Transcendence of special values of L-series

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$$\zeta(2) = \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{\pi^4}{90}$$

$$\zeta(6) = \frac{\pi^6}{945}$$

$$\zeta(8) = \frac{\pi^8}{9450}$$

$$\zeta(10) = \frac{\pi^{10}}{93555}.$$

The general problem

- Given a “zeta function” L(s), what are the special values L(k), when k is an integer ?
- Are these special values transcendental ?
- Do they have a “nice” factorization into an algebraic (or arithmetic) part and a transcendental part ?
- Is it possible to describe the Galois action on the algebraic part?

The Riemann ζ-function

B. Riemann (1826-1866)

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \qquad \sigma = \Re(s) > 1.$$

- In his celebrated paper of 1859, Riemann derived an analytic continuation and functional equation for (s-1)ζ(s) for all complex values of s and indicated its importance in the study of the distribution of prime numbers.

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s),$$

Special values of the Riemann zeta function

- Euler's theorem

$$\zeta(2n) = (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$$

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi$$

Here, π is the transcendental part and the Bernoulli number is the arithmetic part.

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}$$

The values of ζ(2k+1) are still a mystery. Apery (1978) proved that ζ(3) is irrational. Rivoal (2000) showed infinitely many of them are irrational.

Dirichlet L-functions

For any complex-valued character χ mod q, define $L(s,\chi)$ as follows.

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Dirichlet introduced these L-series to show that there are infinitely many primes in a given arithmetic progression.

- In 1880, Hurwitz derived the analytic continuation and functional equation for $L(s,\chi)$.

P.G.L. Dirichlet (1805-1859)

A. Hurwitz (1859-1919)

The Hurwitz zeta function

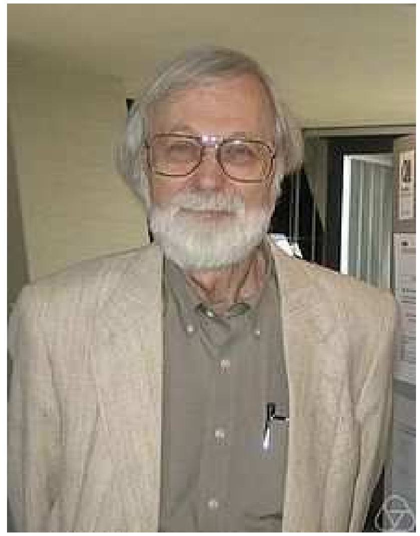
$$\zeta(s,q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^s}.$$

- Hurwitz derived the analytic continuation and functional equation for $\varsigma(s,q)$ using theta series.
- One can write $L(s,\chi)$ as a linear combination of the Hurwitz zeta functions and thereby derive its analytic continuation and functional equation.

Special values of Dirichlet L-series

- Recall that a character χ is called even if $\chi(-1)=1$ and odd if $\chi(-1)=-1$.
- If k and χ have the same parity, then $L(k,\chi)$ is an algebraic multiple of π^k for $k\geq 2$.
- If k and χ have opposite parity, then the nature of $L(k,\chi)$ is still a mystery.
- Nishimoto (2011) has shown that if χ is even then infinitely many of the values $L(2k+1, \chi)$ are irrational. If χ is odd, then infinitely many $L(2k,\chi)$ are irrational.

The Chowla-Milnor conjecture



S. Chowla (1907-1995) John Milnor (1931-)

- Fix $k \geq 2$ and $q \geq 2$. The values $\varsigma(k, a/q)$ are linearly independent over the rationals.

Consequences of the Chowla-Milnor conjecture

- Theorem (S. Gun, R. Murty and P. Rath)
  (1) The Chowla-Milnor conjecture holds for $q=4$ if and only if $\varsigma(2k+1)/\pi^{2k+1}$ is irrational for every $k \geq 1$.
- (2) The Chowla-Milnor conjecture implies that $(\varsigma(2k+1)/\pi^{2k+1})^2$ is irrational for all $k \geq 1$.
- (3) The Chowla-Milnor conjecture holds for either $q=3$ or $q=4$.

The strong Chowla-Milnor conjecture

- The numbers 1, ς(k, a/q) with 1≤a<q and (a,q)=1, are linearly independent over the rationals.
- Theorem (S. Gun, R. Murty and P. Rath) ς(k) is irrational if and only if the strong Chowla-Milnor conjecture holds for either q=3 or q=4.

