Unimodularity of roots of self-inversive polynomials

Matilde Lalín (Université de Montréal)

joint with Mathew Rogers (Université de Montréal),
Chris Smyth (University of Edinburgh)

mlalin@dms.umontreal.ca
http://www.dms.umontreal.ca/~mlalin

3rd Montreal-Toronto Workshop in Number Theory
October 7-9, 2011

Université de Montréal

When do polynomials have their roots on the unit circle?

Theorem (Cohn, 1922)

A polynomial $P(z) \in \mathbb{C}[z]$ has all its roots on the unit circle $\{|z| = 1\}$ iff

- *$P(z)$ is self-inversive and*
- *$P'(z)$ has all its roots in or on the unit circle $\{|z| \leq 1\}$.*

$P(z) \in \mathbb{C}[z]$, $\deg P(z) = d$, self-inversive if

$$P(z) = \varepsilon z^d \overline{P}(1/z) \qquad \text{for some constant } \varepsilon.$$

Then

$$P(z) = \sum_{j=0}^{d} A_j z^j, \qquad A_j = \varepsilon \overline{A_{d-j}}$$

Ramanujan's formula for $\zeta(2k-1)$

$$\frac{(2\pi)^{2k-1}}{2(2k)!}\sum_{j=0}^{k}(-1)^j B_{2j}B_{2k-2j}\binom{2k}{2j}z^{2j}+\frac{\zeta(2k-1)}{2}\left((-1)^k z+z^{2k-1}\right)$$
$$=-\sum_{n=1}^{\infty}\frac{1}{n^{2k-1}}\frac{z^{2k-1}}{e^{2\pi n/z}-1}+(-1)^{k+1}\sum_{n=1}^{\infty}\frac{1}{n^{2k-1}}\frac{z}{e^{2\pi nz}-1}.$$

for $z \notin i\mathbb{Q}$.
Analog of Euler's formula for $\zeta(2k)$:

$$\zeta(2k)=\frac{(-1)^{k+1}B_{2k}(2\pi)^{2k}}{2(2k)!}.$$

Ramanujan's polynomials

$$\frac{(2\pi)^{2k-1}}{2(2k)!}\sum_{j=0}^{k}(-1)^j B_{2j}B_{2k-2j}\binom{2k}{2j}z^{2j}+\frac{\zeta(2k-1)}{2}\left((-1)^k z+z^{2k-1}\right)$$

$$=-\sum_{n=1}^{\infty}\frac{1}{n^{2k-1}}\frac{z^{2k-1}}{e^{2\pi n/z}-1}+(-1)^{k+1}\sum_{n=1}^{\infty}\frac{1}{n^{2k-1}}\frac{z}{e^{2\pi nz}-1}.$$

$$R_{2k-1}(z):=\sum_{j=0}^{k}\frac{B_{2j}B_{2k-2j}}{(2j)!(2k-2j)!}z^{2j}.$$

Theorem (M.R. Murty, Smyth, Wang (2010))

For $k \geq 2$, $R_{2k-1}(z)$

- *Has exactly four distinct real roots. The largest one tends to 2 as k goes to infinity.*
- *All nonreal zeros lie on the unit circle*
- *The only roots of unity that are zeros are*
  - *Both ± 1 if k is odd.*
  - *All four of $\pm e^{\pm 2\pi i/3}$ if $k \equiv 1 (\mathrm{mod} 3)$,*

and no others.

Their motivation comes from Gun, M. R. Murty, and Rath (2010)...

Theorem (Grosswald (1970))

Let

$$\sigma_k(n) = \sum_{d|n} d^k$$

and set

$$F_k(z) = \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^k} e^{2\pi i n z}$$

for $\mathrm{Im}(z) > 0$. *Then*

$$F_{2k-1}(z) - z^{2k-2} F_{2k-1}\left(-\frac{1}{z}\right) = \frac{1}{2}\zeta(2k-1)(z^{2k-2}-1) + \frac{(2\pi i)^{2k-1}}{2z} R_{2k-1}(z).$$

F_{2k-1} is an Eichler integral

$$F_k(z) = \frac{(2\pi i)^k}{(k-1)!} \int_{i\infty}^{z} \left(E_{k+1}(\tau) + \frac{B_{k+1}}{2(k+1)}\right)(\tau - z)^{k-1} d\tau.$$

