all integers $M \in \mathbb{N}$ with $M > \frac{n(2-p)}{4p}$, the spaces $H^p_{L,at,M}(X)$ coincide and their norms are equivalent.

We next describe the notion of a $(p, 2, M, \epsilon)$-molecule associated to an operator L.

Definition

Let $0 < p \leq 1, 0 < \epsilon$ and $M \in \mathbb{N}$. We say that a function $\alpha \in L^2$ is called a $(p, 2, M, \epsilon)$-molecule associated to L if there exist a function $b \in D(L^M)$ and a ball B such that

(i) $\alpha = L^M b$;

(ii) *For every $k = 0, 1, \ldots, M$ and $j = 0, 1, \ldots,$ there holds*

$$\|(r_B^2 L)^k b\|_{L^2(S_j(B))} \leq r_B^{2M} 2^{-j\epsilon} V(2^j B)^{\frac{1}{2}-\frac{1}{p}}.$$

Proposition

Suppose $0 < p \leq 1$ and $M > \frac{n(2-p)}{4p}$. If α is a $(p, 2, M, \epsilon)$-molecule associated to L, then $\alpha \in H^p_L(X)$. Moreover, $\|\alpha\|_{H^p_L(X)}$ is independent of m.

We introduced the Hardy spaces $H^p_{L,S_h}(X)$ for $p \geq 1$. Now consider the case $0 < p < 1$. The space $H^p_{L,S_h}(X)$ is defined as the completion of

$$\{f \in L^2(X) : ||S_h f||_{L^p(X)} < \infty\}$$

under the norms given by the L^p norm of the square function; i.e.,

$$||f||_{H^p_{L,S_h}(X)} = ||S_h f||_{L^p(X)}, 0 < p < 1.$$

Then the "square function" and "atomic" H^p spaces are equivalent, if the parameter $M > \frac{n(2-p)}{4p}$. In fact, we have the following result.

Proposition

Suppose $0 < p \leq 1$ and $M > \frac{n(2-p)}{4p}$. Then we have $H^p_{L,at,M} = H^p_{L,S_h}(X)$ and their norms are equivalent.

Consequently, as in the next definition, one may write $H^p_{L,at}$ in place of $H^p_{L,at,M}$ when $M > \frac{n(2-p)}{4p}$. Precisely, we have the following definition.

Definition

The Hardy space $H^p_L(X), p \geq 1$, is the space

$$H^p_L(X) := H^p_{L,S_h}(X) := H^p_{L,at}(X) := H^p_{L,at,M}(X), \;\; M > n(2-p)/4p.$$

Note that when $L = -\Delta$ on $\mathbb{R}^n$, the space H^p_L coincides with the classical Hardy space.

3. Boundedness of singular integrals and their commutators

To prove that an operator T is bounded on Hardy space H_L^p which possesses an atomic decomposition, it is not enough in general to prove that Ta is uniformly bounded for all atomic functions a. However, if the operator T satisfies extra condition such as being L^2 bounded (or even the weaker condition of weak type $(2,2)$), then the uniform boundedness of Ta does imply the boundedness of T on $H_L^p(X)$. More precisely, we have the following result.

Proposition

Suppose that T is a linear (resp. nonnegative sublinear) operator which maps $L^2(X)$ continuously into $L^{2,\infty}(X)$. The following statements hold:

(i) *If there exists a constant C such that*

$$||Ta||_{L^{1,\infty}} \leq C$$

for all $(1,2,m)$-atoms $a \in H_L^1(X)$, then T extends to a bounded linear (resp. sublinear) operator from $H_L^1(X)$ to $L^{1,\infty}(X)$.

(ii) *If there exists, for $0 < p \leq 1$, a constant C such that*

$$||Ta||_{L^p} \leq C$$

for all $(p,2,m)$-atoms $a \in H_L^p(X)$, then T extends to a bounded linear (resp. sublinear) operator from $H_L^p(X)$ to $L^p(X)$.

The Proposition above is used in the proof of the Main Result.

Proof of Main Theorem (i) It suffices to show that for any $(p,2,m)$-atom a, for $m > \frac{n(2-p)}{4p}$, associated to the ball B, we have $||Ta||_{L^p} \le C$.

