Conference in Harmonic Analysis and Partial Differential Equations in Honour of Eric Sawyer

Fields Institute, University of Toronto, 2011

Convexity and Partial Convexity for Solution of Partial Differential Equations

BAOJUN BIAN

This talk is based on the joint works with Pengfei Guan

Department of Mathematics, Tongji University, China

Talk Outline

- *Introduction*
- *Convexity Problem*
- *Partial Convexity Problem*
- *Main Results*
- *Proof for CRP*
- *Applications*

1. Introduction

□ Heat Equation

- Consider the following Cauchy problem for heat equation

$$u_t - \frac{1}{2}u_{xx} = 0, (x,t) \in R \times (0,\infty) \tag{1}$$

$$u(x,0) = \phi(x), x \in R \tag{2}$$

- **Poisson Formula**

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} \phi(x + \sqrt{2t}\eta) d\eta \tag{3}$$

- **Convexity is Preserved(Improved):** $u(x,t)$ is convex in x if $\phi(x)$ is convex.

□ Linear Equation

- Linear equation

$$u_t - a_{ij}(x,t)u_{x_i x_j} = 0, (x,t) \in R^n \times (0, \infty) \tag{4}$$

$$u(x,0) = \phi(x), x \in R^n \tag{5}$$

- **Question: Is Convexity Preserved ?**

- S.Janson and J.Tysk, Preservation of convexity of solutions to parabolic equations, JDE, 2004

- They solve this problem completely and get the sufficient and necessary condition

□ Sufficient and Necessary Condition

- Convexity Inequality:

$$2a^z_{ij}P^z_{ij} + 2a^z_{ij},_z N^z_{ij} + a^z_{ij},_{zz} M^z_{ij} \geq 0 \tag{6}$$

for all unit vector z and for all $M, N, P \in S^n$ such that

$$\begin{pmatrix} M^z & N^z \\ N^{zT} & P^z \end{pmatrix} \geq 0$$

where

$$f^z = (f^z_{ij}), \quad f^z_{ij} = (\delta_{ik} - z_i z_k) f_{kl} (\delta_{lj} - z_l z_j)$$

$$g,_z = \frac{\partial g}{\partial z}$$

- This condition holds true for $n = 1$ or if

$$a_{ij} = constant$$

- Linear equation of parabolic type

- P.L.Lions, M.Musiela, Convexity of solutions of parabolic equations, C.R.Acad. Sci. Paris, 2006

- They discuss the convexity preserving for linear equations and nonlinear equations by use of PDE approach

□ Stochastic Analysis Approach

- Solution can be expressed by expectation

$$u(x,t) = E(\phi(X_t)|X_0 = x) \tag{7}$$

subject to the SDE

$$dX_t = bdt + \sqrt{2}\sigma(X_t)dW_t,\ X_0 = x$$

where

$$a_{ij} = \sigma_{ik}\sigma_{jk}$$

- **Question: Is Convexity Preserved by Expectation?**

□ Brascamp, Lieb, 1976, J. Functional Anal.

• On Extensions of the Brunn-Minkowski and Pr′ ekopa- Leindler Theorems, Including Inequalities for Log-Concave Functions, and with an Application to the Diffusion Equation

□ C. Borell： Mixing PDE approach and stochastic approach.

◇ Log-concavity for fundamental solutions of diffusion equation.

◇ Log-concavity for first eigenfunction

◇ Brunn-Minkowski inequality for first eigenvalue

□ Related Problems

- Extension to nonlinear equation, such as Bellman equation and Geometric equation

- Elliptic equation: Existence of convex solution

- Convexity problem in bounded domain?

2. Convexity Problem

☐ **Convexity problem for Elliptic PDE**

- Consider the following fully nonlinear elliptic partial differential equation

$$F(\nabla^2 u, \nabla u, u, x) = 0, x \in \Omega \tag{8}$$

$$u = ?, x \in \partial\Omega \tag{9}$$

where Ω is in $\mathbb{R}^n$ or in the manifold, where $F = F(r, p, u, x)$ is a given function in $\mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega$. The ellipticity of this equation is assumed.

☐**Question** Existence of convex solution for problem (8)-(9).

