DEGENERATE ELLIPTIC OPERATORS ON THE TORUS $\mathbb{T}^n$

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Conference in Harmonic Analysis and PDE in honour of Eric Sawyer
Fields Institute, Toronto

July, 2011

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2. This talk concern pseudo-differential operators on the torus

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Pseudo-differential operators on $\mathbb{R}^n$

The euclidean quantization

In order to introduce our theory we need the Fourier transform

$$(\mathcal{F}_{\mathbb{R}^n} u)(\xi) = \hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} u(x)dx.$$

One reason for this : The Fourier transform on $\mathbb{R}$ furnish the spectral decomposition of the derivative operator, and we are interested in problems related with Differential operators.

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Notation

Partial derivatives

$$D_i u = (-i)\frac{\partial u}{\partial x_i}\,; \qquad D^\alpha u = D_n^{\alpha_n} \ldots D_2^{\alpha_2} D_1^{\alpha_1} u\,; \qquad x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

Differential Operators

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

The order of P is m if there is α such that $|\alpha| = m$ and $a_\alpha(x) \neq 0$. The coefficients $a_\alpha(x)$ are considered C^∞.

Basic properties:

Reconstruction formula:

$$f(x) = \int_{\mathbb{R}^n} e^{ix\xi} (\mathcal{F}_{\mathbb{R}^n} f)(\xi) d\xi$$

Another notation $\mathcal{F}_{\mathbb{R}^n} f = \hat{f}$.

Now writing $P(x,D)f$ using the inversion formula

$$P(x,D)f(x) = \int_{\mathbb{R}^n} e^{ix\xi} \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha (\mathcal{F}_{\mathbb{R}^n} f)(\xi) d\xi$$

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Symbols and Classes of Symbols

It is desirable to get more general operators considering a suitable class of functions $P(x, \xi)$ that we will call **Symbols** and the corresponding operators are called **Pseudo-differential operators**

$$\textbf{Symbol} \leftrightarrow \textbf{Operator}$$

The spirit of pseudo-differential calculus lives in the study of this relation. One wants to deduce properties on pseudo-differential operators from properties imposed on the corresponding symbols:

Properties on symbol $\rightarrow$ **Properties on operators**

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Symbols and Classes of Symbols S^m, $S^m_{\rho,\delta}$

Notation:

$$< \xi >= (1+|\xi|^2)^{\frac{1}{2}}$$

Kohn-Nirenberg 1965 Let $m \in \mathbb{R}$, we say that a function $\sigma(x,\xi)$ in $C^\infty(\mathbb{R}^{2n})$ belongs to $S^m(\mathbb{R}^n \times \mathbb{R}^n)$, if for all α, β there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)| \leq C_{\alpha,\beta} < \xi >^{m-|\alpha|}$$

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$$|\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)| \leq C_{\alpha,\beta} < \xi >^{m-\rho|\alpha|+\delta|\beta|}$$

$0 \leq \delta \leq \rho \leq 1$, $\delta < 1$.

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Beals-Fefferman classes $S^{M,m}_{\Phi,\phi}$ and $S(m,g)$ Hörmander Classes

Richard Beals-Charles Fefferman 1974.

$$|\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)| \leq C_{\alpha,\beta} \Phi^{M-|\alpha|}(x,\xi)\phi^{m-\beta|}(x,\xi).$$

Lars Hörmander 1979, Weyl-Hörmander calculus, $S(M,g)$ **classes**

Hörmander's metric g

Hörmander's weight M

$$|a^{(k)}(T_1,...,T_k)| \leq C_k M(X) \prod_{i=1}^{k} g_X^{1/2}(T_i).$$

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Some definitions

Hörmander metric

$X \in \mathbb{R}^{2n}$, g_X positive definite quadratic form on $\mathbb{R}^{2n}$. We say that g is an Hörmander metric if:

1. **Continuity**:
   $\exists C > 0, g_X(Y) \leq C^{-1} \implies C^{-1} \cdot g_{X+Y} \leq g_X \leq C \cdot g_{X+Y}.$
2. **Uncertainty principle**: We define $g_X^\sigma(T) = \sup_{W \neq 0} \frac{[T,W]^2}{g_X(W)}$,
   $$g \leq g^\sigma.$$
3. **Temperancy**: We say that g is temperate if $\exists \overline{C} > 0, J \in \mathbb{N}$ such that
   $$\left(\frac{g_X(\cdot)}{g_Y(\cdot)}\right)^{\pm 1} \leq \overline{C}(1 + g_Y^\sigma(X - Y))^J.$$