Indeed, we have

$$\int_X (Ta)^p d\mu(x) = \sum_{j=0}^{\infty} \int_{S_j(B)} (Ta)^p d\mu(x)$$
$$= \sum_{j=0}^{\infty} K_j.$$

By Jensen and Hölder inequalities and (6), one has, for each j,

$$K_j \le V(S_j(B))^{1-\frac{p}{2}} ||Ta||^p_{L^2(S_j(B))}$$
$$\le CV(2^j B)^{1-\frac{p}{2}} 2^{-2jmp} V(B)^{\frac{p}{2}-1}$$
$$\le C2^{j(n-\frac{np}{2})-2mp}.$$

This together with $m > \frac{n(2-p)}{4p}$ give

$$\int_X (Ta)^p d\mu(x) \le C \sum_{j=0}^{\infty} 2^{j(n-\frac{np}{2})-2mp} \le C.$$

The proof of (i) is complete.

(ii) We now show that there exists a constant $c > 0$ such that

$$\mu\{x \in M : |[b, T]a| > \lambda\} \leq \frac{c}{\lambda}||b||_{BMO}$$

for all $(1, 2, m)$-atom a, $m > \frac{n}{4}$, and all $\lambda > 0$.
Suppose that a is a $(1, 2, m)$-atom associated to the ball B. Setting $b_B = \frac{1}{V(B)} \int_B b d\mu$, we have

$$[b, T]a(x) = [b(x) - b_B]Ta(x) + T([b_B - b]a)(x).$$

Therefore,

$$\begin{aligned}\lambda\mu\{x \in M : |[b, T]a| > \lambda\} \leq \lambda\mu\{x \in M : |[b(x) - b_B]Ta(x)| > \lambda/2\} \\ + \lambda\mu\{x \in M : |T([b_B - b]a)(x)| > \lambda/2\} = E_1 + E_2.\end{aligned}$$

Let us estimate E_2 first. Since T is of weak type $(1, 1)$, one has, by Hölder inequality

$$\begin{aligned}E_2 &\leq C \int_M |[b_B - b(x)]a(x)| d\mu(x) \\ &\leq C||(b_B - b)||_{L^2(B)}||a||_{L^2(B)} \leq C||b||_{BMO} V(B)^{1/2} V(B)^{-1/2} = C||b||_{BMO}.\end{aligned}$$

Now we proceed with the term E_1. Obviously,

$$
\begin{aligned}
E_1 &\leq C \int_X |[b(x) - b_B] Ta(x)| d\mu(x) \\
&= C \sum_{j=0}^{\infty} \int_{S_j(B)} |[b(x) - b_B] Ta(x)| d\mu(x) \\
&\leq C \sum_{j=0}^{\infty} \int_{S_j(B)} |[b(x) - b_{2^j B}] Ta(x)| d\mu(x) + C \sum_{j=0}^{\infty} \int_{S_j(B)} |[b_B - b_{2^j B}] Ta(x)| d\mu(x).
\end{aligned}
$$

By Hölder inequality, (6) and the fact that $|b_B - b_{2^k B}| \leq ck||b||_{BMO}$, we have,

$$
\begin{aligned}
\int_{S_j(B)} |[b(x) - b_{2^j B}] Ta(x)| d\mu(x) &\leq C||[b(x) - b_{2^j B}||_{L^2(2^j B)}||Ta||_{L^2(S_j(B))} \\
&\leq CV(2^j B)^{1/2}||b||_{BMO} 2^{-2mj} V(B)^{-1/2} \\
&\leq C2^{j(\frac{n}{2} - 2m)}||b||_{BMO}.
\end{aligned} \tag{9}
$$

and

$$\int_{S_j(B)} |[b_{2^jB} - b_B]Ta| d\mu(x)$$

$$\begin{aligned} &\leq Cj||b||_{BMO} \int_{S_j(B)} |Ta| d\mu(x) \\ &\leq CjV(2^jB)^{1/2}||b||_{BMO} 2^{-2mj} V(B)^{-1/2} \\ &\leq Cj2^{j(\frac{n}{2}-2m)}||b||_{BMO}. \end{aligned} \tag{10}$$

The estimates (9), (10) together with $m > \frac{n}{4}$ imply $E_1 \leq C$. The proof of (ii) is complete.

