

Norm Inequalities for the Maximal Operator on Variable Lebesgue Spaces

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Trinity College

Conference in Harmonic Analysis and Partial
Differential Equations in honour of Eric Sawyer





Happy Birthday Eric!

Joint work with:

- Alberto Fiorenza
- Christoph Neugebauer
- José María Martell
- Carlos Pérez



Exponent functions

$$p(\cdot) \in \mathcal{P}(\Omega) \quad p(\cdot) : \Omega \rightarrow [1, \infty]$$

$$\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$$

For $E \subset \Omega$

$$p_-(E) = \text{ess inf}\{p(x) : x \in E\}$$

$$p_+(E) = \text{ess sup}\{p(x) : x \in E\}$$

Hereafter: $p_- = p_-(\Omega)$, $p_+ = p_+(\Omega)$



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The modular & norm

Given $p(\cdot) \in \mathcal{P}(\Omega)$

$$\rho_{p(\cdot)}(f) = \rho(f) = \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} dx + \|f\|_{L^\infty(\Omega_\infty)}$$

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1 \right\}$$

The modular & norm

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The space $L^{p(\cdot)}$

Theorem

Given $p(\cdot) \in \mathcal{P}(\Omega)$, $\|\cdot\|_{p(\cdot)}$ is a norm and

$$L^{p(\cdot)}(\Omega) = \{f : \|f\|_{p(\cdot)} < \infty\}$$

is a Banach function space.



Not quite L^p (*Apologies to Lewis Carroll*)



Motivation: PDEs and variational integrals

Minimization problem:

$$\int_{\Omega} |\nabla u(x)|^{p(x)} dx$$

Euler-Lagrange equation

$$\operatorname{div}(p(\cdot)|\nabla u|^{p(\cdot)-2}\nabla u) = 0$$



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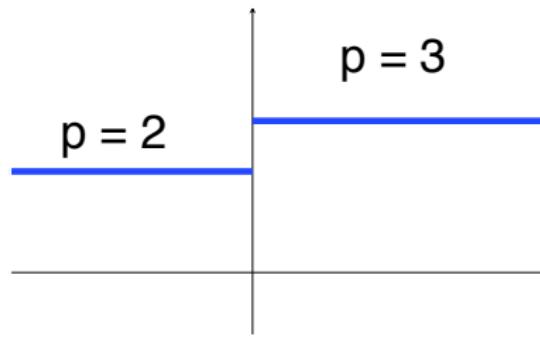
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A simple example

$$p(x) = \begin{cases} 2 & -1 < x \leq 0 \\ 3 & 0 < x < 1. \end{cases}$$



A simple example (continued)

Let $f(x) = |x|^{-1/3} \chi_{(-1,0)}(x)$: $f \in L^{p(\cdot)}((-1, 1))$

For $0 < x < 1$,

$$Mf(x) \geq \int_{-x}^x |f(y)| dy = 3|x|^{-1/3} \notin L^{p(\cdot)}((-1, 1))$$



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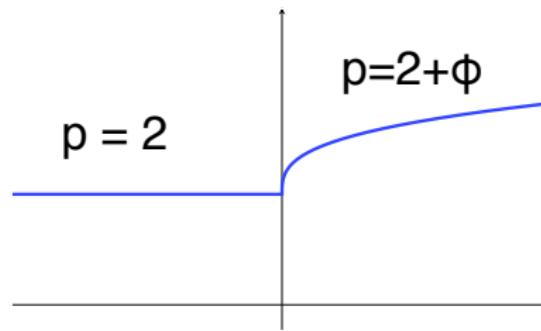
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First question

Do we need $p(\cdot)$ continuous? How much regularity?



Behavior at infinity

Let $p(x) = 3 + \sin(\pi x)$

Define $f(x) = |x|^{-1/3} \sum_k \chi_{[1/4+2k, 3/4+2k]} \in L^{p(\cdot)}(\mathbb{R})$

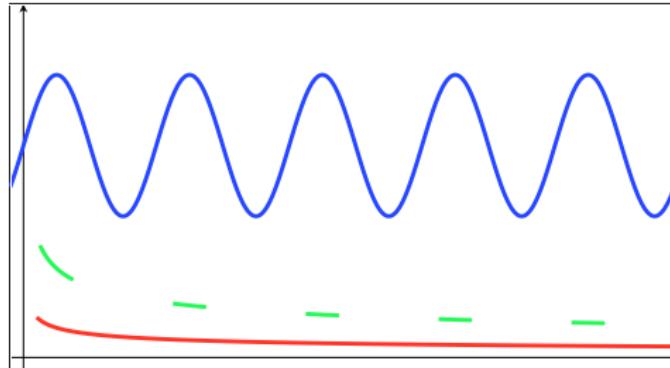
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Second question

Do we need $p(\cdot)$ continuous at infinity? How much regularity?