Some definitions

$g-$weight

We say that a strictly positive function M is a $g-$admissible weight if:

1. M **is** $g-$**continuous**: $\exists C > 0$ such that and $N \in \mathbb{N}$
$$g_X(Y) \leq \frac{1}{\tilde{C}} \implies \left(\frac{M(X+Y)}{M(X)}\right)^{\pm 1} \leq \tilde{C}.$$
2. M **is temperate**: $\exists \tilde{C} > 0$ and $\exists N \in \mathbb{N}$ such that
$$\left(\frac{M(Y)}{M(X)}\right)^{\pm 1} \leq \tilde{C}(1 + g_Y^{\sigma}(X-Y))^N.$$

Examples

(i)

$$g_X^{\rho,\delta} =< \xi >^{2\delta} dx^2 + < \xi >^{-2\rho} d\xi^2, \; X \in \mathbb{R}^{2n},$$

$\rho \leq 1$ is equivalent to the continuity condition. The uncertainty principle is equivalent to the condition $\delta \leq \rho$. The temperancy condition is equivalent to the fact $\delta < 1$.

(ii) The function

$$X = (x, \xi) \mapsto < \xi >,$$

is a weight for the metric $g^{\rho,\delta}$, and $S^m_{\rho,\delta} = S(< \xi >^m, g^{\rho,\delta})$ for all $m \in \mathbb{R}$.

(iii) The function

$$(x, \xi) \mapsto 1$$

is a $g-$weight for every Hörmander's metric g.

Examples

(iv) The Beals-Fefferman classes can be obtained from:

$$g_{(x,\xi)} = dx^2/\varphi(x,\xi)^2 + d\xi^2/\Phi(x,\xi)^2,$$

where the functions φ and Φ satisfy:
(1) $\varphi \leq C$;
(2) $\Phi\varphi \geq c$;
(3) $c \leq \Phi(x,\xi)\Phi(y,\eta)^{-1} \leq C$ and $c \leq \varphi(x,\xi)\varphi(y,\eta)^{-1} \leq C$ if

$$|x-y| \leq c\varphi(x,\xi) \text{ and } |\xi-\eta| \leq c\Phi(x,\xi),$$

(4) $R(x,0) \leq C < x >^c$, where $R(x,\xi) = \Phi(x,\xi)\varphi(x,\xi)^{-1}$;
(5) $c \leq R(x,\xi)R(y,\eta)^{-1} \leq C$, if $|\xi-\eta| \leq cR(x,\xi)^{\delta+\frac{1}{2}}$ and

$$|x-y| \leq cR(x,\xi)^{\delta}R(y,\eta)^{-\frac{1}{2}},$$

where c, C, δ are suitable constants.

Basics on $S^m_{\rho,\delta}$

1. If $a \in S^{m_1}_{\rho,\delta}$ and $b \in S^{m_2}_{\rho,\delta}$ then $ab \in S^{m_1+m_2}_{\rho,\delta}$.
2. If $a \in S^{m_1}_{\rho,\delta}$ and $b \in S^{m_2}_{\rho,\delta}$ then $C(x,D) = a(x,D)b(x,D)$ is a pseudodifferential operator and $C(x,\xi)$ posses an asymptotic expansion

$$C(x,\xi) \sim \sum_{|\alpha|\geq 0} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi a D^\alpha_x b$$

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it means that for every $N \in \mathbb{N}$

$$C(x,\xi) - \sum_{0\leq|\alpha|<N} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi a D^\alpha_x b \in S^{m_1+m_2-(\rho-\delta)N}_{\rho,\delta}$$

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L^2 boundedness

Theorem :

$$a(x,\xi) \in S^0_{\rho,\delta} \Rightarrow a(x,D) : L^2 \to L^2$$

The **Sobolev space** H^s $(s \in \mathbb{R})$ on L^2 can be defined as

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}' : \int_{\widehat{\mathbb{R}^n}} < \xi >^{2s} |\hat{u}(\xi)|^2 d\xi < \infty\}$$

$$\|u\|_{H^s} = (\int_{\widehat{\mathbb{R}^n}} < \xi >^{2s} |\hat{u}(\xi)|^2 d\xi)^{\frac{1}{2}}$$