☐ **Convexity Preserving Problem** This is the counterpart of elliptic convexity problem for solutions of parabolic equations. Consider the following

$$u_t = F(D^2u, Du, u, x, t), \ \ (x,t) \in \Omega_T = \Omega \times (0,T] \tag{10}$$

$$u(x,0) = u_0(x), \ \ x \in \Omega \tag{11}$$

$$u = ?, \ \ x \in \partial\Omega \times (0,T] \tag{12}$$

☐ **Question** Assume that $u_0(x)$ is convex, is $u(x,t)$ convex for $t > 0$?

☐ Hessian flow equation. Curvature flow equation.

• M.C.Caputo, P.Daskalopoulos and N.Sesum, On the evolution of convex hyper-surfaces by the Q_k flow, arXiv:0904.0492v1

☐ Some equations arising from mathematical finance.

□ **Main Points for Convexity**

□ These convexity problems depend on the

- The form of equation, or function $F(r, p, u, x)$.
- The boundary conditions

Some literatures are in manifolds without boundary(B.Guan, P. Guan). If one considers problem in manifolds with boundary or in domain, the boundary condition is important.

☐ Two classes of equations

• Hessian and Hessian quotient equation

$$F(\nabla^2 u, \nabla u, u, x) = \sigma_k(\lambda_1(\nabla^2 u), \cdots, \lambda_n(\nabla^2 u)) = 0, \tag{13}$$

where $\sigma_k(\lambda_1, \cdots, \lambda_n)$ is the $k-th$ elemental symmetry function. It includes Monge-Ampere equation, Hessian quotient equation, Curvature equation.

• Bellman equation

$$F(\nabla^2 u, \nabla u, u, x) = \sup_{\alpha\in A}\{a^{\alpha}_{ij}(x)D_{ij}u + b^{\alpha}_{i}(x)D_i u + c^{\alpha}(x)u - f^{\alpha}(x)\}$$

and Issacs equation arising from stochastic optimal problem and mathematical finance

$$F(\nabla^2 u, \nabla u, u, x) = \sup_{\alpha\in A}\inf_{\beta\in B}\{a^{\alpha,\beta}_{ij}(x)D_{ij}u + b^{\alpha,\beta}_{i}(x)D_i u + c^{\alpha,\beta}(x)u - f^{\alpha,\beta}(x)\}$$

☐ Typical boundary conditions

• Infinite boundary(Neumann, Dirichlet) condition(B. Guan, H. Jian)

$$u = +\infty, \frac{\partial u}{\partial n} = -\infty, \quad x \in \partial\Omega \tag{14}$$

• State constrained condition from stochastic optimal control problems with stochastic viability. It is related to degenerate equation and viscosity solution.

$$F(D^2u, Du, u, x) = 0, x \in \Omega; F(D^2u, Du, u, x) \leq 0, x \in \partial\Omega \tag{15}$$

Alvarez, Lasry and Lions, Convex viscosity solutions and state constraints, J. Math. Pures Appl. **76**(1997), 265-288

• Cauchy problem with growth condition at infinite

$\Omega = R^n, \Omega = R^n_+$

□ **Related convexity problem: Constant Rank Principle(CRP)**

• If u is a smooth convex solution of equation (8), is the rank of $D^2u(x)$, $r(x)$, a constant in Ω?

• If u is a smooth convex solution of equation (10), is the rank of $D^2u(x,t)$, $r(x,t)$, a constant in x and monotone in t?

□ Constant Rank Principle is a powerful tool in the study of convexity, it is particularly useful in producing convex solutions of differential equations via homotopic deformations and by flow.

□ The great advantage of the microscopic convexity principle is that it can treat geometric equations involving tensors on general manifolds.

Related works

☐ The CRP was initially studied by L. Caffarelli, A. Friedman and I. Singer, B. Wong, S.T. Yau, Stephen S.T. Yau in 1985.

☐ There exist a vast of literatures: CRP and Convexity. Guan,

☐ Ekstrom, E., and J. Tysk, The American Put Is Log-Concave in the Log-Price, J. Math. Anal. Appl., 2006

☐ E.Ekstrom and J.Tysk, Properties of option prices in models with jumps, Mathematical Finance, 2007

☐ Y.Giga, S.Goto, H.Ishii, M.H.Sato, Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains, Indiana Univ. Math. J., 1991, 40:443-470.

