Average value problems in differential equations

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ABSTRACT

Let v be a (Borel) measure on $[a, b]$. Using Schauder's fixed point theorem, with the help of an integral representation

$$y(x) - \frac{1}{v[a,b]}\int_a^b y(z)dv(z) = \frac{1}{v[a,b]}\left[\int_a^x v[a,t]y'(t)dt - \int_x^b v[t,b]y'(t)dt\right]$$

in 'Sharp conditions for weighted 1-dimensional Poincaré inequalities', Indiana Univ. Math. J., 49 (2000), 143-175, by Chua and Wheeden, we obtain existence and uniqueness theorems and 'continuous dependence of average condition' for average value problem:

$$y' = F(x, y), \quad \int_a^b y(x)dv = y_0$$

when v is a probability measure under the usual conditions for initial value problem. For example, F is a measurable function on $I \times J$ (rectangle) such that

$$(A)\begin{cases} |F(x,y)| \le M & \text{for } x \in I \text{ and } y \in J \\ |F(x,y_1) - F(x,y_2)| \le w(x)|y_1 - y_2| & \text{for } x \in I, y_1, y_2 \in J. \end{cases}$$

where $w(I) = w[a, b] = \int_a^b w(x)dx < \infty$.

We also extend our existence and uniqueness theorems when v is a signed measure with $v[a, b] \neq 0$ and

$$F : \mathcal{F} \subset C[a,b] \to L^1[a,b] \quad \text{is a continuous operator.}$$

We then further extend this to discuss its application to symmetric solutions of Laplace equations $\Delta u = F(|x|, u)$ with a given average value.

1 Introduction

There are abundant studies on initial value problems :

$$y' = F(x, y), \quad y(a) = y_0, \quad \text{where } a \in \mathbb{R}, y_0 \in \mathbb{R}^n. \tag{1.1}$$

However, not much has been studied for average value problems. Average value problems may seem to be less natural compare to initial value problems. However, in real life, a given initial value is likely to be indeed just an average value; for example, measurement of temperature or speed is indeed average temperature or average speed over a short time interval. Moreover, initial value problems are just special cases of average value problems where the measure are just the Delta measure at a. Indeed, our average value problems also include delay equations. Furthermore, boundary value problems for higher order ordinary differential equation are also average value problems. Nevertheless, as far as we know, there is only one study of such problem [Chua & Wheeden 2000]. After checking through the literature more carefully, we finally realized that indeed functional boundary value problems,

$$y' = F(x, y), \quad l(y) = y_0, \tag{1.2}$$

where l is a continuous function (vector value) from $C([a, b]; \mathbb{R}^n)$ to $\mathbb{R}^n$, have been discussed extensively (see for example [Vidossich 1989]). General boundary value problems was probably first introduced by [Whyburn 1942].

Clearly, average value problems are just special cases of functional boundary value problems. However, none of those studies give parallel result to the case of initial value problems. Indeed, there is no way to obtain such result. Let us look at the following simple example. The (1-dimensional) functional boundary value problem:

$$u' = 0, l(u) = 1$$

will not have solution when l is a bounded linear functional on $C[0,1]$ unless $l(u) \neq 0$ if u is a nonzero constant function. Thus, in order to obtain parallel results as in the case of initial value problems, it is necessary to restrict the choice of the bounded linear functional. Note that in the above example, it has a unique solution when $l(1) \neq 0$. Indeed, bounded linear functionals on $C[0,1]$ with $l(1) \neq 0$ are just the family of signed measures v on $[a,b]$ with $v[a,b] \neq 0$. In this paper we will study such functional value problems which we call them average value problems where we are able to obtain existence and uniqueness results under the same (standard) assumptions for initial value problems (when the linear functional is arising from a probability measure).

We will consider mostly the case $n = 1$ in this paper. Note that this is the case where the intermediate value theorem holds. However, results in this paper can be easily generalized to $n \geq 2$ or vector-valued (infinite dimension) functions provided the intermediate value theorem is not required.

Let us begin with stating the following two 'classical theorems' on initial value problems.

Theorem 1.1 *Let $x_0 \in [a,b] \subset \mathbb{R}$, $K > 0$, $y_0 \in J$, an interval in $\mathbb{R}$. Let $F : [a,b] \times J \longrightarrow \mathbb{R}$ be such that $F(x,\cdot)$ is continuous for each fixed $x \in [a,b]$ and $F(\cdot,y)$ is integrable (on $[a,b]$) for each fixed $y \in J$ (Carathéodory condition). If*

$$\int_a^b \|F(x,\cdot)\|_{L^\infty(J)} \leq K \text{ and } [y_0 - K, y_0 + K] \subset J, \tag{$*$}$$

then the initial value problem:

$$u'(x) = F(x,u(x)) \text{ on } [a,b] \text{ and } u(x_0) = y_0 \tag{1.3}$$

has at least one absolutely continuous solution (Carathéodory solution) taking values in $[y_0 - K, y_0 + K]$.