4. Commutators of BMO functions and Riesz transforms on manifolds

Let X be a complete non-compact connected Riemannian manifold with doubling property, μ the Riemannian measure, ∇ the Riemannian gradient. It is well-known that the Laplace-Beltrami operator Δ satisfies conditions $(H1)$ and $(H2)$. Denote the Hardy space associated to Δ by $H^1_\Delta(X)$.
Let $T = \nabla\Delta^{-1/2}$, the Riesz transform on X, and take $b \in \mathrm{BMO}(X)$ (the space of functions of bounded mean oscillations on X). We define the commutator

$$[b, T]g = bTg - T(bg),$$

where g, b are scalar valued and $[b, T]g$ is valued in the tangent space. In [AM], it was proved that for any $b \in BMO(X)$, if Δ has Gaussian heat kernel bounds, then the commutators $[b, T]$ is bounded on $L^p(X)$ with appropriate weights, for $1 < p < 2$.
Our following theorem gives the endpoint estimate for $[b, T]$ when $p = 1$.

Theorem

Assume that X satisfies the doubling property (1) and $b \in$BMO(X). Then, Riesz transform $T = \nabla\Delta^{-1/2}$ is bounded from H^p_Δ to L^p, for all $0 < p \leq 1$. Moreover, if the Riesz transform $T = \nabla\Delta^{-1/2}$ is of weak type $(1, 1)$ then the commutator $[b, T]$ maps $H^1_\Delta(X)$ continuously into $L^{1,\infty}(X)$.

Remark: It is known that if Δ has Gaussian heat kernel bounds, then $T = \nabla\Delta^{-1/2}$ is of weak type $(1, 1)$.

Consider the following versions of the square functions

$$\mathcal{G}f(x) := \Big(\int_0^\infty t|\nabla e^{-t\sqrt{\Delta}} f(x)|^2 dt \Big)^{1/2},$$

$$\mathcal{H}f(x) := \Big(\int_0^\infty |\nabla e^{-t\Delta} f(x)|^2 dt \Big)^{1/2},$$

$$gf(x) := \Big(\int_0^\infty t|\sqrt{\Delta} e^{-t\sqrt{\Delta}} f(x)|^2 dt \Big)^{1/2},$$

and

$$hf(x) := \Big(\int_0^\infty |\Delta e^{-t\Delta} f(x)|^2 dt \Big)^{1/2}.$$

We have

Theorem

(i) *$\mathcal{G}, \mathcal{H}, g$ and h are bounded from $H^p_\Delta(X)$ to $L^p(X)$ for any $0 < p \leq 1$.*

(ii) *If Δ has Gaussian heat kernel bounds, then the commutators of a BMO function b with either $\mathcal{G}$ or $\mathcal{H}$ or g or h are bounded from H^1_Δ to $L^{1,\infty}$.*

5. Commutators of BMO functions and Riesz transforms associated with magnetic Schrödinger operators

Consider magnetic Schrödinger operators as follows. Let the real vector potential $\vec{a} = (a_1, \cdots, a_n)$ satisfy

$$a_k \in L^2_{\mathrm{loc}}(\mathbb{R}^n), \qquad \forall k = 1, \cdots, n, \tag{11}$$

and an electric potential V with

$$0 \leq V \in L^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{12}$$

Let $L_k = \partial/\partial x_k - i a_k$. We define the form Q by

$$Q(f,g) = \sum_{k=1}^{n} \int_{\mathbb{R}^n} L_k f \overline{L_k g}\, dx + \int_{\mathbb{R}^n} V f \overline{g}\, dx$$

with domain $D(Q) = \mathcal{Q} \times \mathcal{Q}$ here

$$\mathcal{Q} = \{f \in L^2(\mathbb{R}^n), L_k f \in L^2(\mathbb{R}^n) \text{ for } k = 1, \cdots, n \text{ and } \sqrt{V} f \in L^2(\mathbb{R}^n)\}.$$

It is well known that this symmetric form is closed and this form coincides with the minimal closure of the form given by the same expression but defined on $C_0^\infty(\mathbb{R}^n)$ (the space of C^∞ functions with compact supports).