3. Partial Convexity Problem

□ **Elliptic Partially Convex Solution** Existence of convex solution of elliptic equation in partial variables

$$F(D^2u, Du, u, x) = 0, \quad x \in \Omega \tag{16}$$

$$u =, \quad x \in \partial\Omega \tag{17}$$

□ **Question** Let $x = (x', x'')$. Is there solution $u(x) = u(x', x'')$ of equation (16) which is convex in x'?

□**Partial Constant Rank Principle** If u is a smooth solution of equation (16) and convex in x', is the rank of $D^2_{x'}u(x)$ a constant in Ω?

□ **Partial Convexity Problem for Parabolic Equation** Convexity preserving for solutions of parabolic equations in partial variables

$$u_t = F(D^2u, Du, u, x, t), \ \ (x,t) \in \Omega_T = \Omega \times (0,T] \tag{18}$$

$$u(x,0) = u_0(x), \ \ x \in \Omega \tag{19}$$

$$u =, \ \ x \in \partial\Omega \times (0,T] \tag{20}$$

□ **Question** Let $x = (x', x^{''})$, assume that $u_0(x', x^{''})$ is convex in x', is solution $u(x,t)$ convex in x'?

□ **Partial Constant Rank Principle** If u is a smooth solution of equation (18) and convex in x', is the rank of $D^2_{x'}u(x,t)$ a constant in x and monotone in t in Ω_T?

□Motivations for Partial Convexity

□ The partial convexity of solutions to fully nonlinear equations in the form has significant geometric implications. In particular, it is important to understand this property for solutions of Monge-Ampere type equations(P.Guan's lecture in Fudan University, Aug. 2010).

□ Curvature flow equation.

□ Models in mathematical finance.

□ Example: Optimal Investment in CEV Model

- Constant Elasticity of Variance model

$$\mathrm{d}S_t = S_t(\mu \mathrm{d}t + \sigma S_t^{\beta} \mathrm{d}W_t) \tag{21}$$

- Value function $V(x, s, t)$

$$V(x, s, t) = \sup_{\pi. \in A} E(e^{r(t-T)} U(X_T) | X_t = x, S_t = s) \tag{22}$$

- Bellman Equation

$$\frac{\partial V}{\partial t} + \sup_{\pi \geq 0} \{ \frac{1}{2} \sigma^2 \pi^2 x^2 s^{2\beta} \frac{\partial^2 V}{\partial x^2} + \sigma^2 \pi x s^{2\beta+1} \frac{\partial^2 V}{\partial x \partial s} + (r + (\mu - r)\pi) x \frac{\partial V}{\partial x} \}$$

$$+ \frac{1}{2} \sigma^2 s^{2\beta+2} \frac{\partial^2 V}{\partial s^2} + \mu s \frac{\partial V}{\partial s} - rV = 0, x, s > 0, t < T$$

- Terminal condition

$$V(x, s, T) = U(x), x > 0 \tag{23}$$

□ **Optimal Investment Strategy**

- If value function $V(x, s, t)$ is smooth and strictly concave in x, then we can construct optimal control

- Suppose $U(x)$ is (strictly) concave, discuss the concavity for value function $V(x, s, t)$ in x

4. Main Results

□ **Constant Rank Theorem**

◇ [Bian, Guan, 2009]Constant Rank Theorem for elliptic and parabolic equations.

◇ [Bian, Guan, 2010]Improvement of 2009's results and Partial Constant Rank Theorem for elliptic and parabolic equations.

□ **Convexity Preserving for Parabolic Equation** Convexity preserving for solutions of parabolic equations

◇ [Bian, Guan, 2008]Convexity preserving for parabolic equation and Bellman equation.

◇ []Partial convexity preserving for parabolic equation.