Theorem 1.2 *Under the above assumption, instead of assuming $(*)$, suppose F satisfies the following generalized Lipschitz condition:*

$$|F(x,y_1) - F(x,y_2)| \leq w(x)|y_1 - y_2|, \ \ y_1, y_2 \in J, x \in [a,b] \tag{1.4}$$

where $w[a,b] = \int_a^b w dx < \infty$. If y and z are absolutely continuous solutions of the differential equation $u'(x) = F(x,u(x))$ on $[a,b]$ such that $z(x), y(x) \in J$ for all $x \in [a,b]$, then

$$\|y - z\|_{L^\infty[a,b]} \leq |y(x_0) - z(x_0)| \exp(w[a,b]). \tag{1.5}$$

In particular, if $y(x_0) = z(x_0)$, then we know the solution of the initial value problem (1.3) is unique.

Remark 1.3.

1. The two theorems above are usually stated with F being continuous (instead of just measurable) and with $w(x)$ equals to a constant L. Note that when F is continuous, it is easy to check that Carathéodory solutions are just classical solutions, that is, $u'(x) = F(x, u(x))$ everywhere instead of almost everywhere since it is now clear that u is continuously differentiable.

2. When F is given to be continuous on $[a, b] \times \mathbb{R}$, then Peano's theorem asserts that the initial value problem has at least a solution.

In 2000, Chua and Wheeden obtained the following existence and uniqueness result on average value problems.

Theorem 1.4 *[Chua & Wheeden 2000, Theorem 4.2]*

Let $1 < p < \infty$, $M, K > 0$, $-\infty < a < b < \infty$ and $|b - a| \leq K/M$. Let σ be a nonnegative weight and v be a measure on $[a, b]$ such that $v[a, b] > 0$. Suppose F is a measurable function on $[a, b] \times [y_0 - K, y_0 + K] = I \times J$ such that

$$(A) \begin{cases} |F(x, y)| \leq M & \text{for } x \in [a, b] \text{ and } y \in J \text{ and} \\ |F(x, y_1) - F(x, y_2)| \leq w(x)|y_1 - y_2| & \text{for } x \in I, y_1, y_2 \in J. \end{cases}$$

(B) $\lambda = \frac{1}{v[a,b]} || \left(\mu[\cdot,b]v[a,\cdot]^{p'} + \mu[a,\cdot]v[\cdot,b]^{p'}\right)^{1/p'} ||_{L^\infty[a,b]} ||w||_{L^p_\sigma[a,b]} < 1, \quad \mu = \sigma^{1-p'}.$

Then the ordinary differential equation

$$u'(x) = F(x, u(x)), x \in [a,b],$$

has a unique absolutely continuous solution

$$u : [a,b] \to [y_0 - K, y_0 + K] \ \ \textit{such that} \ \int_a^b u dv / v[a,b] = y_0.$$

The proof of Theorem 1.1 in [Chua & Wheeden 2000] is basically just a modification of Picard's iterations using the integral representation. In case v is just the Delta measure at a (and hence the average value problem is just an initial value problem), $w(x) = L$, then

$$\lambda = \mu[a,b]^{1/p'} L\sigma[a,b]^{1/p} < 1$$

will imply $|b-a| < 1/L$ (by the Hölder inequality) which is certainly not required in standard theorem on initial value problems. Thus condition (B) is too restrictive. We will extend Theorems 1.1 and 1.2 to average value problems, that is, existence and uniqueness theorems for average value problems still holds under the same assumption for initial value problems. Our methods is quite elementary that involved mostly Schauder's fixed point theorem, intermediate value theorem and the integral representation.

Theorem 1.5 *Let v be a probability measure on $[a, b]$. Under the assumption of Theorem 1.1, the average value problem:*

$$y'(x) = F(x, y(x)) \quad \text{on } [a, b] \text{ and } y_0 = \int_a^b y dv \tag{1.6}$$

has at least an absolutely continuous solution taking values in $[y_0 - K, y_0 + K]$.

Similarly, we also have the uniqueness and continuous dependence of average value as follows:

Theorem 1.6 *Let v be a probability measure on $[a, b]$. Under the assumptions of Theorem 1.2, we have*

$$\|y - z\|_{L^\infty[a,b]} \leq |\int_a^b (y - z) dv| \exp(w[a, b]). \tag{1.7}$$

Remark 1.7.

1. Note that $w[a, b] < \infty$ whenever $\lambda < \infty$ (see condition (B) of Theorem 1.4) since there exists $\alpha \in [a, b]$ such that $v[a, \alpha], v[\alpha, b] > 0$ and by Hölder's inequality we have

$$w[\alpha, b] \leq \|w\|_{L^p_\sigma[\alpha,b]} (\sigma^{1-p'}[\alpha, b])^{1/p'} < \infty$$

and $w[a, \alpha] < \infty$ similarly.

