

Multiple-copy State Discrimination

Thinking Globally, Acting Locally

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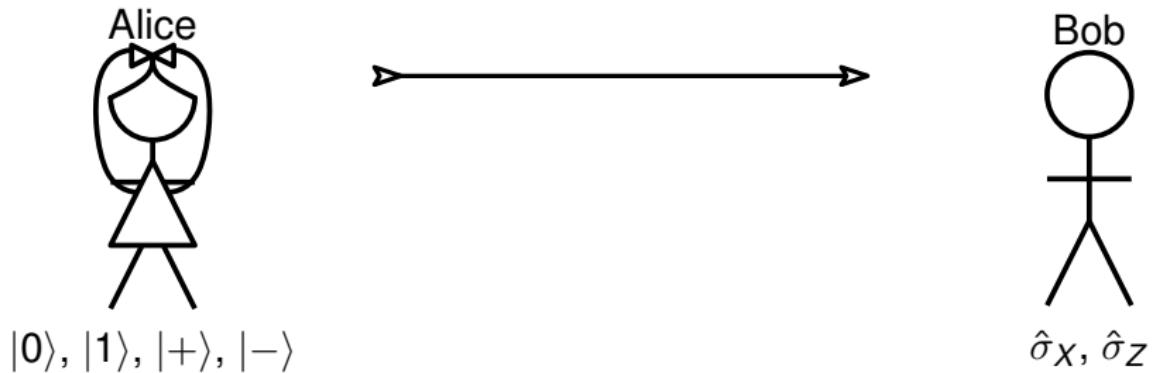
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CQIQC — 9th Aug 2011