Université de Montréal

$$\frac{(2\pi)^{2k-1}}{2(2k)!}\sum_{j=0}^{k}(-1)^j B_{2j}B_{2k-2j}\binom{2k}{2j}z^{2j}+\frac{\zeta(2k-1)}{2}\left((-1)^k z+z^{2k-1}\right)$$
$$=-\sum_{n=1}^{\infty}\frac{1}{n^{2k-1}}\frac{z^{2k-1}}{e^{2\pi n/z}-1}+(-1)^{k+1}\sum_{n=1}^{\infty}\frac{1}{n^{2k-1}}\frac{z}{e^{2\pi nz}-1}.$$

Case $k=2$, left-hand side yields

$$z^4+5z^2+1-\frac{90\zeta(3)}{\pi^3}\left(z^3+z\right)=0$$

has all of its zeros on the unit circle.
Is this true in general?

Theorem

$$P_k(z) := \frac{(2\pi)^{2k-1}}{(2k)!} \sum_{j=0}^{k} (-1)^j B_{2j} B_{2k-2j} \binom{2k}{2j} z^{2j}$$
$$+ \zeta(2k-1)\left(z^{2k-1} + (-1)^k z\right)$$
$$Q_k(z) := \left(2^{2k}+1\right) P_k(z) - 2^{2k} P_k(z/2) - P_k(2z),$$
$$W_k(z) := \left(2^{2k-1}+2\right) P_k(z) - 2^{2k} P_k(z/2) - P_k(2z),$$
$$S_k(z) := \sum_{j=0}^{k} E_{2j} E_{2k-2j} \binom{2k}{2j} z^{2j}.$$

have all its nontrivial roots on the unit circle.

Université de Montréal

$$Q_k(z) := \frac{(2\pi)^{2k-1}}{(2k)!} \sum_{j=0}^{k} (-1)^j B_{2j} B_{2k-2j} (2^{2j}-1)(2^{2k-2j}-1) \binom{2k}{2j} z^{2j}$$
$$+ \zeta(2k-1)(2^{2k-1}-1)((-1)^k z + z^{2k-1})$$
$$W_k(z) := \frac{(2\pi)^{2k-1} 2^{2k}}{(2k)!} \sum_{j=0}^{k} (-1)^j B_{2j} B_{2k-2j} (1-2^{1-2j})(1-2^{1-2k+2j}) \binom{2k}{2j} z^{2j}$$

The polynomial $S_k(z)$ appears in another identity of Ramanujan:

$$\frac{(\pi/2)^{2k+1}}{2(2k)!} S_k(iz) = z^{2k} \sum_{n=1}^{\infty} \frac{\chi_{-4}(n) \sec(\pi n/2z)}{n^{2k+1}}$$

$$+(-1)^k \sum_{n=1}^{\infty} \frac{\chi_{-4}(n) \sec(\pi nz/2)}{n^{2k+1}}.$$

Proof for $S_k(z)$

Uses

Theorem (Lakatos, Losonczi, Schinzel, 2002-09)

If

$$f(z) = \sum_{j=0}^{k} A_j z^j$$

is reciprocal and there is a $c \in \mathbb{C}$ *such that*

$$|A_k| \geq \sum_{j=0}^{k} |cA_j - A_k|,$$

then $f(z)$ *has all of its zeros on the unit circle. If inequality is strict, zeros are simple.*

Proof for $S_k(z)$

The proof uses the Euler polynomial binomial convolution

$$\sum_{j=0}^{n} \binom{n}{j} E_j(v) E_{n-j}(w) = 2(1-w-v)E_n(v+w) + 2E_{n+1}(v+w).$$

and inequalities

$$\frac{4^{k+1}(2k)!}{\pi^{2k+1}} > |E_{2k}| > \frac{4^{k+1}(2k)!}{\pi^{2k+1}(1+3^{-1-2k})}.$$

$$c = \frac{\pi}{4(1+3^{-1-2k})}$$

Université de Montréal

Understanding the proof

$$f(z) = \sum_{j=0}^{k} A_j z^{2j}, \qquad |A_k| \geq \sum_{j=0}^{k} |cA_j - A_k|$$