2. It is essential that $w[a, b] < \infty$. For example, in the case where $w \notin L^1[0, 1]$, the equation $y' = w(x)y$ will not have an absolutely continuous solution on [0,1] with $y(0) \neq 0$.

3. In case v is just the Delta measure at a, Theorems 1.5 and 1.6 are just the classical theorems: Theorems 1.1 and 1.2.

Combining the above two theorems, we have

Corollary 1.8 *Under the assumptions of both Theorem 1.5 and 1.6, the average value problem (1.6) has a unique absolutely continuous solution taking values in* $[y_0 - K, y_0 + K]$.

In the case where the generalized Lipschitz condition (1.4) holds for $J = \mathbb{R}$, we also have:

Theorem 1.9 *Let* v *be a probability measure on* $[a, b]$. *Let* $F : [a, b] \times \mathbb{R} \to \mathbb{R}$ *be such that* $F(\cdot, y)$ *is integrable (on* $[a, b]$*) for each fixed* $y \in \mathbb{R}$ *and satisfies the generalized Lipschitz condition (1.4) with* $J = \mathbb{R}$. *Then for any* $y_0 \in \mathbb{R}$, $u'(x) = F(x, u(x))$ *has a unique absolutely continuous solution* y *on* $[a, b]$ *such that* $\int_a^b y dv = y_0$. *Moreover, if* $z'(x) = F(x, z(x))$ *on* $[a, b]$ *such that* $\int_a^b z dv = z_0$, *then*

$$\|y - z\|_{L^\infty[a,b]} \le |y_0 - z_0| \exp(w[a, b]). \tag{1.8}$$

Remark 1.10. In Theorem 1.9, we did not assume $\int_a^b \|F(x, \cdot)\|_{L^\infty(\mathbb{R})} dx < \infty$, as in Theorems 1.1 or 1.5. Thus the existence of the solution is not a consequence of Theorem 1.5.

Moreover, we will prove the following existence and uniqueness theorem in the case where v is just a signed measure. Instead of assuming F being a function that satisfies the Carathéodory condition (see Theorem 1.1), we will assume that $F : \mathcal{F} \subset C[a,b] \to L^1[a,b]$.

Theorem 1.11 *Let $x_0 \in\subset [a,b] \subset \mathbb{R}$, $y_0 \in \mathbb{R}$. Let $1 \leq p \leq \infty$ and μ a weight on $[a,b]$ such that $\mu > 0$ a.e.. Let v be a signed measure on $[a,b]$ with $v[a,b] \neq 0$. Let $F : \mathcal{F} \subset C[a,b] \to L^1[a,b]$. Suppose F satisfies the following Lipschitz type condition:*

$$\|F[y] - F[z]\|_{L^1[a,b]} \leq L\|y - z\|_{L^q_\mu[a,b]} \text{ for all } y, z \in \mathcal{F}. \tag{1.9}$$

Let

$$v_0 = \frac{1}{|v[a,b]|} \| \left(\mu[\cdot,b]|v[a,\cdot]|^{p'} + \mu[a,\cdot]|v[\cdot,b]|^{p'}\right)^{1/p'} \|_{L^\infty[a,b]} < \infty. \tag{$**$}$$

Suppose either (1) $\mathcal{F} = C[a,b]$, or (2) $\mathcal{F}_0 = \{y \in C[a,b] : \|y - y_0\|_{L^p_\mu[a,b]} \leq v_0 K\} \subset \mathcal{F}$ and $\int_a^b |F[y]|dx \leq K$ for all $y \in \mathcal{F}_0$. If $v_0 L < 1$, then the average value problem:

$$y'(x) = F[y](x) \quad \text{on } [a,b] \text{ and } y_0 v[a,b] = \int_a^b y dv \tag{1.10}$$

has a unique absolutely continuous solution (in $\mathcal{F}_0$) on $[a,b]$.

Remark 1.12.

1. $(**)$ gives the sharp constant for the following Poincaré inequality:

$$\|u - \frac{1}{|v[a,b]|}\int_a^b u dv\|_{L^p_\mu[a,b]} \leq C\|u'\|_{L^1[a,b]}$$

for all absolutely continuous functions u. When $p = \infty$ and $\mu = 1$, we have

$$v_0 = \frac{1}{|v[a,b]|} \| \max\{|v[a,\cdot]|, |v[\cdot,b]|\| \|_{L^\infty[a,b]}.$$

Indeed, we have by [Chua & Wheeden 2000],

$$\| \frac{1}{v[a,b]} \left(\int_a^x v[a,t] f(t) dt - \int_x^b v[t,b] f(t) dt \right) \|_{L^p_\mu[a,b]} \leq v_0 \|f\|_{L^1[a,b]} \tag{1.11}$$

for all $f \in L^1[a,b]$.

2. Theorem 1.11 can be extended to the case when the measure v is defined only on $[a', b']$ and (1.9) holds only on $[a', b']$ provided condition (4.1) (which is a generalization of (1.4) also holds so as to extend solution from $[a', b']$ to $[a, b]$.