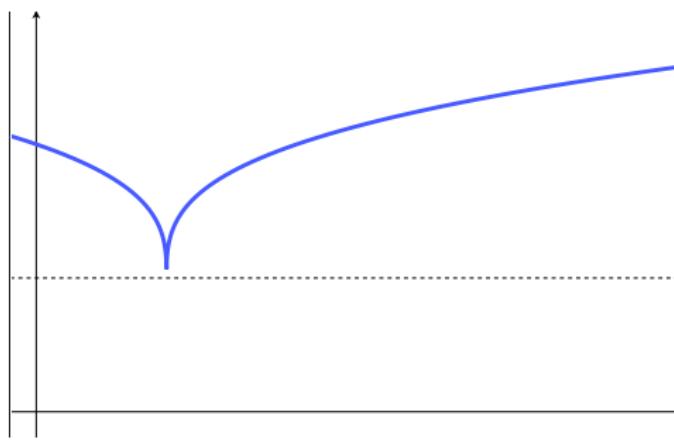
Two more questions

Can $p(\cdot) = \infty$ on parts of the domain?

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Can $p(\cdot) = \infty$ on parts of the domain?

Can $p(\cdot)$ get arbitrarily close to 1?



Local log-Hölder continuity

Given $p(\cdot)$, we say $1/p(\cdot) \in LH_0$ if

$$\left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| \leq \frac{C_0}{-\log(|x-y|)}, \quad |x-y| < \frac{1}{2}$$

If $p_+ < \infty$, $p(\cdot) \in LH_0$ iff $1/p(\cdot) \in LH_0$.

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$L^{p(\cdot)}$ estimates for the maximal operator

Theorem (DCU,AF,CJN)

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $1/p(\cdot) \in LH$. Then for all $\lambda > 0$,

$$\|\lambda \chi_{\{x: Mf(x) > \lambda\}}\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)}$$

If in addition $p_- > 1$, then

$$\|Mf\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)}.$$



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Outline of proof

- Assume f non-negative, bounded and compact support, $\|f\|_{p(\cdot)} = 1$
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An application: Extrapolation in $L^{p(\cdot)}$

Theorem (DCU,AF,JMM,CP)

Given an operator T , suppose that for some p_0 ,

$1 \leq p_0 < \infty$, and all $w \in A_1$,

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w(x) dx \leq C([w]_{A_1}) \int_{\mathbb{R}^n} |f(x)|^{p_0} w(x) dx.$$

Let $p(\cdot)$ be such that $p_0 \leq p_- \leq p_+ < \infty$ and $p(\cdot) \in LH$ (so M is bounded on $L^{(p(\cdot)/p_0)'}).$ Then

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$p_- > 1$ necessary

Theorem (DCU,AF,CJN)

Given any $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, if $p_- = 1$, then the maximal operator is not bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

If $p_- = 1$, then there exists a sequence of balls B_k and exponents $s_k \searrow 1$ such that

$$\{x \in B_k : 1 < p(x) \leq s_k\}$$

$$f_k(x) = |x - x_k|^{-n + \frac{1}{k+1}} \chi_{B_k}(x) \in L^{p(\cdot)}$$

$$Mf_k(x) \geq c(n)(k+1)f_k(x).$$



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LH_0 and LH_∞ pointwise sharp

Take any $\phi(\cdot) : [0, \infty) \rightarrow [0, 1]$, increasing, smooth
 $\phi(x) = 0$ and

$$\lim_{x \rightarrow 0^+} \phi(x) \log(x) = -\infty.$$

Define

$$p(x) = \begin{cases} 2 + \phi(x) & x \geq 0 \\ 2 & x < 0 \end{cases}$$

Then $p(\cdot) \notin LH_0$ and M not bounded on $L^{p(\cdot)}(\mathbb{R})$.

Similar construction holds for LH_∞ .