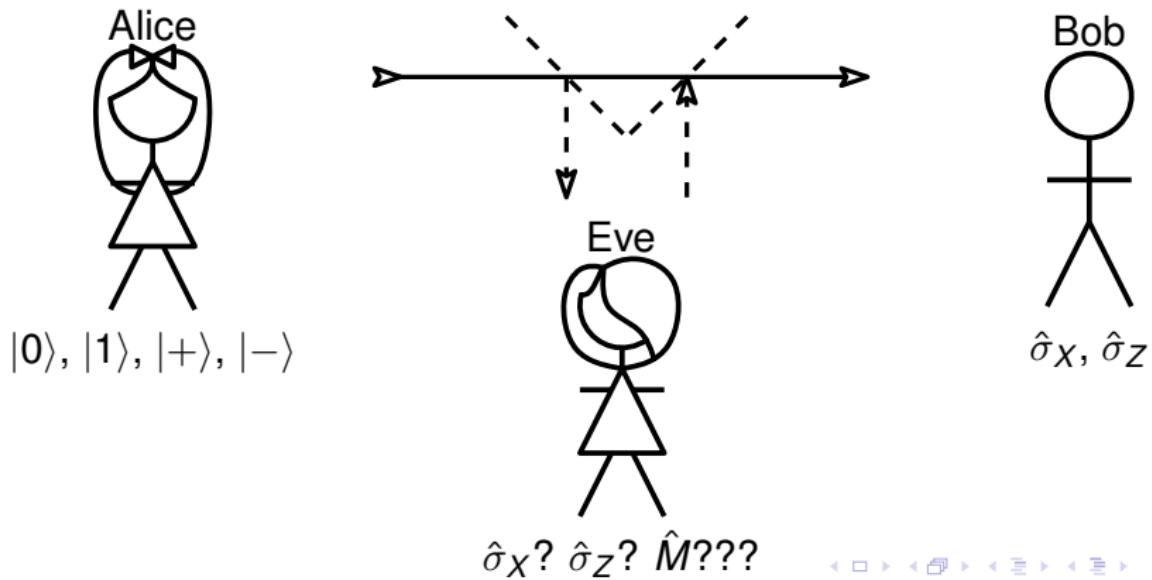
State discrimination

An example: Quantum key distribution

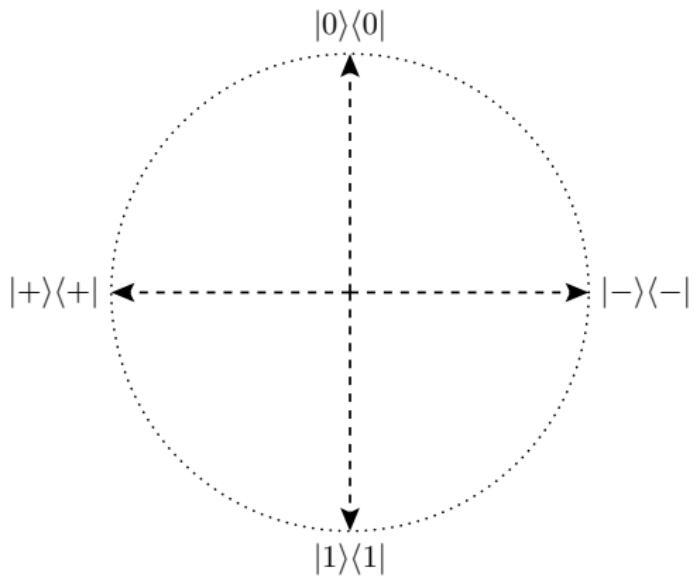


State discrimination

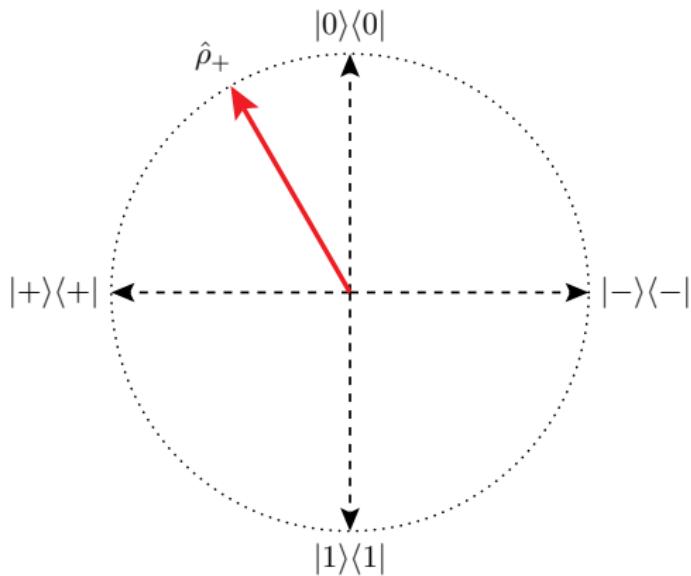
An example: Quantum key distribution



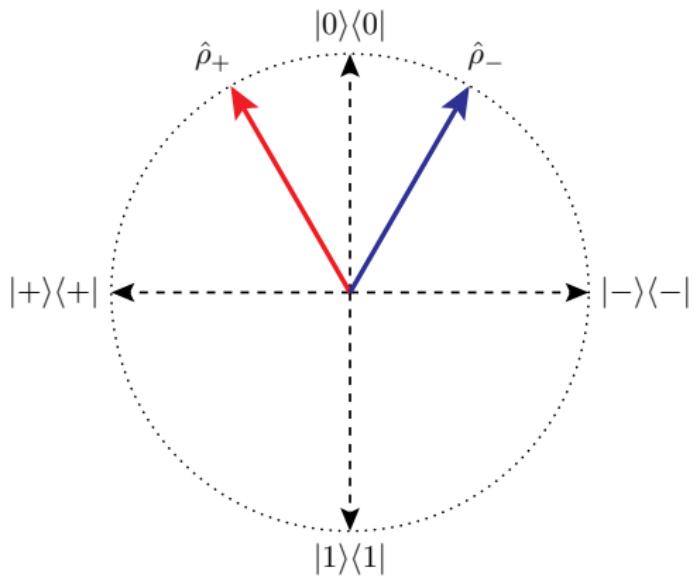
The problem



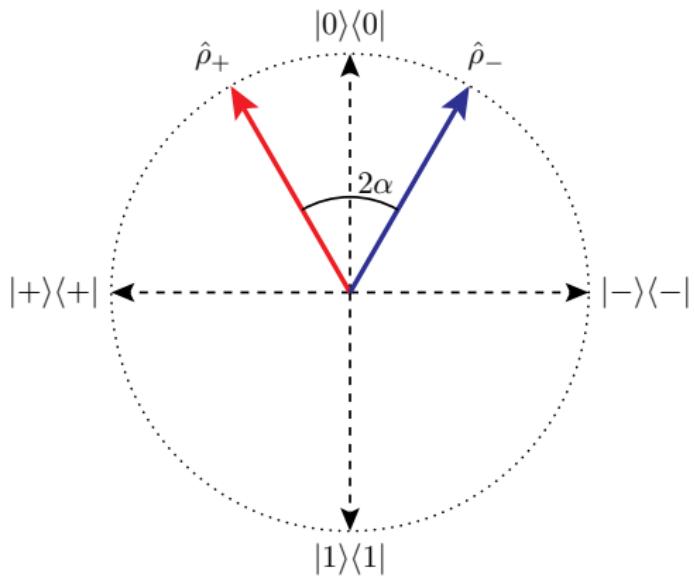
The problem



