Embeddability for CR 3-Manifolds and CR Yamabe Invariants

Sagun Chanillo

Rutgers University

Conference for Eric Sawyer

This is joint work with Paul Yang and Hung-Lin Chiu.

A theme that runs in our work is there is a close connection between 4 dimensional Conformal Geometry and 3 dimensional CR Geometry.
This point of view is not new: Fefferman-Graham, Fefferman-Hirachi and in String Theory the AdS-CFT correspondence.

Definitions

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3. $\dim_C \mathcal{V} = n$

We call $\mathcal{V}$ the CR (Cauchy-Riemann)bundle. Let $H = Re(\mathcal{V} \oplus \overline{\mathcal{V}})$. Let us also assume that there is a non-vanishing section of $H^\perp \subset T^\star M$, where $T^\star M$ is the co-tangent bundle and let us call this section (a real 1 -form)θ. The Levi form is given by,

$$L_\theta(V, W) = -id\theta(V \wedge \overline{W}),\ \ V, W \in \mathcal{V}$$

The Contact Form 1

From now on $n = 1$, and our manifold M^3 will be **compact** with no boundary.

We shall assume that the Levi form is positive definite (strongly pseudo-convex case). We choose a frame so that $\theta(T) = 1($ normalization), and $d\theta(T, \cdot) = 0$, and a frame for CTM:

$$\{T, Z_1, Z_{\bar{1}}\}, Z_1 \in \mathcal{V}, Z_{\bar{1}} \in \bar{\mathcal{V}}$$

Dual frame is:

$$\{\theta, \theta^1, \theta^{\bar{1}}\}$$

The form θ satisfies:

$$\theta \wedge d\theta \neq 0$$

Such a form is called a Contact form.

The Contact Form 2

The contact form satisfies:

$$d\theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}}, \;\; h_{1\bar{1}} > 0$$

$h_{1\bar{1}}$ is the Levi form that can be normalized(by choosing Z_1 to have "length" 1) to be 1 and is used to raise and lower symbols. Also:

$$dV = \theta \wedge d\theta \neq 0$$

dV will be the volume element for our manifold M.

One can view the CR Structure as $J : H \to H$ an endomorphism with $J^2 = -I$, $H = \ker \theta$ with $J\mathcal{V} = i\mathcal{V}$. J is called the almost complex structure.

The CR Structure-Another View

The CR structure is given by a triple (M, θ, J). We almost always will never change θ but for the fixed θ and M we will change J. This will be called deforming the CR structure. So deformation here is **NOT** in the sense of Elasticity. M the manifold is **NOT** getting distorted.
If we do change θ it will be by multiplying θ by a conformal factor:

$$e^{2f}\theta$$

This change still preserves $H = \ker\theta$.

Webster Curvature and Torsion

Webster defined a connection in studying the CR equivalence problem:

$$d\theta^1 = \theta^1 \wedge \theta_1^1 + A_{\bar{1}}^1 \theta \wedge \theta^{\bar{1}}.$$

$$d\theta_1^1 = R\theta^1 \wedge \theta^{\bar{1}} + \cdots$$

R is called the Tanaka-Webster curvature, $A_{\bar{1}}^1$ is called the Webster torsion.

We are in $n = 1$ so R is a function(think of Gauss curvature).

Webster: J. Differential Geom., 1978.

Well-known Example

The Heisenberg group with coordinates on it $(x, y, t) \in R^3$ is a CR 3-manifold. Contact form is:

$$\theta = dt + xdy - ydx$$

This is a non-compact CR manifold. Its the boundary of a complex manifold U(Siegel upper half plane):

$$U = \{(z, w) : Im\ w > |z|^2,\ (z, w) \in \mathbf{C}^2,\ z = x + iy,\ w = t + is\}$$

$$Z_{\bar{1}} = \frac{\partial}{\partial \bar{z}} - i\frac{z}{2}\frac{\partial}{\partial t},\ z = x + iy$$

More Examples

Some examples of CR manifolds are:

Heisenberg Group(Here Tor$=A^{11}=0$, R=0, R is the Webster curvature tensor)

Hypersurfaces in $\mathbf{C}^{n+1},\ n \geq 1$ with positive-definite Levi form.

Manifolds with negative Webster curvature can be created out of the unit tangent bundle of a compact Riemann surface of genus $g \geq 2$.

CR Manifolds typically arise as boundaries of Complex manifolds.

Odd Dimensional Spheres S^{2n+1}

All odd dimensional spheres are CR manifolds. For $n = 1$, with $u = 0$ the defining function of S^3,

$$\theta = \frac{i}{2}(\bar{\partial}u - \partial u)|_{S^3}, u = |z_1|^2 + |z_2|^2 - 1, \ (z_1, z_2) \in S^3 \subset \mathbf{C}^2$$

where ∂ and $\bar{\partial}$ are the Cauchy-Riemann operators on $\mathbf{C}^2$.

$$Z_1 = \bar{z}_2 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial z_2}$$

$$\theta^1 = z_2 dz_1 - z_1 dz_2$$

With this choice of θ, Webster curvature $R = 2$, $A^{11} = 0$.