Theorem :

$$p(x,\xi) \in S^m_{\rho,\delta} \Rightarrow p(x,D) : H^s \to H^{s-m}$$

L^2 boundedness

A more general version

Theorem :

$$a(x,\xi) \in S(1,g) \Rightarrow a(x,D) : L^2 \to L^2$$

The **Sobolev space** $H(m,g)$ can be defined as

$$H(m,g) = \{u \in \mathcal{S}' : a(x,D)u \in L^2, \forall a(x,D) \in OpS(m,g)\}.$$

$$\|u\|_{H(m,g)} = inf\{\|a(x,D)u\|_{L^2} : a(x,D) \in OpS(m,g)\}$$

Theorem : For two $g-$ *weights* m_1, m we have

$$a(x,\xi) \in S(m,g) \Rightarrow a(x,D) : H(m_1,g) \to H(m_1/m,g).$$

Pseudo-differential operators on the torus

The idea of treating pseudo-differential operators on the torus with symbols on $\mathbb{T}^n \times \mathbb{Z}^n$ was first suggested by M.S. Agranovich in 1979.

Special features on the torus:

1. Estimates about pseudo-differential operators on $\mathbb{T}^n$ can be extended to smooth manifolds diffeomorphic to $\mathbb{T}^n$.
2. Results on $\mathbb{T}^n$ can be reduced to the study of $n = 1$.
3. Periodic integral operators are a source of applications for the theory of pseudo-differential operators on the torus.

The Fourier transform on the torus

$$(\mathcal{F}_{\mathbb{T}^n} u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{T}^n} e^{-ix\xi} u(x) dx.$$

$$f(x) = \sum_{\xi} e^{ix\xi} (\mathcal{F}_{\mathbb{T}^n} f)(\xi)$$

On the discrete group $\mathbb{Z}^n$ we shall define the partial difference operator. Let $\sigma : \mathbb{Z}^n \to \mathbb{C}$. Let $e_j \in \mathbb{N}^n, (e_j)_j = 1$, and $(e_j)_i = 0$ if $i \neq j$. We define the **partial difference operator** Δ_{ξ_j} by

$$\Delta_{\xi_j} \sigma(\xi) := \sigma(\xi + e_j) - \sigma(\xi)$$

and

$$\Delta_{\xi}^{\alpha} = \Delta_{\xi_1}^{\alpha_1} \cdots \Delta_{\xi_n}^{\alpha_n}, \ \alpha \in \mathbb{N}_0^{\mathbb{N}}.$$

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Symbol classes on the torus

The symbol $\sigma(x,\xi)$ is in the class $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$ for $1 < \rho, 0 \leq \delta \leq 1$ if

$$|\partial_x^\beta \Delta_\xi^\alpha \sigma(x,\xi)| \leq C_{\alpha,\beta} < \xi >^{m-\rho|\alpha|+\delta|\beta|} .$$

The operator Δ enjoys of good properties:

1. Leibnitz formula
2. Summation by parts

An $L^2(\mathbb{T}^n)$ theorem

Theorem (Ruzhansky and Turunen 2007) : Let $k \in \mathbb{N}$ and $k > \frac{n}{2}$. Let $a : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ be such that

$$|\partial_x^\beta a(x,\xi)| \leq C_\beta,$$

for all $|\beta| \leq k$. Then

$$a(x,D) : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n).$$

Symbol classes on the torus

The equivalence of Global and Local definitions of Periodic Pseudodifferential operators was established by W. McLean (Math.Nachr, 1991) by directly studying charts of the torus.

More recently a different approach based on extension and periodisation techniques the equivalence has been proved by M. Ruzhansky and V. Turunen (Modern trends in pseudodifferential operators, 2007.)

Theorem. Let $0 \leq \delta \leq 1$, $0 < \rho \leq 1$. The symbol $\tilde{a} \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$ is a toroidal symbol if and only if there exists a Euclidean symbol $a \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{R}^n)$ such that $\tilde{a} = a|_{\mathbb{T}^n \times \mathbb{Z}^n}$. Moreover, this extended symbol a is unique modulo $S^{-\infty}(\mathbb{T}^n \times \mathbb{R}^n)$.