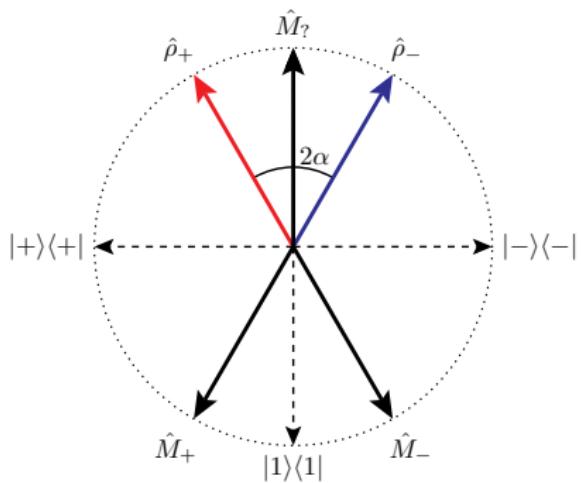
The problem



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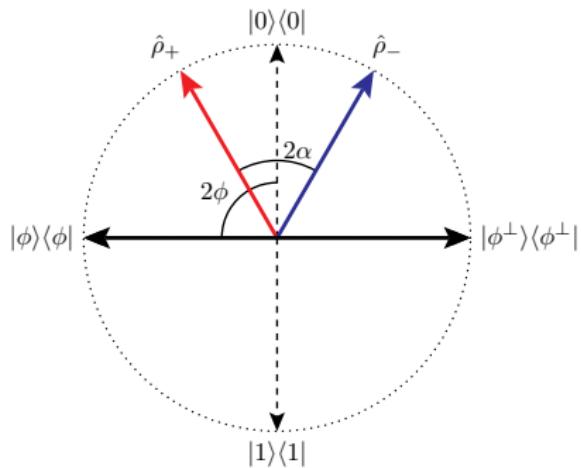


The problem



■ Unambiguous, 3-outcome
POVM: \hat{M}_+ , \hat{M}_- , $\hat{M}_?$

The problem

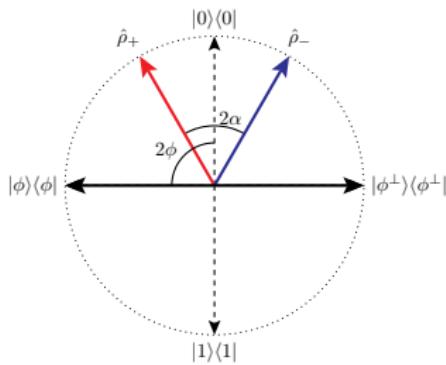


- Unambiguous, 3-outcome POVM: \hat{M}_+ , \hat{M}_- , $\hat{M}_?$
- Conclusive 2-outcome (projective),