Let A be the self-adjoint operator associated with Q. Its domain is given by

$$\mathcal{D}(A) = \left\{ f \in \mathcal{D}(Q), \exists g \in L^2(\mathbb{R}^n) \text{ such that } Q(f,\varphi) = \int_{\mathbb{R}^n} g\bar{\varphi}dx, \ \forall \varphi \in \mathcal{D}(Q) \right\},$$

and A is given by the expression

$$Af = \sum_{k=1}^{n} L_k^* L_k f + Vf. \tag{13}$$

Formally, we write $A = -(\nabla - i\vec{a}) \cdot (\nabla - i\vec{a}) + V$. For $k = 1, \cdots, n$, the operators $L_k A^{-1/2}$ are called the Riesz transforms associated with A. It is easy to check that the operators $L_k A^{-1/2}$ and $V^{1/2}A^{-1/2}$ are bounded on $L^2(\mathbb{R}^n)$. Using the Gaussian heat kernel bounds, one can prove that for each $k = 1, \cdots, n$, the Riesz transforms $L_k A^{-1/2}$ and $V^{1/2}A^{-1/2}$ are bounded on $L^p(\mathbb{R}^n)$ for all $1 < p \leq 2$.

Theorem

(i) The Riesz transforms $L_k A^{-1/2}$ and $V^{1/2}A^{-1/2}$ are bounded from H_A^p to L^p for all $0 < p \leq 1$.

(ii) Let $b \in BMO(\mathbb{R}^n)$. Then the commutators $\left[b, V^{1/2}A^{-1/2}\right]$ and $\left[b, L_k A^{-1/2}\right]$ maps H_A^1 continuously into $L^{1,\infty}(X)$.

6. Application to spectral multipliers

Assume that L is non-negative, self-adjoint and satisfies conditions $(H1)$. Let

$$F(L)f = \int_0^\infty F(\lambda)dE_L(\lambda)f$$

be the spectral multiplier $F(L)$ defined by using the spectral resolution of L. Our main result on spectral multipliers is the following.

Proposition

Assume that L satisfies conditions $(H1)$ and $(H2)$. Let F be a bounded function defined on $(0,\infty)$ such that for some real number $\alpha > \frac{n(2-p)}{2p} + \frac{1}{2}$ and any non-zero function $\eta \in C_c^\infty(\frac{1}{2}, 2)$ there exists a constant C_η such that

$$\sup_{t>0} \|\eta(\cdot)F(t\cdot)\|_{W^{\alpha,2}(\mathbb{R}^+)} \leq C_\eta \tag{14}$$

where $\|F\|_{W^{p,\alpha}(\mathbb{R})} = \|(I - d^2/dx^2)^{\alpha/2}F\|_{L^p}$. Then the multiplier operator satisfies the following estimate

$$\left(\int_{S_j(B)} |F(\sqrt{L})a|^2 dx\right)^{\frac{1}{2}} \leq C2^{-j\delta}V(B)^{\frac{1}{2}-\frac{1}{p}} \tag{15}$$

for some $\delta > \frac{n(2-p)}{4p}$, for any $(p,2,m)$-atom a supported in B and sufficiently large m.

Theorem

(i) *Assume that L satisfies conditions $(H1)$ and $(H2)$. Let F be a bounded function defined on $(0,\infty)$ such that for some real number $\alpha > \frac{n(2-p)}{2p} + \frac{1}{2}$ and any non-zero function $\eta \in C_c^\infty(\frac{1}{2}, 2)$, the condition (14) is satisfied. Then the multiplier operator $F(L)$ is bounded from $H_L^p(X)$ to $L^p(X)$ for $0 < p < 1$.*

(ii) *Under the same assumptions as (i), the operators $F(L)$ is bounded from $H_L^p(X)$ to H_L^p for all $0 < p \leq 1$.*

(iii) *Assume that L satisfies $(H1)$ and $(H3)$. Let F be a bounded function defined on $(0,\infty)$ such that for some real number $\alpha > \frac{n}{2} + \frac{1}{2}$ and any non-zero function $\eta \in C_c^\infty(\frac{1}{2}, 2)$ there exists a constant C_η such that*

$$\sup_{t>0} \|\eta(\cdot)F(t\cdot)\|_{W^{\alpha,\infty}(\mathbb{R}^+)} \leq C_\eta. \tag{16}$$

Then the commutator of $F(L)$ and a BMO function b is bounded from $H_L^1(X)$ to $L^{1,\infty}(X)$.

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