□ Main Theorem

• **Theorem 4.1**(CRP, Bian, Guan, 2009) Suppose $F = F(r, p, u, x) \in C^{2,1}$ and F satisfies condition

$$F(A^{-1}, p, u, x) \quad \text{is locally convex in } (A, u, x) \text{ for each } p \text{ fixed.}$$

If $u \in C^{2,1}(\Omega)$ is a convex solution of

$$F(D^2u, Du, u, x) = 0, \quad x \in \Omega \tag{24}$$

then the rank of Hessian $(\nabla^2 u(x))$ is constant in Ω.

□ Improvement on structural condition for CRP

- Denote $\mathcal{S}^n_+$ the space of positive definite real symmetric $n \times n$ matrices, for each fixed $p \in \mathbb{R}^n$, define the zero sub-level set

$$\Gamma_F = \{(A, u, x) \in \mathcal{S}^n_+ \times \mathbb{R} \times \Omega | F(A^{-1}, p, u, x) \leq 0\}. \tag{25}$$

- **Theorem 4.2**(CRP, Bian, Guan, 2010) Let $F = F(r, p, u, x) \in C^{2,1}(\mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega)$ and let $u \in C^{2,1}(\Omega)$ be a convex solution of (13). Suppose F satisfies condition and at $(\nabla^2 u(x), \nabla u(x), u(x), x)$ for each $x \in \Omega$. If for each $x \in \Omega$ and $p = \nabla u(x)$,

$$\Gamma_F \text{ is locally convex at } (A, u(x), x), \tag{26}$$

then the rank of the hessian $(\nabla^2 u(x))$ is constant in Ω.

- **Remark** We can prove this theorem under weaker condition

□ **Example: Poisson Equation**

- Consider

$$\Delta u - f(x) = 0, \quad x \in \Omega \tag{27}$$

◇ The condition in Theorem 4.2 is that $\frac{1}{f(x)}$ is convex. This is the condition in L. Caffarelli, A. Friedman, 1985.

□ **Parabolic Constant Rank Principle**(Bian, Guan, 2009)

• **Theorem 4.3**(PCRP, 2009) Let $u \in C^{2,1}(\Omega \times [0,T))$ be a convex solution of the equation

$$\frac{\partial u}{\partial t} = F(\nabla^2 u, \nabla u, u, x, t), \tag{28}$$

and assume

$$F(A^{-1}, p, u, x, t) \quad \text{is locally convex in } (A, u, x) \text{ for each } (p, t) \text{ fixed.} \tag{29}$$

Suppose For each $T > t > 0$, let $l(t)$ be the minimal rank of $(\nabla^2 u(x,t))$ in Ω. Then, the rank of $(\nabla^2 u(x,t))$ is constant for each $T > t > 0$ and $l(s) \le l(t)$ for all $s \le t < T$.

□ **Example: Linear Parabolic Equations**

- Consider

$$-u_t + a_{ij}(x)D_{ij}u = 0, \quad x \in \Omega \tag{30}$$

◇ The condition in Theorem 4.2 is the condition in S.Janson and J.Tysk, 2004.

□ Notations for Partial Convexity

- Let us write $x = (x^{'}, x^{''}) \in \Omega$ and $p = (p^{'}, p^{''}) \in R^N$ with $p^{'} \in R^{N^{'}}$, $p^{''} \in R^{N^{''}}$ and split a matrix $A \in \mathcal{S}^N$ into $\begin{pmatrix} a & b \\ b^T & c \end{pmatrix}$ with $a \in \mathcal{S}^{N^{'}}$, $b \in \mathbb{R}^{N^{'} \times N^{''}}$ and $c \in \mathcal{S}^{N^{''}}$. Let

$$\mathcal{S}^{N,\oplus} = \{A \in \mathcal{S}^N | A = \begin{pmatrix} a & b \\ b^T & c \end{pmatrix}, a \in \mathcal{S}_+^{N^{'}}\}$$

- Define for $(A, p, u, x) \in \mathcal{S}^{N,\oplus} \times \mathbb{R}^N \times \mathbb{R} \times \Omega$

$$\tilde{F}(A, p^{''}, u, x^{'}) = F(\begin{pmatrix} a^{-1} & a^{-1}b \\ (a^{-1}b)^T & c + b^T a^{-1} b \end{pmatrix}, p, u, x)$$