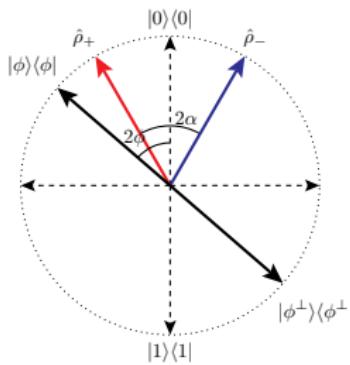
$$|\phi\rangle = \cos\phi|0\rangle + \sin\phi|1\rangle$$

minimal error: $\min_{\phi} C$

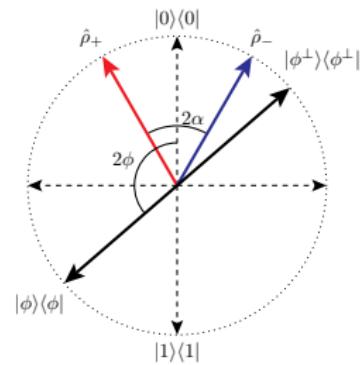
Optimal solutions



$$q = 1/2$$



$$q = 3/4$$



$$q = 1/4$$

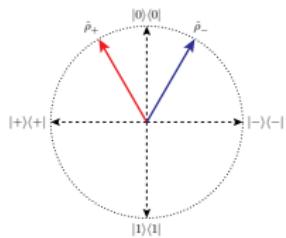
q : Likelihood of preparing $\hat{\rho}_+$

$$\phi(q) = \frac{1}{2} \operatorname{arccot}[(2q - 1) \cot \alpha]$$

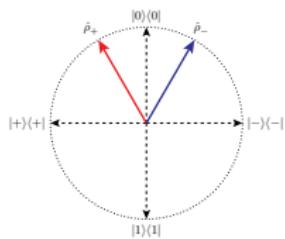
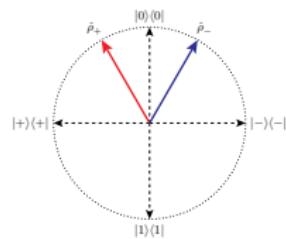
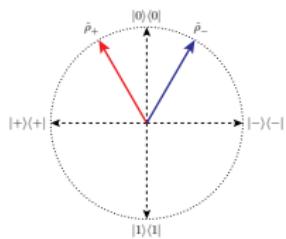
$$\Rightarrow C = \frac{1}{2}[1 - \sqrt{1 - c^2}]$$

where $c = \cos \alpha$

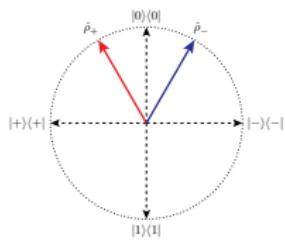
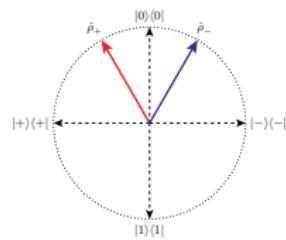
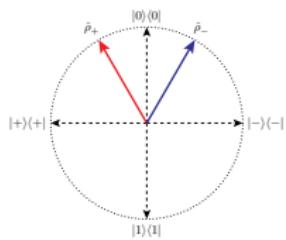
Fixed and adaptive schemes



Fixed and adaptive schemes



Fixed and adaptive schemes



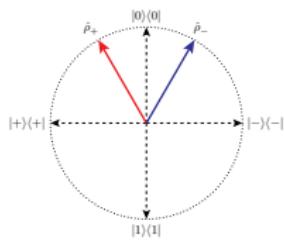
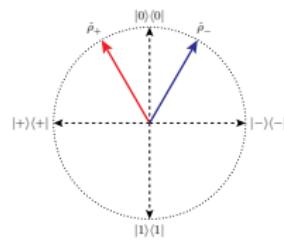
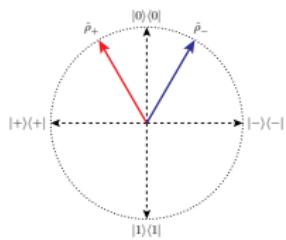
...