This is called the Standard CR structure on S^3.

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3. CR Paneitz operator(set $f_1 = Z_1 f$):

$$P_0 f = \frac{1}{8}\big((\overline{\Box}_b \Box_b + \Box_b \overline{\Box}_b) f + 8 Im(A^{11} f_1)_1\big).$$

It follows that P_0 is a real and symmetric operator.

The CR Paneitz Operator

For $n \geq 2$, the CR Paneitz operator is non-negative, (R. Graham and J. Lee, Duke Math. J., 1988).
For $n = 1$ non-negativity is a condition. Non-negativity is a CR invariant condition (does not depend on the choice of contact form). Changing the contact form(by a conformal change)(Hirachi):

$$\theta_1 = e^{2f}\theta, \quad \theta_1 \wedge d\theta_1 = e^{4f}\theta \wedge d\theta$$

$$P_{0,new} = e^{-4f}P_0$$

So,

$$\int_M P_{0,new} f\bar{f}\theta_1 \wedge d\theta_1 = \int_M P_0 f\bar{f}\theta \wedge d\theta \geq 0$$

Also note if Torsion$=A^{11} = 0$, then $P_0 \geq 0$.

Embeddability Problem

We are concerned with global Embeddability.
Given an abstract compact, strongly pseudo-convex CR manifold, can we embed it into C^N for some N by functions that are CR holomorphic. That is the functions f satisfy,

$$\bar{\partial}_b \, f = 0$$

The embedded manifold in $\mathbf{C}^N$ is smooth and bounds a complex variety. But the variety may **not** be a smooth complex variety(Complex Plateau Problem) a deep result of Harvey-Lawson. (Annals of Math. 1975)

Some Previous results

1. Boutet de Monvel: For $n \geq 2$, Global embedding is always possible.
2. Boutet de Monvel/ Kohn: For $n \geq 1$, Global embeddability is equivalent to any one of the following:

1. $\bar{\partial}_b$ has closed range.
2. $\Box_b$ has closed range. ($\Box_b$ is NOT Elliptic, but sub-elliptic)
3. Zero is not a limit point of the spectrum of $\Box_b$.

Idea of Proofs: Use the invertibility of $\Box_b$ to manufacture lots of point separating CR holomorphic functions.

Kohn: Duke Math. J. 1985.

Grauert-Andreotti-Siu-Rossi Examples

For $n = 1$ a small deformation of an embeddable structure fails to embed. How do the embeddable structures sit in the space of all CR structures?

Theorem: On S^3 for any $\tau \neq 0$, the structure given by

$$Z_{\bar{1}}^{\tau} = Z_{\bar{1}} + \tau Z_1, \ \tau \neq 1$$

fails to embed. (Note we have NOT changed the contact form)

The CR holomorphic functions for the new structure take the same values at anti-podal points of the sphere. Local embedding is still possible.

There are deformations of the Standard structure of S^3 that fail to embed.

Local Embedding

Kuranishi's theorem: Local embedding is true for $n \geq 4$. (Annals of Math.)

Akahori: Local embedding true for $n = 3$

Nirenberg: In general false for $n = 1$

Treves: Local embeddability fails when n=1, generically(Inventionnes, 1983).

Open for $n = 2$ (dimension=5).

Nirenberg gives an excellent exposition of his theorem in his NSF-CBMS lecture notes.

CR Yamabe Constant

Given a CR manifold with fixed background contact form θ, the CR Yamabe constant $Y(M,\theta)$ is defined as: Let

$$\theta_1 = e^{2f}\theta,\ \theta_1 \in [\theta]$$

θ_1 is in the same conformal class as θ. R_1 is the Webster curvature for θ_1:

$$Y(M,\theta) = \inf_{\theta_1} \frac{\int_M R_1\theta_1 \wedge d\theta_1}{\int_M \theta_1 \wedge d\theta_1}.$$

$Y(M,\theta)$ obviously does not depend on the choice of contact form to describe the given CR structure, and is a CR invariant.

Main Theorem (2010)

(a) Let $P_0 \geq 0$ (CR Paneitz non-negative) and let $R > 0$, i.e. Webster curvature is positive. Then,

$$\lambda_1(\Box_b) \geq \min R > 0$$

(b) Let $P_0 \geq 0$ and $Y(M) > 0$, then M globally embeds in some C^N.

The conditions (b) are CR invariant conditions and not functional analysis type of conditions on the range. (a) is a sharp lower bound and achieved for the Standard CR structure on S^3.