3. The differential equation in (1.10) will certainly include differential equations such as

$$y'(x) = f(x, y(\frac{x + x_0}{2})), \ y(x_0) = y_0.$$

Our idea can also be modified to include delay-differential equations:

$$y'(x) = f(x, y(x - r(x)), \ y(x_0) = y_0,$$

see [Walter 1998, p.82] for more detail.

Finally, we would like to discuss an application/extension of our results to symmetric (rotational) solution of the Laplace equation:

$$\Delta u = F(|x|, u), \int_0^a u(r) dv(r) = u_0 \quad (r = |x|).$$

Theorem 1.13 *Let $b \geq b' > 0$ and $y_0 \in \mathbb{R}$. Let v be a signed measure on $[0, b']$ such that $v[0, b'] \neq 0$. Let $v_0 > 0$ be such that $|v[0, t]|, |v[t, b']| \leq v_0 |v[0, b']|$ for almost all $t \in [0, b']$. Let $F : \mathcal{F} \subset C[0, b] \to L^1[0, b]$ such that $F[y_1] = F[y_2]$ on $[0, b']$ whenever $y_1 = y_2$ on $[0, b']$ and satisfies a Lipschitz-type condition*

$$\int_0^x |F[y_1](t) - F[y_2](t)| dt \leq w(x) \|y_1 - y_2\|_{L^\infty[0,x]} \quad y_1, y_2 \in \mathcal{F}, \ \text{for } x \in [0, b] \tag{1.12}$$

with $w[0, b] < \infty$. Suppose either (1) $\mathcal{F} = C[a, b]$ or (2) there exists $K > 0$ such that

$$\mathcal{F}_0 = \{y \in C[0, b] : \|y - y_0\|_{L^\infty[0,b]} \leq v_0 K\} \subset \mathcal{F}$$

and $\int_0^b |F[y]| dx \leq K/b$ for all $y \in \mathcal{F}_0$. If $v_0 w[0, b'] < 1$, then the following partial differential equation

$$\Delta u = F(|x|, u), \ \text{for all } x \in \mathbb{R}^n, |x| \leq b$$

has a unique (rotational) symmetric solution $u(x) = y(r)$, $r = |x|$, such that $y \in C^1[0, b]$ and y' is absolutely continuous, with $\int_0^{b'} y dv = y_0 v[0, b']$.

2 Preliminaries

We shall collect a few useful results here. First, let us state a theorem on compact operators: Schauder's fixed point theorem.

Theorem 2.1 *[GT, Corollary 11.2] Let X be a Banach space. If $\mathcal{D} \subset X$ is closed and convex such that $T : \mathcal{D} \to \mathcal{D}$ is continuous and $T(\mathcal{D})$ is precompact, then there exists $x_0 \in \mathcal{D}$ such that $T(x_0) = x_0$.*

We will now use the above fixed point theorem to prove existence of fixed points of some integral operators.

Lemma 2.2 *Let $[a,b] \subset \mathbb{R}$, $K > 0$, $y_0 \in \mathcal{F} \subset C[a,b]$. Let $1 \le p \le \infty$ and μ be a weight on $[a,b]$. Let $F : \mathcal{F} \to L^1[a,b]$ be such that F is continuous. Let v and v_0 be as in Theorem 1.11. Let*

$$Tu(x) = y_0(x) + \frac{1}{v[a,b]} \left(\int_a^x v[a,t]F[u](t)dt - \int_x^b v[t,b]F[u](t)dt \right) \quad \textit{for } x \in [a,b].$$

Suppose $\mathcal{F}_0 = \{y \in C[a,b] : \|y - y_0\|_{L^p_\mu[a,b]} \le v_0 K\} \subset \mathcal{F}$
and $\int_a^b |F[y](x)|dx \le K$ for all $y \in \mathcal{F}_0$.
If there exists $f \in L^1[a,b]$ such that $|F[y]| \le f$ a.e. for all $y \in \mathcal{F}_0$, then the operator T has a fixed point in $\mathcal{F}_0$.

Proof. First, there exists $\tilde{v}_0 > 0$ such that $|v[a,x]|, |v[x,b]| \le \tilde{v}_0 |v[a,b]|$ a.e.. Next since

$$|Tu(x) - Tu(z)| \le |\int_x^z 2\tilde{v}_0 |f(t)|dt| + |y_0(x) - y_0(z)|, \tag{2.1}$$

it is clear that Tu is continuous for $u \in \mathcal{F}_0$ since $F[u] \in L^1[a,b]$.