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Not necessary

LH_0 and LH_∞ not necessary

Define

$$\frac{1}{p(x)} = \begin{cases} \frac{1}{2} + \log(|x|)^{-1/2} & |x| < e^{-9} \\ 5/6 & |x| \geq e^{-9}. \end{cases}$$

Then M bounded on $L^{p(\cdot)}(\mathbb{R})$ but $p(\cdot) \notin LH_0$.

Similar example holds at infinity.



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Continuity not necessary

For $\alpha > 0$ small, define

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Averaging operators on cubes

Define

$$\mathcal{Y} = \{\mathcal{Q} = \{Q\} : Q \text{ pairwise disjoint cubes}\}.$$

Given $\mathcal{Q} \in \mathcal{Y}$ define the averaging operator

$$A_{\mathcal{Q}} f(x) = \sum_{Q \in \mathcal{Q}} \int_Q |f(y)| dy \chi_Q(x).$$



Equivalent conditions

Theorem (Diening, 2005)

Given $p(\cdot)$ such that $1 < p_- \leq p_+ < \infty$, TFAE:

- M bounded on $L^{p(\cdot)}$
- A_Q uniformly bounded for all $Q \in \mathcal{Y}$
- M bounded on $L^{p'(\cdot)}$
- For some $s > 1$, $M(|\cdot|^s)^{1/s}$ bounded on $L^{p(\cdot)}$



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An open question: duality

Find a direct proof of the equivalence of

- M bounded on $L^{p(\cdot)}$
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Special case: Prove this for classical Lebesgue spaces.



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Local condition

Definition

$p(\cdot) \in K_0$ if $\sup_Q |Q|^{-1} \|\chi_Q\|_{p(\cdot)} \|\chi_Q\|_{p'(\cdot)} < \infty$.

Theorem (Kopaliani, Lerner, Diening)

If M is bounded on $L^{p(\cdot)}$, then $p(\cdot) \in K_0$.

Conversely, if $p(\cdot) \in K_0$, then M is bounded on $L^{p(\cdot)}(\Omega)$ for bounded Ω .

There exist $p(\cdot) \in K_0$ such that M is not bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.



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Variable A_p weights

Definition

A weight $w \in A_{p(\cdot)}$ if

$$\sup_Q |Q|^{-1} \|w\chi_Q\|_{p(\cdot)} \|w^{-1}\|_{p'(\cdot)} < \infty.$$

When $p(\cdot)$ constant, equivalent to $w^p \in A_p$.



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Theorem (DCU,AF,CJN—Diening, Hästö)

Given $p(\cdot) \in LH$, suppose $w \in A_{p(\cdot)}$. Then

$$\|(Mf)w\|_{p(\cdot)} \leq C\|fw\|_{p(\cdot)}.$$

Conversely, if M is bounded, $w \in A_{p(\cdot)}$.



Outline of proof

- Prove that $W = w^{p(\cdot)}$, $\sigma = w^{-p'(\cdot)}$ in A_∞ .
- Split f into $f \geq 1$ and $0 \leq f < 1$
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A final question: the two-weight problem

Given a pair of weights (u, v) , find a necessary and sufficient condition for

$$\|(Mf)u\|_{p(\cdot)} \leq C\|fv\|_{p(\cdot)}.$$

Conjecture: for every cube Q ,

$$\|M(v(\cdot)^{-p'(\cdot)}\chi_Q)u\|_{p(\cdot)} \leq C\|v(\cdot)^{1-p'(\cdot)}\chi_Q\|_{p(\cdot)}.$$

Partial results (with strong regularity assumptions) due to Kokilashvili and Meskhi.



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$$\|M(v(\cdot)^{-p'(\cdot)}\chi_Q)u\|_{p(\cdot)} \leq C\|v(\cdot)^{1-p'(\cdot)}\chi_Q\|_{p(\cdot)}.$$

Partial results (with strong regularity assumptions) due to Kokilashvili and Meskhi.



A final question: the two-weight problem

Given a pair of weights (u, v) , find a necessary and sufficient condition for

$$\|(Mf)u\|_{p(\cdot)} \leq C\|fv\|_{p(\cdot)}.$$

Conjecture: for every cube Q ,

$$\|M(v(\cdot)^{-p'(\cdot)}\chi_Q)u\|_{p(\cdot)} \leq C\|v(\cdot)^{1-p'(\cdot)}\chi_Q\|_{p(\cdot)}.$$

Partial results (with strong regularity assumptions) due to Kokilashvili and Meskhi.