For each fixed $x^{''}$ and $p^{'} \in \mathbb{R}^{N^{'}}$, define the zero sub-level set

$$\Gamma_F = \{(A, p^{''}, u, x^{'}) \in \mathcal{S}^{N,\oplus} \times \mathbb{R}^{N^{''}} \times \mathbb{R} \times \mathbb{R}^{N^{'}} | \tilde{F}(A, p^{''}, u, x^{'}) \leq 0\}. \quad (31)$$

Introduction
Convexity Problem
Partial Convexity...
Main Results
Proof for CRP
Applications
Home Page
Title Page
Page 29 of 44
Go Back
Full Screen
Close
Quit

□ **Elliptic Partial Constant Rank Principle**(Bian, Guan, 2010)

• **Theorem 4.4**(PCRP, 2010) Let $F = F(r, p, u, x) \in C^{2,1}(\mathcal{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega)$ and let $u \in C^{2,1}(\Omega)$ be a partial convex solution of (13). Suppose F satisfies condition

$$\Gamma_F \text{ is locally convex at } (A, p^{''}, u, x^{'}),$$

then the rank of the hessian $(\nabla^2_{x'} u(x))_{N' \times N'}$ is constant in $x = (x', x") \in \Omega$. If l is the rank of $(\nabla^2_{x'} u(x))_{N' \times N'}$, then $\forall x_0 \in \Omega$, there exist a neighborhood U of x_0 and $(N^{'} - l)$ fixed directions $V_1, \dots, V_{N'-l} \in \mathbb{R}^{N^{'}}$ such that $(\nabla^2_{x'} u(x))_{N' \times N'} V_j = 0$ for all $1 \leq j \leq N^{'} - l$ and $x \in U$.

$\square$ **Parabolic Partial Constant Rank Principle**(Bian, Guan, 2010)

- **Theorem 4.5**(PPCRP, 2010) Let $u \in C^{2,1}(\Omega \times [0,T))$ be a smooth and partially convex(in x') solution of the equation

$$\frac{\partial u}{\partial t} = F(\nabla^2 u, \nabla u, u, x, t), \tag{32}$$

and assume

$$\tilde{F}(A^{-1}, p'', u, x', t) \quad \text{is locally convex in } (A, u, x) \text{ for each } (p', x'', t) \text{ fixed.} \tag{33}$$

Suppose For each $T > t > 0$, let $l(t)$ be the minimal rank of $(\nabla^2_{x'} u(x,t))$ in Ω. Then, the rank of $(\nabla^2_{x'} u(x,t))$ is constant for each $T > t > 0$ and $l(s) \leq l(t)$ for all $s \leq t < T$.

□ Example: Partial Convexity Preserving for Linear Parabolic Equations

- Consider

$$u_t = \sum_{i,j=1}^{N} a_{ij}(x) D_{ij} u, \quad x \in \Omega$$

- The condition in Theorem 4.5 is

$$a_{ij,x_k} = 0,\ 1 \le k \le N',\ N' + 1 \le i, j \le N$$

$$a_{ij,x_k x_l} = 0,\ 1 \le i, k, l \le N',\ N' + 1 \le j \le N$$

and the convexity inequality

$$\sum_{i,j=1}^{N}[a^z_{ij},_{zz} M^z_{ij} + 2a^z_{ij},_z N^z_{ij} + 2a^z_{ij} P^z_{ij}] \geq 0$$

for all unit vector $z \in R^N$ with $z_k = 0, N' + 1 \leq k \leq N$ and for all M, N, P such that

$$\begin{pmatrix} M^z & N^z \\ (N^z)^T & P^z \end{pmatrix} \geq 0$$

5. Proof for CRP

Key points in proof

☐ To consider function $\sigma_{l+1}(\nabla^2_{x'}u)$ here l the minimal rank of $\nabla^2_{x'}u$. $\nabla^2_{x'}u$ is of constant rank is equivalent to $\sigma_{l+1}(\nabla^2_{x'}u) \equiv 0$.

☐ When deal with general equation, linear terms of third order derivatives of u (i.e., the gradient of the symmetric tensor $\nabla^2 u$) will appear. How to control them is the major challenge.