Next, it is easy to see that

$$\|Tu - y_0\|_{L^\infty[a,b]} \leq v_0 \|F[u]\|_{L^1[a,b]} \leq v_0 K$$

and hence $T(\mathcal{F}_0) \subset \mathcal{F}_0$. Moreover $\mathcal{F}_0$ is clearly closed and convex. Now, let us check that T is continuous on $\mathcal{F}_0$. Suppose $u_n \to u$ in $C[a,b]$ such that $u_n, u \in \mathcal{F}_0$. We have

$$\lim_{n\to\infty} \|Tu_n - Tu\|_{L^\infty[a,b]} \leq \lim_{n\to\infty} \int_a^b \tilde{v}_0 |F[u_n](x)) - F[u](x)| dx = 0$$

since F is a continuous operator. It is then clear that T is continuous.

Moreover, note that by (2.1), the family of functions $\{T(\mathcal{F}_0)\}$ is equi-continuous. Also, clearly the family of functions $\{T(\mathcal{F}_0)\}$ is uniformly bounded. Thus by the Ascoli Arzela theorem, we know $\{T(\mathcal{F}_0)\}$ is pre-compact. Hence, by Schauder's fixed point theorem, the operator T must have at least one fixed point.

Remark 2.3.

Suppose F satisfies the Carathéodory condition as in Theorem 1.1, and $\|F(x,\cdot)\|_{L^\infty(J)} \in L^1[a,b]$. If we define

$$\tilde{F}[y](x) = F(x, y(x)) \text{ for } x \in [a,b], y \in C[a,b],$$

then $\tilde{F}$ is clearly a continuous operator from $C[a,b]$ to $L^1[a,b]$.

Next, let us recall a simple fact on Laplace operator from [Walter 1998].

Proposition 2.4 *Let $a > 0$ and $B_a(0) = \{x \in \mathbb{R}^n : |x| \leq a\}$. If $u \in C^1(B_a(0))$ is (rotational) symmetric function, i.e., $u(x) = y(r), r = |x|$, and y' is absolutely continuous on $[0, b]$ then*

$$\Delta u(x) = y''(r) + \frac{n-1}{r} y'(r), \text{ for almost all } x \text{ and } y'(0) = 0.$$

3 Average value problems with signed measure

In this section, we will study existence and uniqueness theorems for average value problem when the probability measure is being replaced by just a signed measure. Of course, we can only obtain a weaker result under this assumption. However, on the other hand, our assumption on F can be relaxed.

First of all, recall that the space of bounded linear functionals on $C[a, b]$ is just the collection of all signed (Borel) measures on $[a, b]$. Here are our main theorems in this section.

Theorem 3.1 *Let $[a, b] \subset \mathbb{R}$, $y_0 \in \mathbb{R}$. Let μ be a weight and $1 \leq p \leq \infty$. Let v be and $v_0 > 0$ be as in Theorem 1.11. Let $F : \mathcal{F} \subset C[a, b] \to L^1[a, b]$ be such that $F[y_n] \to F[y]$ in $L^1[a, b]$ whenever $\|y_n - y\|_{L^\infty[a,b]} \to 0$. Let $\mathcal{F}_0 = \{y \in C[a, b] : \|y - y_0\|_{L^p_\mu[a,b]} \leq v_0 K\}$ and $\int_a^b |F[y]| dx \leq K$ for all $y \in \mathcal{F}_0$. If $\mathcal{F}_0 \subset \mathcal{F}$ and there exists $f \in L^1[a, b]$ such that $|F[y]| \leq f$ a.e.*

for all $y \in \mathcal{F}_0$*, then the average value problem:*

$$y'(x) = F[y](x) \quad \text{on } [a,b] \text{ and } y_0 v[a,b] = \int_a^b y dv \tag{3.1}$$

has at least an absolutely continuous solution.

Theorem 3.2 *Let* v *and* v_0 *be as before. Let* $F : \mathcal{F} \subset C[a,b] \to L^1[a,b]$ *such that*

$$\|F[y_1] - F[y_2]\|_{L^1[a,b]} \leq L\|y_1 - y_2\|_{L^p_\mu[a,b]} \tag{3.2}$$

for all $y_1, y_2 \in \mathcal{F}$*. If* $Lv_0 < 1$*, then*

$$\|y - z\|_{L^p_\mu[a,b]} \leq \frac{\mu[a,b]^{1/p} |\int_a^b (y-z)dv|}{|v[a,b]|(1 - Lv_0)}. \tag{3.3}$$

Before we prove these two theorems, let us use Theorem 3.1 to provide a quick proof to a consequence of [Vidossich 1989, Theorem 2].

Corollary 3.3 *Let* v *be a signed measure on* $[a,b]$*. If the functional boundary value problem:*

$$u' = 0, \ \int_a^b u dv = y_0 v[a,b]$$

has a unique solution $u \in C[a,b]$ *for every* $y_0 \in \mathbb{R}$*, then for every bounded continuous function* $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ *and every* $y_0 \in \mathbb{R}$*, the functional boundary value problem:*

$$u'(x) = g(x, u(x)), \ \int_a^b u dv = y_0 v[a,b]$$

has at least one solution.