$\hat{\phi}$

$\hat{\phi}$

$\hat{\phi}$

Fixed and adaptive schemes



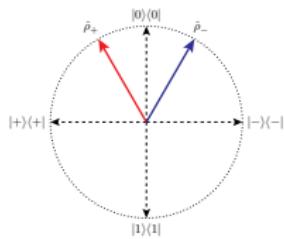
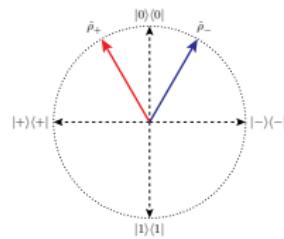
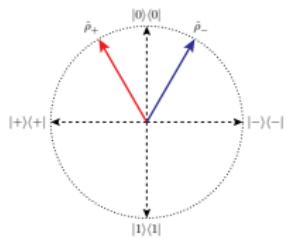
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$\hat{\phi}_1 \Rightarrow$

$\Rightarrow \hat{\phi}_2 \Rightarrow$

$\Rightarrow \hat{\phi}_3 \Rightarrow$

Fixed and adaptive schemes



...

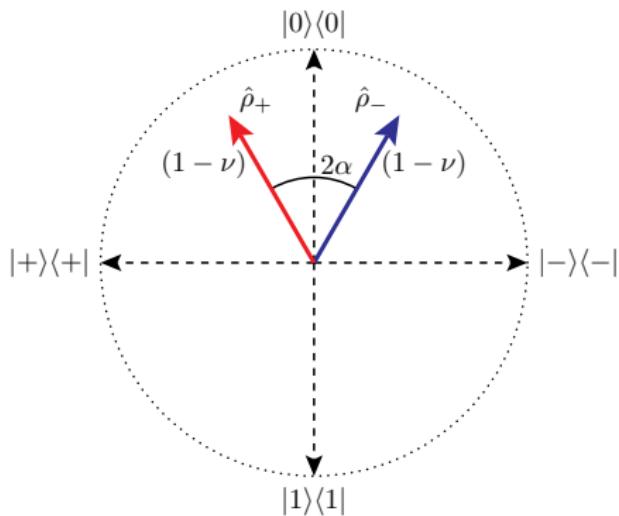
$\hat{\phi}_1 \Rightarrow$

$\Rightarrow \hat{\phi}_2 \Rightarrow$

$\Rightarrow \hat{\phi}_3 \Rightarrow$

Optimal $C!$
(For pure
states...)

Pure and mixed states



Mixture $\nu \in [0, 1]$

$\nu = 0$: Pure

$\nu = 1$: Depolarized

Minimal single-copy error:

$$C^{\text{OSM}} = \frac{1}{2}[1 - (1 - \nu)\sqrt{1 - c^2}]$$

$$\hat{\rho}_{\pm} = \frac{1}{2}[\hat{\sigma}_I + (1 - \nu)(\hat{\sigma}_Z \cos \alpha \pm \hat{\sigma}_X \sin \alpha)]$$