☐ We introduce a new auxiliary function which is composed as a quotient of elementary symmetric functions $\frac{\sigma_{l+2}(\nabla^2_{x'}u)}{\sigma_{l+1}(\nabla^2_{x'}u)}$ near points where $\nabla^2_{x'}u(x)$ is of minimal rank l. We show $\frac{\sigma_{l+2}(\nabla^2_{x'}u)}{\sigma_{l+1}(\nabla^2_{x'}u)}$ has optimal $C^{1+1,}$ regularity.

Sketch of proof

□ Step 1: Auxiliary function

Define for $W = (\nabla^2_{x'} u) \in \mathcal{S}^{N'}$

$$\phi = \sigma_{l+1}(W) + q(W) \tag{34}$$

here

$$q(W) = \begin{cases} \frac{\sigma_{l+2}(W)}{\sigma_{l+1}(W)}, & \text{if } \ \sigma_{l+1}(W) > 0 \\ 0, & \text{if } \ \sigma_{l+1}(W) = 0 \end{cases} \tag{35}$$

• **Lemma 4.1** Let $u \in C^{3+1,1+1}(\Omega)$ be a convex function in x' and $W(x) = (\nabla^2_{x'} u), x \in \Omega$. Let $l = \min_{x \in \Omega} \mathrm{rank}(W(x))$, then the functions $q(x) = q(W(x))$ and $\phi(x)$ is in $C^{1+1,1+0}(\Omega)$.

□ Step 2: Differential inequality

• **Lemma 4.2** Suppose that the function F satisfies conditions, let $u \in C^{3+1,1+1}(\Omega)$ is a partially convex solution. If $(\nabla^2_{x'}u)$ attains minimum rank l at certain point $x_0 \in \Omega$, then there exist a neighborhood $\mathcal{O}$ of x_0 and a positive constant C independent of ϕ), such that

$$\sum_{\alpha,\beta=1}^{N} F^{\alpha\beta}\phi_{\alpha\beta}(x', x'') \leq C(\phi + |\nabla\phi|), \quad \forall x = (x', x'') \in \mathcal{O}. \tag{36}$$

In turn, $(\nabla^2_{x'}u)$ is of constant rank in $\mathcal{O}$.

• We prove inequality (36) from partial differential equation, structure condition and estimates.

□ Proof of Lemma 4.2:

- Let $W(x) = (\nabla^2 u(x))_{N' \times N'}$ and $l = \min_{x\in\Omega} \mathrm{rank} W(x)$. Since $l = N^{'}$ is of full rank, only $l \le N^{'} - 1$ is of interest.

- For each $x_0 \in \Omega$ where W is of minimal rank l. Pick an open neighborhood $\mathcal{O}$ of x_0, for any $x \in \mathcal{O}$, let $\lambda_1(x) \le \lambda_2(x) ... \le \lambda_{N'}(x)$ be the eigenvalues of W at x. There is a positive constant $C > 0$ depending only on $\|u\|_{C^{3,1}}$, $W(z_0)$ and $\mathcal{O}$, such that $\lambda_{N'}(x) \ge \lambda_{N'-1}(x) ... \ge \lambda_{N'-l+1}(x) \ge C$ for all $x \in \mathcal{O}$. Let $G = \{N^{'} - l + 1, N^{'} - l + 2, ..., N^{'}\}$ and $B = \{1, ..., N^{'} - l\}$ be the "good" and "bad" sets of indices respectively.

☐ Differentiate equation in x_i and then x_j

$$F(\nabla^2 u, \nabla u, u, x) = 0$$

we obtain by the definition of ϕ

$$\sum_{\alpha,\beta=1}^{N} F^{\alpha\beta}\phi_{\alpha\beta}$$

$$= O(\phi + \sum_{i,j\in B} |\nabla u_{ij}|) - \sum_{\alpha,\beta=1}^{N} F^{\alpha\beta}[\frac{\sum_{i\in B} V_{i\alpha}V_{i\beta}}{\sigma_1^3(B)} + \frac{\sum_{i,j\in B, i\neq j} u_{ij\alpha}u_{ji\beta}}{\sigma_1(B)}]$$