Multiple Γ-functions

- The multiple Γ-function is defined recursively as follows.
- $\Gamma_{m+1}(z+1) = \Gamma_{m+1}(z)/\Gamma_m(z)$. $\Gamma_0(z)=1/z$.
- Thus $\Gamma_1(z)$ is the classical Γ-function.
- One can show that $1/\Gamma_m(z)$ extends to an entire function of order m and has an explicit Hadamard factorization.
- $\Gamma_2(z)$ was first studied by Barnes in 1900 and sometimes denoted by G(z).

$$G(z+1) = (2\pi)^{z/2} \exp(-(z(z+1)+\gamma z^2)/2) \times \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right)^n \exp(-z + z^2/(2n)) \right],$$

Some conjectures and results

- We know that $\Gamma(1/2)=\sqrt{\pi}$.
- We expect $\Gamma_m(1/2)$ to be transcendental for every $m \geq 2$.
- In fact, we expect the numbers $\Gamma_m(1/2)$, $m \geq 1$, to be algebraically independent.
- Theorem (S. Gun, R. Murty, and P. Rath) The number $[\zeta(3)/\pi^3]^2$ is irrational or the number $\Gamma_2(1/2)\Gamma_3(1/2)^{-1}$ is transcendental.

Multiple zeta values

$$\zeta(a_1,\ldots,a_k)=\sum_{n_1>n_2>\cdots>n_k>0}\frac{1}{n_1^{a_1}\cdots n_k^{a_k}}=\sum_{n_1>n_2>\cdots>n_k>0}\prod_{i=1}^{k}\frac{1}{n_i^{a_i}},$$

- We define V_m to be the Q-vector space spanned by the multizeta values $\varsigma(a_1, \ldots, a_k)$ with $a_1 + \ldots + a_k = m$, and $k \geq 1$.
- Zagier's conjecture: Let d_m be the dimension of V_m over Q. Set $d_0=1$, $d_1=0$. Then, for $m \geq 2$, $d_m = d_{m-2} + d_{m-3}$.
- This implies that d_m grows exponentially.
- Not a single value of m is known where $d_m > 1$!

The value of $L(1, \chi)$

- Using Gauss sums, one can evaluate $L(1,\chi)$ explicitly as an algebraic linear combination of logarithms of algebraic numbers.
- More precisely, for χ primitive, $\tau(\chi)(L(1,\chi^*) = -\Sigma_{a<q}\, \chi(a)\log(1-\varsigma^a)$, where ς is a primitive q-th root of unity and $\tau(\chi)$ is a Gauss sum.
- It is interesting to note that we may replace the logarithmic term by $\log (1-\varsigma^a)/(1-\varsigma)$ so that the right hand side is not only a linear combination of logarithms of algebraic numbers, logarithms of units in the cyclotomic field.

Baker's theorem

- Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be non-zero algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the rationals. Then $1, \log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the field of algebraic numbers.
- Baker's theorem implies that $L(1,\chi)$ is transcendental.

Alan Baker (1939 -)

Schanuel's conjecture

- Suppose that $x_1, \ldots, x_n$ are linearly independent over the rationals. Then the transcendence degree of K over Q is at least n. Here

$$K = \mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$$

Schanuel's conjecture implies the following strengthening of Baker's theorem: if $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the rationals, then they are algebraically independent over the field of algebraic numbers.

Some new consequences of Schanuel's conjecture

- Theorem (S. Gun, R. Murty and P. Rath) Assume Schanuel's conjecture. Then either $\zeta(3)$ and π are algebraically independent or number $\Gamma_2(1/2)\Gamma_3(1/2)^{-1}$ is transcendental.
- Recall that we proved unconditionally that either the number $[\zeta(3)/\pi^3]^2$ is irrational or the number $\Gamma_2(1/2)\Gamma_3(1/2)^{-1}$ is transcendental.

The Frobenius automorphism and the Artin symbol

- Let K be an algebraic number field and consider a finite Galois extension F/K with group G.
- For each place v of K, let w be a place of F lying above v. Let σ_w denote the Frobenius automorphism at w, which is well-defined modulo the inertia group of w.
- As one ranges over the places w above a fixed place v, the σ_w's describe a conjugacy class of G(well defined modulo inertia at w) called the Artin symbol at v and denoted σ_v.

Artin L-series

- Given a complex linear representation $\rho: G \to GL(V)$, where V is a d-dimensional vector space over the complex numbers, we define the Artin L-series as follows.
- $L(s, \rho, F/K) = \Pi_v \det(1-\rho(\sigma_v)Nv^{-s}|V^I)^{-1}$.
- Sometimes we simply write $L(s,\rho)$.
- This product converges absolutely for $Re(s)>1$ and thus is analytic in this region.
- It is clear that if $\rho=\rho_1 +\rho_2$, then $L(s, \rho_1)L(s,\rho_2)$.

Artin L-series as generalizations of Dirichlet's L-series

- If K is the field of rational numbers and F is the q-th cyclotomic field, then Gal(F/K) is isomorphic to the group of coprime residue classes mod q.
- The characters of this Galois group are precisely the Dirichlet characters and the Artin L-series attached to these characters are Dirichlet's L-series.
- If 1 is the trivial representation, then $L(s,1)=\varsigma_K(s)$, the Dedekind zeta function of K.