Probability of error, C_N

$$\nu = 0.3$$

0.1

Fixed

2

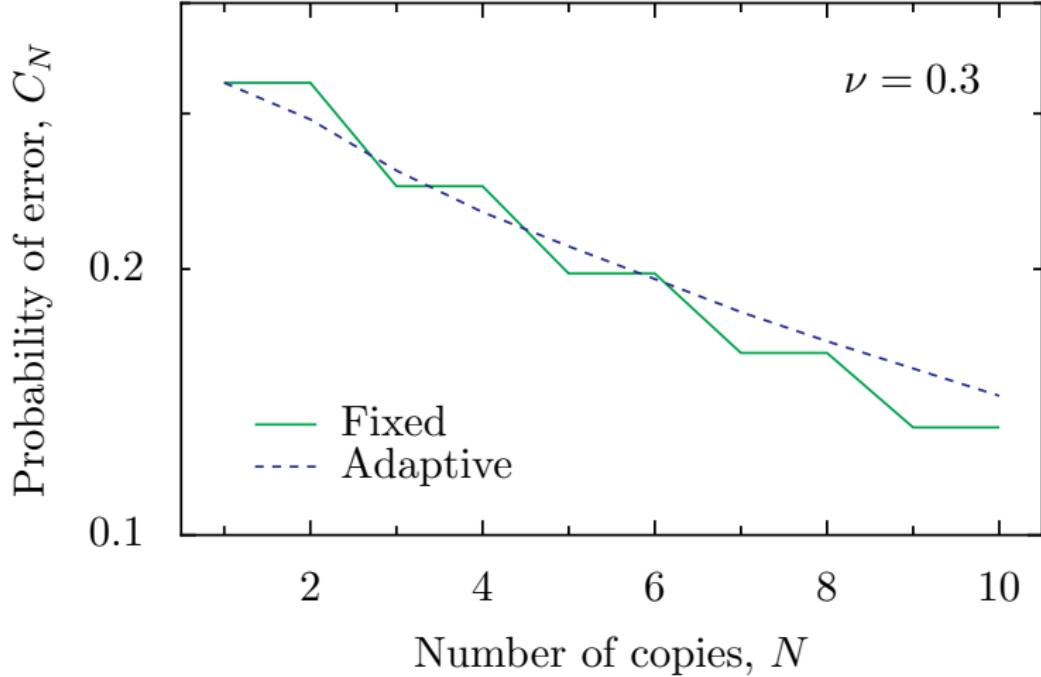
4

6

8

10

Number of copies, N



Optimizing adaptively ends up *worse* than staying fixed!

- Locally optimizing
vs. globally optimizing

- Locally optimizing
vs. globally optimizing
- Fixed schemes
vs. adaptive schemes

- Locally optimizing
vs. globally optimizing
- Fixed schemes
vs. adaptive schemes
- Moderate N
vs. large N scaling

Locally optimal fixed (LOF) scheme

$$\hat{\rho}_\pm$$

$$\hat{\rho}_\pm$$

$$\hat{\rho}_\pm$$

$$\phi : \min_\phi C$$

,

$$\phi : \min_\phi C$$

,

$$\phi : \min_\phi C$$

,

...

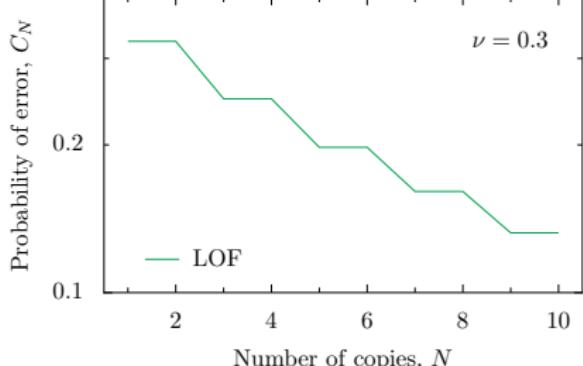
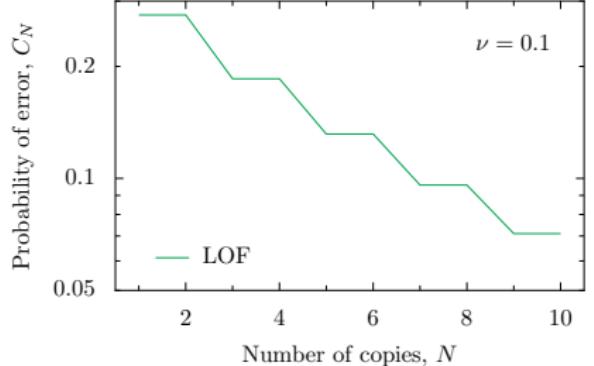
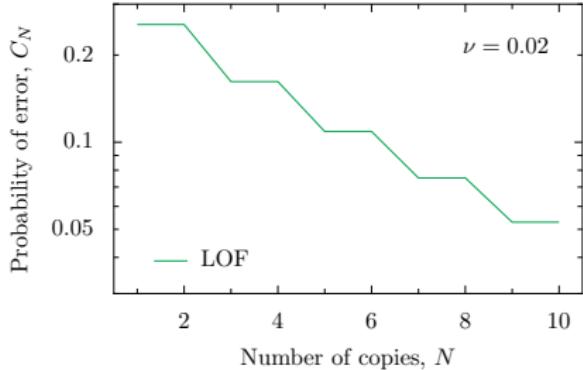
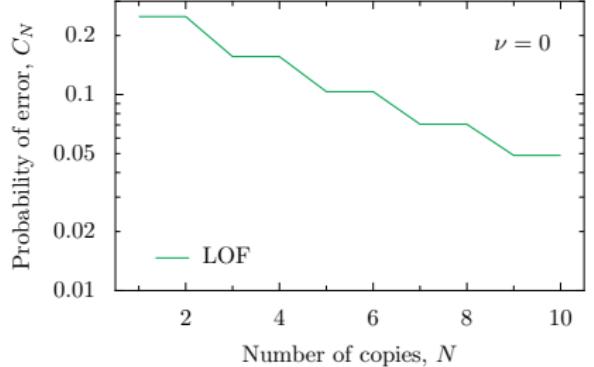
$$q = 1/2$$