implies

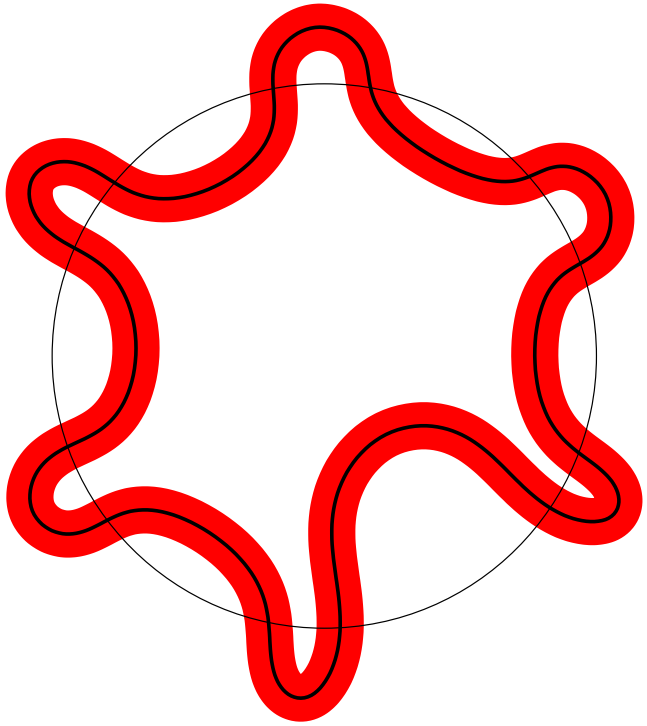
$$\left| \frac{c}{A_k} f(z) - \frac{z^{2(k+1)} - 1}{z^2 - 1} \right| < 1$$

$z = e^{i\theta}$,

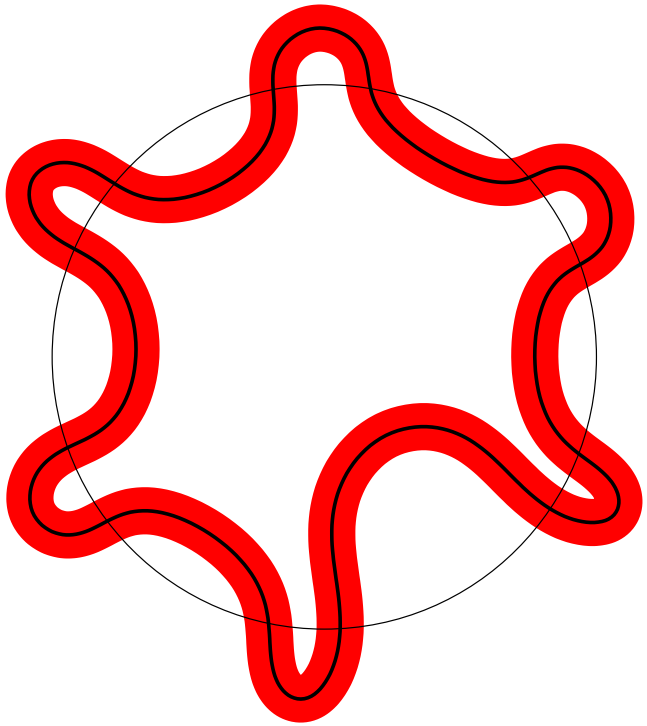
$$\frac{\sin\left((k+1)\theta\right)}{\sin\left(\theta\right)}$$

has $2k$ extremes of absolute value ≥ 1 and alternating sign in the unit circle.

$$\frac{(2j+1)\pi}{2(k+1)}, \qquad j = -k, \ldots, 0, \ldots, k-1$$



We can replace $\frac{z^{2(k+1)}-1}{z^2-1}$...



Proof of $W_k(z)$

$$W_k(z) := \frac{(2\pi)^{2k-1}2^{2k}}{(2k)!}\sum_{j=0}^{k}(-1)^j B_{2j}B_{2k-2j}(1-2^{1-2j})(1-2^{1-2k+2j})\binom{2k}{2j}z^{2j}$$

we will compare it with

$$w_k(z) = (z^k + z^{-k}) + \frac{\pi^2}{6}(z^{k-2} + z^{2-k}) + \frac{2}{(1-2^{1-2k})}\frac{z^{k-3} - z^{3-k}}{z - z^{-1}}.$$