Artin's conjecture

- If ρ is irreducible and $\neq 1$, then $L(s,\rho)$ extends to an entire function.
- Artin's reciprocity theorem: If ρ is one-dimensional, then there is a Hecke L-series $L_K(s,\psi)$ such that $L(s,\rho)=L_K(s,\psi)$.
- By virtue of the analytic continuation of Hecke L-series, we derive Artin's conjecture in this case.
- Brauer's induction theorem allows us to write any character as an integral linear combination of inductions of one-dimensional characters. Thus, Artin's reciprocity allows us to derive the meromorphic continuation of any Artin L-series.
- These L-series also satisfy a functional equation relating s to 1-s.

The two-dimensional reciprocity law

- Theorem (Khare-Wintenberger, 2009) If ρ is 2-dimensional and odd (that is, det $\rho(c) = -1$, where c is complex conjugation), then $L(s,\rho) = L(s,\pi)$ for some automorphic form of $GL(2, A_K)$.

C. Khare

P. Winterberger

Stark's conjecture on $L(1,\chi, F/K)$

Harold Stark (1939 -)

- There are algebraic numbers $W(\chi)$ with $|W(\chi)|=1$ and $\theta(\chi)$ such that $L(1,\chi, F/K) = W(\chi)2^a\pi^b\theta(\chi)R(\chi)$, where $R(\chi)$ is the determinant of a "regulator" matrix whose entries are linear forms in logarithms of units in the ring of integers of F.

Transcendence of $L(1, \chi, F/K)$

- Theorem (S. Gun, R. Murty and P. Rath) Schanuel's conjecture implies the transcendence of $L(1, \chi, F/K)$ if χ is a rational character. If in addition, we assume Stark's conjecture, then $L(1, \chi, F/K)$ is transcendental.

Artin L-series at integer arguments

- Given an Artin representation $\rho:G\rightarrow GL(V)$, we can decompose V via the action of complex conjugation. Thus, $V = V^{+}+V^{-}$ where V^{+} is the +1 eigenspace and V^{-} is the (-1)-eigenspace.
- We will say ρ (or V) has Hodge type (a,b) if $\dim V^{+} = a$ and $\dim V^{-} = b$.
- The precise nature of $L(k,\rho, F/K)$ will depend partly on the Hodge type (a,b).

The Siegel-Klingen theorem (1962)

- Let K be a totally real field. Then $\zeta_K(2n)$ is rational multiple of $\pi^{2n[K:Q]}$.
- The proof uses the theory of Hilbert modular forms.

Carl Ludwig Siegel (1896-1981)

Helmut Klingen

The Coates-Lichtenbaum Theorem (1973)

- Suppose that $L(-n, \rho) \neq 0$. Then, $L(-n,\rho)$ is an algebraic number lying in the field generated over Q by the values of the character of ρ. Moreover, for any Galois automorphism σ, we have $L(-n,\rho)^{\sigma} = L(-n, \rho^{\sigma})$.
- This means that if the Hodge type of ρ is (a,0), then $L(2k, \rho)$ is an algebraic multiple of a power of π, by virtue of the functional equation.
- If the Hodge type is (0,b) then $L(2k+1, \rho)$ is an algebraic multiple of a power of π.

The polylogarithm

- We define $L_k(z)=\Sigma_{n\geq 1}\, z^n/n^k$, for $|z|<1$.
- For $k=1$, this is $-\log(1-z)$.
- For $k\geq 2$, the series converges in the closed disc $|z|\leq 1$.
- One can show that these functions extend to the cut complex plane $C-[1,\infty)$.
- The polylog conjecture: If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are algebraic numbers such that $L_k(\alpha_1), \ldots, L_k(\alpha_n)$ are linearly independent over Q, then they are linearly independent over the field of algebraic numbers.

The Chowla-Milnor conjecture revisited

- Theorem (S. Gun, R. Murty and P. Rath) The polylog conjecture implies the Chowla-Milnor conjecture for all q and all $k \geq 2$.
- The Chowla-Milnor conjecture implies that the vector space spanned by multiple zeta values of weight $4k+2$ has $\mathbb{Q}$-dimension at least 2.

Zagier's conjecture

- $L(k,\chi, F/K)$ is an algebraic multiple of a power of π and $R_K(\chi)$ where $R_K(\chi)$ is the determinant of a matrix whose entries are linear forms in polylogarithms evaluated at algebraic arguments.
- Theorem (Goncharov) If $\chi=1$ and $k=2,3$, then Zagier's conjecture is true.