Proof. First note that the condition will imply that $v[a,b] \neq 0$. Next, if g is a bounded continuous function on $[a,b]\times\mathbb{R}$, then there exists $M>0$ such that $|g(x,y)| \leq M$ on $[a,b]\times\mathbb{R}$ and hence $\int_a^b |g(x,y(x))|dx \leq M|b-a|$. Since v is a signed measure and $v[a,b] \neq 0$, there exists $\tilde{v}_0 > 0$ such that

$$|v[a,t]|, |v[t,b]| \leq \tilde{v}_0 |v[a,b]|.$$

It is now clear Corollary 3.3 follows from Theorem 3.1 with $p=\infty$ and $\mu = 1$.

Proof of Theorem 3.1.

The key idea of the proof is to deduce the integral representation.

First let us show that $y \in \mathcal{F}$ is an absolutely continuous solution of the average value problem (3.1) if and only if

$$y(x) - y_0 = \frac{1}{v[a,b]}\left[\int_a^x v[a,t]F[y](t)dt - \int_x^b v[t,b]F[y](t)dt\right]. \tag{3.4}$$

Clearly, (3.4) will imply $y'(x) = F[y](x)$ for almost all $x \in [a,b]$ and y is absolutely continuous since $F[y] \in L^1[a,b]$. Next, observe that if y is absolutely continuous on $[a,b]$, we have by Fubini's theorem

$$\begin{aligned}
& y(x) - \frac{1}{v[a,b]}\int_a^b y(z)dv(z) = \frac{1}{v[a,b]}\int_a^b (y(x)-y(z))dv(z) \\
= & \frac{1}{v[a,b]}\int_a^b \int_z^x y'(t)dtdv(z) \\
= & \frac{1}{v[a,b]}\left[\int_a^x \int_z^x y'(t)dtdv(z) - \int_x^b \int_x^z y'(t)dtdv(z)\right] \\
= & \frac{1}{v[a,b]}\left[\int_a^x v[a,t]y'(t)dt - \int_x^b v[t,b]y'(t)dt\right].
\end{aligned} \tag{3.5}$$

It is now easy to see that y satisfies (3.4) will imply that y is a solution of the average value problem (3.1) and conversely, y satisfies (3.4) if it is an absolutely continuous solution of (3.1).

Thus, we need only to find a fixed point of the following operator

$$Ty(x) = y_0 + \frac{1}{v[a,b]}\left[\int_a^x v[a,t]F[y](t)dt - \int_x^b v[t,b]F[y](t)dt\right].$$

By Schauder's fixed point theorem, the operator T has a fixed point $u \in C[a,b]$. Moreover, it is clear that Tu is absolutely continuous and so is u. This completes the proof of Theorem 3.1.

We will now prove Theorem 3.2

Proof of Theorem 3.2. First it is clear that we have

$$y(x) - z(x) = \frac{1}{v[a,b]}\int_a^b (y-z)dv +$$
$$\frac{1}{v[a,b]}\left[\int_a^x v[a,t]\{F[y](t) - F[z](t)\}dt - \int_x^b v[t,b]\{F[y](t) - F[z](t\}dt\right]$$

and hence by the triangle inequality, 'Poincaré inequality' (1.11) and (3.2),

$$\begin{aligned}\|y-z\|_{L^p_\mu[a,b]} &\leq \mu[a,b]^{1/q}|\frac{1}{v[a,b]}\int_a^b y - zdv| + v_0\||F[y] - F[z]\|_{L^1[a,b]}\\ &\leq |\frac{\mu[a,b]^{1/q}}{v[a,b]}\int_a^b y - zdv| + v_0L\|y-z\|_{L^p_\mu[a,b]}\end{aligned}$$

Theorem 3.2 is now clear.

Remark 3.4. When conditions in both Theorems 3.1 and 3.2 hold, Picard's iteration can also be used to obtain the solution of the average value problem. Just let $u_0(x) = y_0$ and

$$u_n(x) = y_0 + \frac{1}{v[a,b]}\left[\int_a^x v[a,t]F[u_{n-1}](t)dt - \int_x^b v[t,b]F[u_{n-1}](t)dt\right]$$

for $n = 1, 2, 3, \cdots$.

4 Proof of main theorems

First, note that Theorems 1.5 and 1.1 are clearly special cases of Theorem 3.1.

Next, let us prove a stronger result instead of Theorem 1.2.