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Symbol classes on the torus

Theorem. (Equality of operator classes). For $0 \leq \delta \leq 1$, $0 < \rho \leq 1$ we have

$$OpS^{m}_{\rho,\delta}(\mathbb{T}^n \times \mathbb{R}^n) = OpS^{m}_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n),$$

i.e., classes of 1-periodic pseudodifferential operators with euclidean (Hörmander's) symbols in $S^{m}_{\rho,\delta}(\mathbb{T}^n \times \mathbb{R}^n)$ and toroidal symbols in $S^{m}_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$ coincide.

Degenerate elliptic operators on $\mathbb{R}^n$

Cancelier-Chemin-Xu studied (Ann. Inst. Fourier, vol 43, no.4, 1993) the existence of parametrices for sublaplacians

$$\Delta_{(P)} = \sum_{j=1}^{q} P_j^* P_j$$

associated to a system of smooth real vector fields satisfying the Hörmander condition of order 2.

$$g_X(dx, d\xi) = m^{-2}(X)(<\xi>^2 dx^2 + d\xi^2),$$

where m is the Hörmander's weight

$$m(X) = m(x,\xi) = (a(x,\xi) + <\xi>)^{\frac{1}{2}},$$

and $a(x,\xi) = \sum_{j=1}^{q} P_j^0(x,\xi)^2$.

Non-homogeneous classes

$\sigma \in S(m^t, g)$:

$$|\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)| \leq C_{\alpha,\beta} m^{t-|\alpha|-|\beta|}(x,\xi) < \xi >^{|\beta|}$$

Example on $\mathbb{R}^2$

$$\partial_{x_1}^2 + \tilde{x}_1^2 \partial_{x_2}^2$$

$\tilde{}$ is in $\mathbb{C}^\infty(\mathbb{R})$, $\tilde{\lambda} = \lambda$ for $|\lambda| \leq 2$, $\tilde{\lambda}$ is increasing, $\tilde{\lambda} = 4sgn(\lambda)$ if $|\lambda| \geq 4$.

- J. Delgado. *Estimations L^p pour une classe d'opérateurs pseudo-différentiels dans le cadre du calcul de Weyl-Hörmander*. Journal d'Analyse mathématique, 2006.
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Non-homogeneous classes

More general, on $\mathbb{R}^{n+m}$ for $n+m \geq 2$

$$\Delta_x + |\tilde{x}|^{2\ell}\Delta_t,$$

where $\Delta_x = \sum_{i=1}^{n} D_{x_i}^2$ and $\Delta_t = \sum_{i=1}^{m} D_{t_i}^2$, $\ell = 1, 2, \ldots$.

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• L. Maniccia and M. Mughetti. *Parametrix Construction for a Class of Anisitropic Operators.* Ann. Univ. Ferrara. 2003.

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Non-homogeneous classes

$$
\begin{aligned}
M(x,t,\xi,\tau) &= (\sum_{i=1}^{n}\xi_i^2+\sum_{i=1}^{n}\tilde{x}_i^{2\ell}\sum_{i=1}^{m}\tau_i^2+<\tilde{\xi}>^{2\delta})^{\frac{1}{2}} \\
G_{x,t,\xi,\tau}(dx,dt,d\xi,d\tau) &= M^{-2/\ell}(x,t,\xi,\tau)<\xi,\tau>^{2/\ell}\sum_{i=1}^{n}dx_i^2+\sum_{i=1}^{m}dt_i^2 \\
&\quad +M^{-2/\ell}(x,t,\xi,\tau)<\xi,\tau>^{-2\delta(1-1/\ell)}\sum_{i=1}^{n}d\xi_i^2 \\
&\quad +<\xi,\tau>^{-2}\sum_{i=1}^{m}d\tau_i^2,
\end{aligned}
$$

where $<\xi,\tau>=(1+\mid\xi,\tau\mid^2)^{1/2}$ and $\delta=\frac{1}{1+\ell}$.