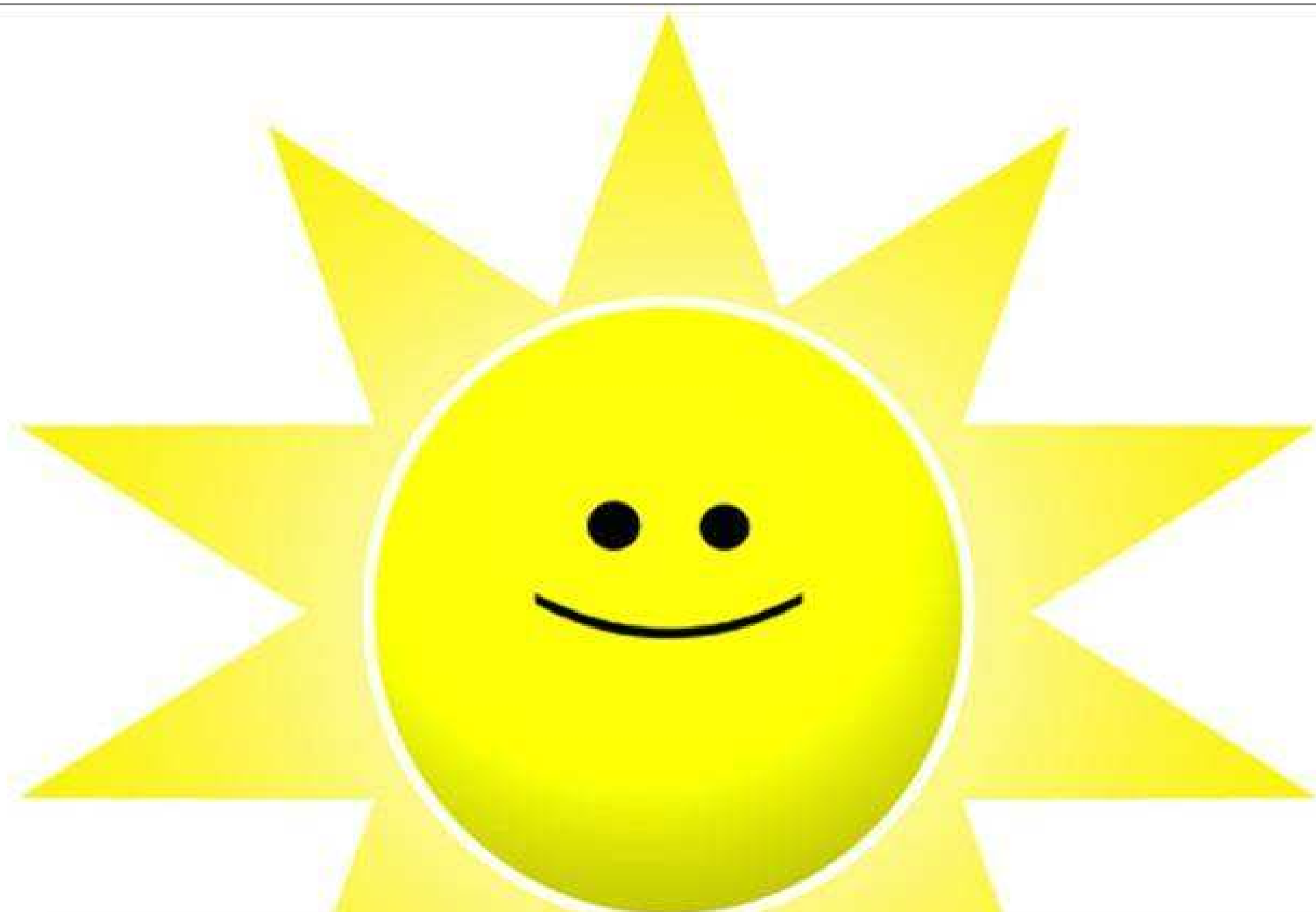
Don Zagier (1951-)

The case k=1 revisited.

- If K is an imaginary quadratic field and F is its Hilbert class field, then $L_K(1,\chi)$ where χ is a character of the ideal class group, is the special value of an Artin L-function, by Artin's reciprocity law.
- Exercise: $L_K(1,\chi) = L_K(1,\chi^*)$ where χ^* is the complex conjugate of χ.
- Theorem: The set of values $\{L_K(1,\chi):\chi \in CL_K^{\wedge}\}/^*$ are linearly independent over the field of algebraic numbers.

Future research

- Develop Baker's theory for the dilogarithms of algebraic numbers and more generally polylogarithms.
- Understand the relationship of multiple zeta values and the special values of the Riemann ς-function.
- Brown's theorem (2011): The vector space of MZV's of weight m is spanned by $\varsigma(a_1, \ldots, a_k)$ with $a_i \leq 3$.
- Develop a theory of multiple zeta values with "twists" and relate these to special values of Dirichlet L-series.



THANK YOU!