$$\therefore \phi = \pi/4$$

$$C_N^{\text{LOF}} = \sum_{n=0}^{\lfloor N/2 \rfloor} \binom{N}{n} \left(1 - C^{\text{OSM}}\right)^n \left(C^{\text{OSM}}\right)^{N-n}$$

for odd N .

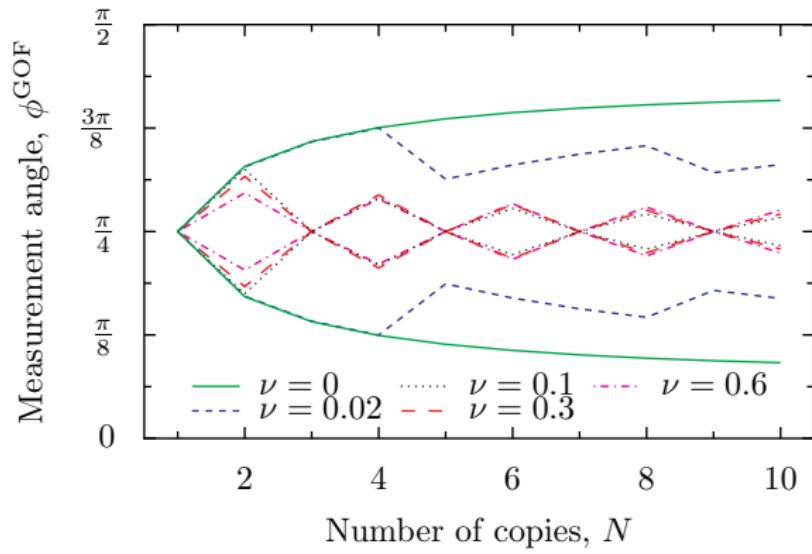
Locally optimal fixed (LOF) scheme



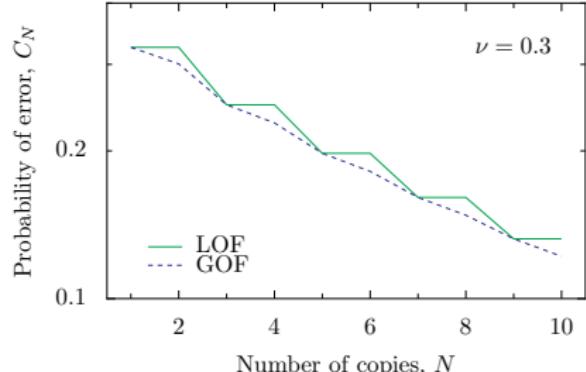
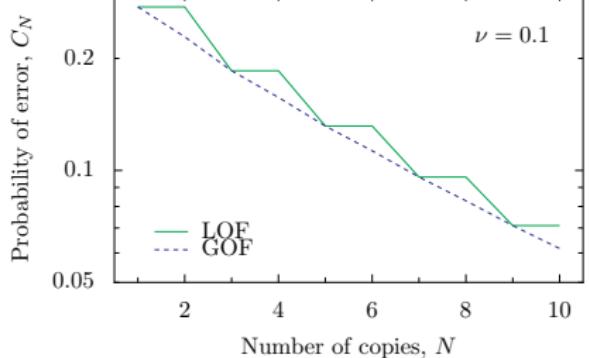
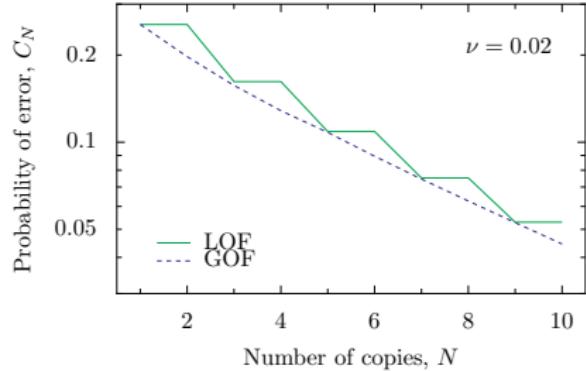