Proof of $W_k(z)$

$$W_k(iz) = \sum_{j=0}^{k} A_j z^{2j}$$

$$w_k(z) = (z^k + z^{-k}) + \frac{\pi^2}{6}(z^{k-2} + z^{2-k}) + \frac{2}{(1-2^{1-2k})}\frac{z^{k-3} - z^{3-k}}{z - z^{-1}}.$$

The A_j have all sign $(-1)^k$.

$$\frac{\pi^2}{6} = \lim_{k\to\infty} \frac{A_1}{A_0} \qquad \frac{A_j}{A_0} < \frac{2}{(1-2^{1-2k})}$$

Combining with the Bernoulli polynomial binomial convolution,

$$\left| \frac{z^{-k} W_k(iz)}{A_0} - w_k(z) \right| < 0.3$$

Université de Montréal

Proof of $W_k(z)$

$z = e^{i\theta}$

$$w_k(\theta) = \cos((k-1)\theta)\cos(\theta)\left(\frac{\pi^2}{3} + 2 - \frac{4}{(1-2^{1-2k})}\right)$$

$$+\sin((k-1)\theta)\left(\left(\frac{\pi^2}{3} - 2 - \frac{4}{(1-2^{1-2k})}\right)\sin\theta + \frac{2}{(1-2^{1-2k})}\frac{1}{\sin\theta}\right)$$

If we evaluate at

$$\theta = \begin{cases} \frac{j\pi}{k-1} & \frac{j}{k-1} \notin (\alpha, 1-\alpha) \cup (\alpha-1, -\alpha) \\ \frac{(2j-1)\pi}{2(k-1)} & \frac{(2j-1)\pi}{2(k-1)} \in (\alpha, 1-\alpha) \cup (\alpha-1, -\alpha) \end{cases}$$

with certain α.

We obtain $2k$ points of absolute values > 0.3 and alternating sign.

Université de Montréal

This method works as well with $Q_k(z)$...

... but not with $P_k(z)$.

A "generalization" of Cohn's result

Proposition

$h(z)$ monic, $\deg h = n$, all its zeros in $|z| \leq 1$.
$d > n$ and any λ such that $|\lambda| = 1$, then

$$P^{(\lambda)}(z) = z^{d-n} h(z) + \lambda h^*(z)$$

has all its zeros on the unit circle.
Conversely, $P(z)$ monic self-inversive with roots on the unit circle, we can always find $h(z)$ ($h(z) = \frac{P'(z)}{d}$).

Here $h^*(z) = z^n h(1/z)$.

Université de Montréal

Corollary

If

$$f(z) = \sum_{j=0}^{k} A_j z^j$$

is reciprocal and

$$2\,|A_k| \geq \sum_{j=0}^{k-1} |A_j - A_{j+1}|,$$

then $f(z)$ has all of its zeros on the unit circle.

Proof by looking at $(z-1)f(z)$.
This implies Schinzel's result.

Proof of $P_k(z)$

Lemma

Let h be as before, with $|h(z)| \geq c > 0$ for $|z| = 1$. Let $e(z)$ be another (not necessarily monic) polynomial $\deg e = m$ such that $|e(z)| \leq c$ for $|z| = 1$. Then for $k > m, n$ the self-inversive polynomial

$$z^{2k-n}h(z) + z^k e(z) + \lambda(h^*(z) + z^{k-m}e^*(z))$$

has all its zeros on the unit circle.

We can write,

$$-\frac{\pi}{\zeta(2k)}(z^2+1)P_k(z) = H_r(z) + E_r(z)$$

$$H_4(z) = z^{2k-6}h_4(z) + (-1)^k h_4^*(z),$$
$$E_4(z) = z^{k+2}e_4(z) + (-1)^k z e_4^*(z),$$
$$|e_4(z)| \leq 0.019, \quad k \geq 11, \quad |z| = 1$$

$$h_4(z) = z^8 - 3.141592654z^7 + 4.289868136z^6 - 3.141592654z^5 + 1.125221668z^4 - 0.129960342z^2 + 0.026531411$$

Zeros satisfy $|z| < 1$ and

$$0.02146485500 \leq |h_4(z)|, \text{ in } \{|z| = 1\}$$

Lemma to h_4 and e_4 with $c = 0.019$.

Further questions

- Explore consequences of the new criterion.
- Can we say anything about the arithmetic nature of $\zeta(2k-1)$?

Thank you!
Merci!