Idea of the proof (a) A Bochner identity. (b) Under $Y(M) > 0$ we can always find a conformal contact form with $R > 0$ and then use (a).

Some Comments

1. For the Rossi example $R \to \infty$ as $\tau \to 1$, but the CR Paneitz operator for the structures are all negative as soon as $\tau \neq 0$.

2. Our proof works also when $n > 1$, we get a proof of Boutet's result if $Y(M) > 0$. By Jack Lee and Graham: $P_0 \geq 0$ when $n = 2$.

3. The embeddable CR structures on S^3 form some sort of thin set in the moduli space of all CR structures on S^3, but the precise description is still not understood well. For example is it true the embeddable structures form a connected set?

A Partial Converse (2011)

Consider the sphere S^3 equipped with the standard CR structure. Now deform the structure that is the new CR vector fields are

$$Z_{\bar{1}} + \tau\phi(\cdot)Z_1$$

and $|\tau| < \varepsilon$. Assume the new structure is embeddable as the boundary of a domain in $\mathbf{C}^2$. Then the CR Paneitz operator for the new structure P_0^τ is **non-negative**.

The Proof uses Lempert's Stability theorem for CR functions.

Related Question: Characterize the Kernel of P_0. All CR pluri-harmonic functions(real parts of CR holomorphic functions) are in the kernel, what else is there? Is the kernel of P_0 stable?

Condition (BE) Burns-Epstein

BE: J. Amer Math. Soc. 1990
Bland: Acta Math. 1994.

On S^3 one can characterize embeddable structures. First define

$$H_{p,q} = \{h = \sum_{a,b,c,d} c_{a,b,c,d} z^a \bar{z}^b w^c \bar{w}^d, a + c = p,\ b + d = q,\ \Delta_{R^4} h = 0\}$$

These are the bi-graded spherical harmonics. For $\phi \in C^\infty(S^3)$ assume:

(Condition (BE)): Projection of ϕ onto $H_{p,q}$ vanishes provided

$$p < q + 4, q = 0, 1, 2, \cdots$$

Burns-Epstein-Bland Theorem

Let us consider on S^3 the CR structure,

$$Z_{\bar{1}}^{\tau} = Z_{\bar{1}} + \tau \phi Z_1, \ |\tau| < \epsilon$$

Then the structure embeds if and only if ϕ satisfies (BE).

Here note the CR structure on S^3 is changing i.e. J is changing but the contact form(which is the standard one on S^3) remains fixed as τ moves.

Summary Theorem/Small Deformations

Let us consider the three sphere S^3 and a CR structure J_τ obtained as a small perturbation of the standard CR structure on S^3 and whose CR vector field is given by $Z_{\bar{1}}^{\tau}$ above. Then the following are equivalent.

1. The CR structure embeds in $\mathbb{C}^2$.
2. $\Box_b^{\tau}$, the Kohn Laplacian for the deformed structure has closed range.
3. The deformation function $\phi(\cdot)$ used to define the CR vector field $Z_{\bar{1}}^{\tau}$, satisfies the Burns-Epstein condition (BE).
4. The CR Paneitz operator P_0^{τ} for the deformed structure is non-negative and the Yamabe constant for the deformed structure is positive.

Additional Comments, Condition (BE)

1. First note that the Rossi example fails the (BE) condition as $\phi \equiv 1$.

2. Also our sufficiency theorem holds on any compact CR manifold, not necessarily S^3 and the sufficient condition is **non-perturbative**.

3. Second variation of P_0^τ and condition (BE)...

Applications: CR Positive Mass Theorem

(J.-H. Cheng, A. Malchiodi, P. Yang) Expansion of the Green function of the Sub-elliptic Laplacian.

ADM Mass-Positive Mass theorem

Consider a manifold (M^3, g) which is asymptotically flat. Outside a compact set, the metric satisfies,

$$g_{ij} = \delta_{ij} + O(|x|^{-1}), \quad \partial_x g_{ij} = O(|x|^{-1-\gamma}), \gamma > 0$$

Then the ADM mass is

$$M = lim_{r\to\infty} \int_{S_r} (g_{ii,j} - g_{ij,i})\nu^j$$

Schoen-Yau: If the scalar curvature $S \geq 0$, then $M \geq 0$.

Yamabe problem

Riemannian Yamabe Problem: Can we change the metric by a conformal factor $u^{4/(n-2)}g$ so that the new metric has constant scalar curvature.

Schoen/Aubin: Yes: Central point

$$Y(M) < Y(S^3) = \text{Sobolev constant}$$

CR Yamabe problem: Can we change the contact form $e^f\theta$ so that the new contact form has constant Webster curvature.

Yes: Jerison-Lee, Gamarra, Cheng-Malchiodi-Yang

$$Y(M) < Y(S^3, \theta)$$