$$-\sum_{i\in B}[\sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}]J_i$$

- where

$$V_{i\alpha} = u_{ii\alpha}\sigma_1(B) - u_{ii}(\sum_{j\in B} u_{jj\alpha})$$

$\square$

$$J_i = \sum_{\alpha,\beta,\gamma,\eta\notin B} F^{\alpha\beta,\gamma\eta} u_{i\alpha\beta} u_{i\gamma\eta} + 2\sum_{\alpha,\beta\notin B} F^{\alpha\beta} \sum_{j\in G} \frac{1}{\lambda_j} u_{ij\alpha} u_{ij\beta}$$

$$+2\sum_{\alpha,\beta\notin B} \Big(\sum_{k=N'+1}^{N} F^{\alpha\beta,p_k} u_{\alpha\beta i} u_{ik} + F^{\alpha\beta,u} u_{i\alpha\beta} u_i + F^{\alpha\beta,x_i} u_{i\alpha\beta} \Big)$$

$$+ \sum_{k,l=N'+1}^{N} F^{p_k,p_l} u_{ik} u_{il} + 2\sum_{k=N'+1}^{N} \left(F^{p_k,u} u_{ik} u_i + F^{p_k,x_i} u_{ik}\right)$$

$$+F^{u,u} u_i^2 + 2F^{u,x_i} u_i + F^{x_i,x_i},$$

- We have from structural conditions

$$J_i \geq -C(\phi + \sum_{i,j\in B} |\nabla u_{ij}|)$$

$$C \geq \sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)} \geq 0$$

- Hence the inequality is reduced to

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} \le C(\phi + \sum_{i,j\in B} |\nabla u_{ij}|)$$

$$-\sum_{\alpha,\beta} F^{\alpha\beta}(\frac{\sum_{i\in B} V_{i\alpha}V_{i\beta}}{\sigma_1^3(B)} + \frac{\sum_{i,j\in B, i\neq j} u_{ij\alpha}u_{ji\beta}}{\sigma_1(B)})$$

- Estimate for term $\sum_{i,j\in B} |\nabla u_{ij}|$

$$\sum_{i,j\in B} |\nabla u_{ij}| \le \delta(\frac{\sum_{i\in B} V_{i\alpha}V_{i\beta}}{\sigma_1^3(B)} + \frac{\sum_{i,j\in B, i\neq j} u_{ij\alpha}u_{ji\beta}}{\sigma_1(B)}) + \frac{C}{\delta}(\phi + |\nabla\phi|)$$

6. Applications

□ Convexity Preserving for Parabolic Bellman Equations

- Bian,Guan, 2008

- Integro-differential equation

$$u_t = F(\nabla^2 u, \nabla u, u, x, t) + Bu, \quad (x,t) \in R^n \times [0,T], \tag{37}$$

where Bu is a integro-differential operator

$$Bu = \int_0^1 (u(x + \psi(x,t,\eta), t) - u(x,t)$$
$$-\psi(x,t,\eta) \cdot \nabla u(x,t))d\eta$$

- These equations are from finance problem in jump diffusion model

□ We obtain Macro nature convexity preserving by using constant rank theorem

□ **Theorem 6.1** Assume

$$F(A, p, u, x, t) \quad \text{is locally convex in } (A, u, x) \text{ for each } (p, t) \text{ fixed.} \tag{38}$$

If $u \in C^{4,2}$ is a solution of equation, then $u(x,t)$ is convex provided the initial date $u(x,0)$ is convex.

- Our result works for the Bellman equations

$$F(\nabla^2 u, \nabla u, u, x, t) = \sup_{\alpha \in A}\{a_{ij}^{\alpha}(x) D_{ij} u + b_i^{\alpha}(x) D_i u + c^{\alpha}(x) u - f^{\alpha}(x)\}$$

□ **Partial convexity: CEV Model**

- Terminal condition

$$V(x, s, T) = U(x), x > 0 \tag{39}$$

- We prove that $V(x, s, t)$ is smooth and strictly concave in x if $U(x)$ is concave and Inada, then get optimal investment strategy.

□ Related work on Black-Schoeles model: By use of Full convexity preserving

- Bian,Miao,Zheng, Smooth Value Functions for a Class of Nonsmooth Utility Maximization Problems, SIAM J. Financial Mathematics, to appear

Thank you!