Proposition 4.1 *Let $x_0 \in [a,b] \subset \mathbb{R}$, $y_0 \in \mathbb{R}$. Let $F : \mathcal{F} \subset C[a,b] \to L^1[a,b]$ such that F satisfies the following Lipschitz type condition:*

$$\begin{aligned} |F[u](x) - F[v](x)| &\leq w(x)\|u - v\|_{L^\infty[x_0,x]} \text{ for } x \in (x_0, b] \text{ and} \\ |F[u](x) - F[v](x)| &\leq w(x)\|u - v\|_{L^\infty[x,x_0]} \text{ for } x \in [a, x_0), \end{aligned} \quad (4.1)$$

for all $u, v \in \mathcal{F}$ with $w[a,b] < \infty$. If $y, z \in \mathcal{F}$ are solutions of the equation $u' = F[u]$, then

$$\|y - z\|_{L^\infty[a,b]} \leq |y(x_0) - z(x_0)|(\max\{\exp w[x_0,b], \exp w[a,x_0]\}). \quad (4.2)$$

Moreover, if $\mathcal{F} = C[a,b]$, then for each $\alpha \in \mathbb{R}$, there exists a unique absolutely continuous solution of the initial value problem:

$$u' = F[u], \ u(x_0) = \alpha.$$

Proof of Proposition 4.1. Since $y, z \in \mathcal{F}$ are solutions to the differential equation, we have

$$y(x) = y(x_0) + \int_{x_0}^{x} F[y](t)dt \quad \text{and}$$
$$z(x) = z(x_0) + \int_{x_0}^{x} F[z](t)dt.$$

To simplify the computation, let us assume $x_0 = a$. Then,

$$\begin{aligned} |y(x) - z(x)| &\leq |y(x_0) - z(x_0)| + \int_a^x |F[y](t) - F[z](t)|dt \\ &\leq |y(x_0) - z(x_0)| + \int_a^x w(t)\|y - z\|_{L^\infty[a,t]}dt \qquad (4.3) \\ &\leq |y(x_0) - z(x_0)| + w[a,x]\|y - z\|_{L^\infty[a,x]}. \end{aligned}$$

(4.2) is then just a consequence of Gronwall's inequality. Indeed,

$$\|y - z\|_{L^\infty[a,x]} \leq |y(x_0) - z(x_0)| + w[a,x]\|y - z\|_{L^\infty[a,x]}, \qquad (4.4)$$

we can repeat our previous argument. Let $\varepsilon = |y(x_0) - z(x_0)|$. Then

$$\begin{aligned} & |y(x) - z(x)| \\ \leq\ & \int_a^x w(t)(w[a,t]\|y - z\|_{L^\infty[a,t]} + \varepsilon)dt + \varepsilon \quad \text{(by (4.3) and (4.4))} \\ \leq\ & \int_a^x w(t)(w[a,t]\|y - z\|_{L^\infty[a,x]} + \varepsilon)dt + \varepsilon \\ =\ & \|y - z\|_{L^\infty[a,x]} w[a,x]^2/2 + \varepsilon w[a,x] + \varepsilon. \end{aligned}$$

Repeat it n times, we have

$$\|y - z\|_{L^\infty[a,x]} \leq (w[a,x]^n/n!)\|y - z\|_{L^\infty[a,x]} + \varepsilon \exp(w[a,x])$$

and hence,

$$\|y - z\|_{L^\infty[a,x]} \leq \varepsilon \exp(w[a,x]).$$

This completes the proof of the first part of the proposition and Theorem 1.2 is now clear. The second part of the proof is indeed again just a simple modification of the classical argument, we put them here simply because we are unable to find a reference. Again, we will just prove the case $x_0 = a$. We will prove it by Banach contraction theorem on $C[a,b]$ with norm

$$\|u\|_\alpha = \sup_{x\in[a,b]} \|u\|_{L^\infty[a,x]} e^{-\alpha w[a,x]}, \quad \alpha > 1$$

Let $Tu(x) = y(a) + \int_a^x F[y](t)dt$. Note that if $a \leq x' \leq x$, we have

$$|Tu(x') - Tz(x')| \leq \int_a^x |F[u] - F[z]|dt \leq \int_a^x w(t)\|u - z\|_{L^\infty[a,t]}dt.$$

Hence

$$\begin{aligned}\|Tu - Tz\|_{L^\infty[a,x]} &\leq \int_a^x w(t)\|u - z\|_{L^\infty[a,t]} e^{-\alpha w[a,t]} e^{\alpha w[a,t]} dt \\ &\leq \|u - z\|_\alpha \int_a^x w(t) e^{\alpha w[a,t]} dt \leq \|u - z\|_\alpha e^{\alpha w[a,x]}/\alpha.\end{aligned}$$

Thus

$$\|Tu - Tz\|_\alpha \leq \|u - z\|/\alpha.$$

Hence by the Banach contraction theorem, T has a unique fixed point and it is easy to see that the fixed point is the solution.

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. Note that by the fact that $y-z$ is continuous, we can find $x_0 \in [a,b]$ such that

$$\int_a^b (y-z)dv = y(x_0) - z(x_0).$$

Since y and z are absolutely continuous solutions of the differential equation $u' = F(x,u)$, by (1.5) in Theorem 1.2, we conclude that

$$\|y-z\|_{L^\infty[a,b]} \le |y(x_0) - z(x_0)| \exp(w[a,b]) = |\int_a^b (y-z)dv| \exp(w[a,b]).$$

This completes the proof of our main theorem.

Proof of Theorem 1.9.