Non-homogeneous classes

One can also prove a version for $S(m, g)$ classes:

$$OpS_{m^t,g}(\mathbb{T}^n \times \mathbb{Z}^n) = OpS_{m^t,g}(\mathbb{T}^n \times \mathbb{R}^n)$$

In order to ilustrate we shall consider a particular case on $\mathbb{R}^2$

$$\partial_{x_1}^2 + \tilde{x}_1^2 \partial_{x_2}^2$$

the metric

$$g_X(dx, d\xi) = (\xi_1^2 + \tilde{x}_1^2 \xi_2^2 + < \xi >)^{-1} (< \xi >^2 \, dx^2 + d\xi^2),$$

and the weight

$$m(X) = m(x, \xi) = (\xi_1^2 + \tilde{x}_1^2 \xi_2^2 + < \xi >)^{\frac{1}{2}}.$$

Non-homogeneous classes

$$|\partial_x^\beta \Delta_\xi^\alpha a(x,\xi)| \leq C_{\alpha,\beta} m^{t-|\alpha|-|\beta|}(x,\xi) < \xi >^{|\beta|}$$

(a first step of the proof) Let $a \in S_{m^t,g}(\mathbb{T}^n \times \mathbb{R}^n)$. If $|\alpha| = 1$ and applying the Lagrange Mean Value Theorem there exists η on the line from ξ to $\xi + \alpha$ such that

$$\begin{aligned} \Delta_\xi^\alpha \partial_x^\beta \tilde{a}(x,\xi) &= \Delta_\xi^\alpha \partial_x^\beta a(x,\xi) \\ &= \partial_\xi^\alpha \partial_x^\beta a(x,\xi)|_{\xi=\eta}. \end{aligned}$$

Hence

$$\begin{aligned} |\Delta_\xi^\alpha \partial_x^\beta \tilde{a}(x,\xi)| &\leq C_{\alpha,\beta} m^{t-|\alpha|-|\beta|}(x,\eta) < \eta >^{|\beta|} \\ &\leq ?? C_{\alpha,\beta} m^{t-|\alpha|-|\beta|}(x,\xi) < \xi >^{|\beta|} . \end{aligned}$$

Non-homogeneous classes

The last inequality is explained by

Lemma

If $\alpha \in \mathbb{N}_0^2$ there exist a constant $C_\alpha > 0$ such that for all $x \in \mathbb{T}^2, \xi \in \mathbb{R}^2$ and $\eta \in Q := [\xi_1, \xi_1 + \alpha_1] \times [\xi_2, \xi_2 + \alpha_2]$

$$\left(\frac{m(x,\eta)}{m(x,\xi)}\right)^{\pm 1} \leq C.$$

Non-homogeneous classes

For the proof of the existence one needs:

Lemma

There exists a constant $C > 0$ *such that for all* $\xi, \eta \in \mathbb{R}^2$ *and* $x \in \mathbb{T}^2$ *we have*

$$\begin{aligned} i) m(x, \xi + \eta) &\leq C m(x, \xi) m(x, \eta) \\ ii) m(x, \xi) &\leq C m(x, \xi + \eta) m(x, \eta). \end{aligned}$$

Non-homogeneous classes

Lemma

If $\alpha \in \mathbb{N}_0^2$ there exist a constant $C_\alpha > 0$ such that for all $x \in \mathbb{T}^2, \xi \in \mathbb{R}^2$ and $\eta \in Q := [\xi_1, \xi_1 + \alpha_1] \times [\xi_2, \xi_2 + \alpha_2]$

$$\left(\frac{m(x,\eta)}{m(x,\xi)}\right)^{\pm 1} \leq C.$$

Lemma

There exists a constant $C > 0$ such that for all $\xi, \eta \in \mathbb{R}^2$ and $x \in \mathbb{T}^2$ we have

$$\begin{aligned} i) m(x, \xi + \eta) &\leq C m(x,\xi) m(x,\eta) \\ ii) m(x,\xi) &\leq C m(x, \xi + \eta) m(x, \eta). \end{aligned}$$

Lemma

Let $a : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ *be a measurable function such that for some* $\gamma > 0$, $|a(x,\xi)| \leq C < \xi >^{-(n+\gamma)}$. *Then*

$$a(x,D) : L^p(\mathbb{T}^n) \to L^p(\mathbb{T}^n);\ 1 \leq p \leq \infty. \tag{1}$$