We will provide an elementary proof instead of using Schauder's fixed point theorem. We will prove the theorem by 2 steps. First we show that the theorem holds if $w[a,b] < 1$. We then make use of Proposition 4.1 and step 1 to conclude the proof.

Step 1. We will first prove that the theorem holds when $w[a,b] < 1$ using Picard's iterations.

Step 2. Since $w \in L^1[a,b]$, it is easy to see that there exists $\delta > 0$ such that if $[a',b'] \subset [a,b]$ such that $b' - a' < \delta$, then $w[a',b'] < 1$ and hence (1.6) has a unique solution y on $[a',b']$ such that $\int_{a'}^{b'} y(x)dv/v[a',b'] = y_0$ provided $v[a',b'] > 0$ by step 1 as $w[a',b'] < 1$.

We will decompose $[a,b]$ into 2^m ($m \in \mathbb{N}$) equal closed subintervals so that the average value problem (1.6) has a solution on each subintervals with average y_0 (provided the v-measure of that subinterval is > 0). All we need to do next is to find a way to "combine" solutions on any two neighboring subintervals of same length. We will show that this is always possible and then our proof will be complete. For this step we will need the fact that solution is continuously depending on the initial data and intermediate value theorem.

This completes the proof of our main theorem.

Proof of Theorem 1.11.

First, by Picard's iteration, we see that (1.10) has an absolutely continuous solution. Next, since for any $y_1, y_2 \in C[a,b]$,

$$\|y_1' - y_2'\|_{L^1[a,b]} = \|F[y_1] - F[y_2]\|_{L^1[a,b]}$$

$$\leq L\|y_1 - y_2\|_{L^q_\mu[a,b]} \leq Lv_0\|y_1' - y_2'\|_{L^1[a,b]}.$$

Since $Lv_0 < 1$, $y_1 = y_2$ μ-a.e. and hence $y_1 = y_2$ as they are both continuous and $\mu > 0$ a.e.. This conclude the proof of Theorem 1.11.

Next, instead of proving Theorem 1.13, we will prove a slightly more general proposition.

Proposition 4.2 *Let $b \geq b' > 0$ and $y_0 \in \mathbb{R}$. Let $v_0 \geq 1$. Let v be a signed measure on $[0, b']$ such that $v[0, b'] \neq 0$ and $|v[0,t]|, |v[t,b']| \leq v_0|v[0,b']|$ for almost all $t \in [0, b']$. Let $F : \mathcal{F} \subset C[0,b] \to L^1[0,b]$ such that $F[y_1] = F[y_2]$*

on $[0, b']$ *whenever* $y_1 = y_2$ *on* $[0, b']$ *and let*

$$G[y](t) = t^{-\alpha} \int_0^t s^\alpha F[y](s) ds$$

satisfy the following Lipschitz type condition

$$|G[y_1](x) - G[y_2](x)| \le w(x) \|y_1 - y_2\|_{L^\infty[0,x]} \quad \textit{for } x \in [0, b], \ y_1, y_2 \in \mathcal{F}. \tag{4.5}$$

Suppose either (1) $\mathcal{F} = C[0, b]$ *or (2) there exists* $K > 0$ *such that*

$$\mathcal{F}_0 = \{y \in C[0, b] : \|y - y_0\|_{L^\infty[0,b]} \le v_0 K\} \subset \mathcal{F}$$

and $\int_0^b |G[y]| dx \le K$ *for all* $y \in \mathcal{F}_0$*. If* $v_0 w[0, b'] < 1$*,* $\alpha \ge 0$*, then the following second order linear ordinary differential equation*

$$y'' + \frac{\alpha}{x} y' = F[y] \tag{4.6}$$

has a unique solution y *such that* y' *is absolutely continuous,* $y'(0) = 0$ *and* $\int_0^{b'} y dv = y_0$*.*

Proof. Exercise.

Finally, let us prove an extension of a classical Peano's uniqueness theorem.

Theorem 4.3 *Let* $F(x, y)$ *be continuous in* $[a, b] \times [y_0 - K, y_0 + K]$ *and nonincreasing in* y *for each fixed* x*. If* v *is a probability measure on* $[a, b]$*,* $|F(x, y)| \le M$ *and* $M(b - a) \le K$*, then the average value problem (1.6)* *has at most one absolutely continuous solution* $u : [a,b] \to [y_0 - K, y_0 + K]$ *with* $\int_a^b u dv = y_0$.

Final Remark. Consider the following boundary value problem:

$$u'' = F(x,u) \text{ on } [0,1], \; u(0) = u(1) = 0. \tag{4.7}$$

Note that this is indeed a special case of our average value problem (in $\mathbb{R}^2$). It is known that if F is continuous and

$$|F(x,y_1) - F(x,y_2)| \le L|y_1 - y_2| \;\; \text{for } x \in [0,1], y_1, y_2 \in \mathbb{R},$$

then this boundary value problem has a unique solution if $L < \pi^2$. However, there are counter examples when $L = \pi^2$ (see [Walter 1998, p.254]).

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