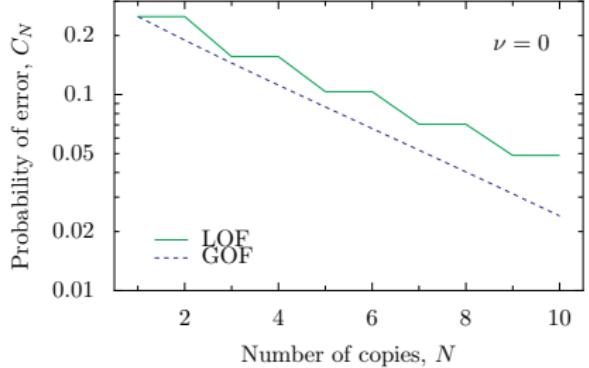
$$\alpha = \pi/6$$

Globally optimal fixed (GOF) scheme

$$\min_{\phi} C \Rightarrow \hat{\rho}_{\pm}, \phi, \hat{\rho}_{\pm}, \phi, \hat{\rho}_{\pm}, \phi, \dots$$



Globally optimal fixed (GOF) scheme



$$\alpha = \pi/6$$

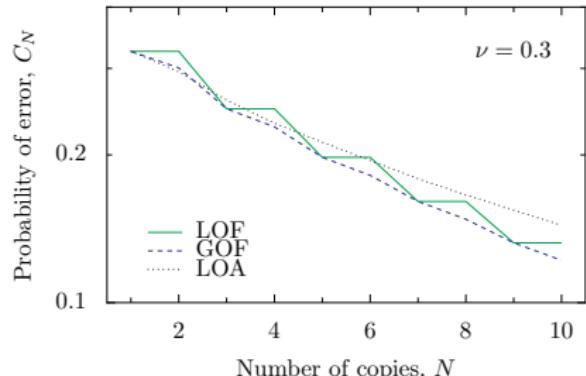
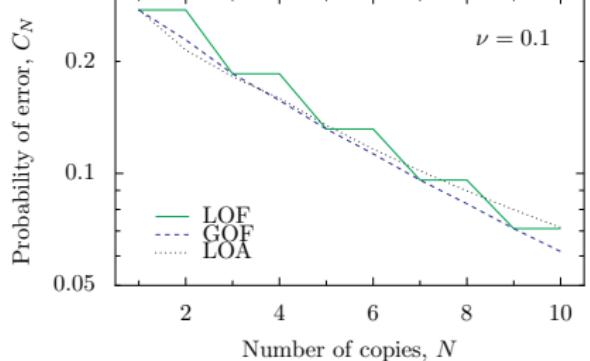
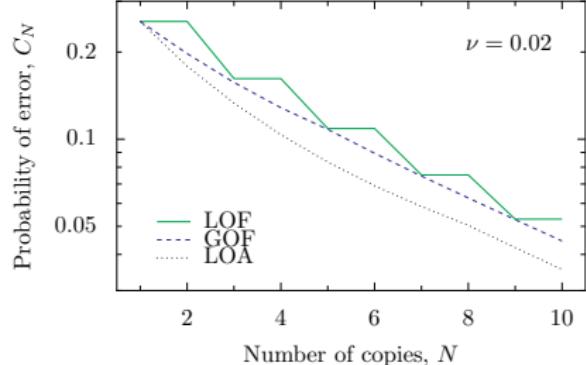
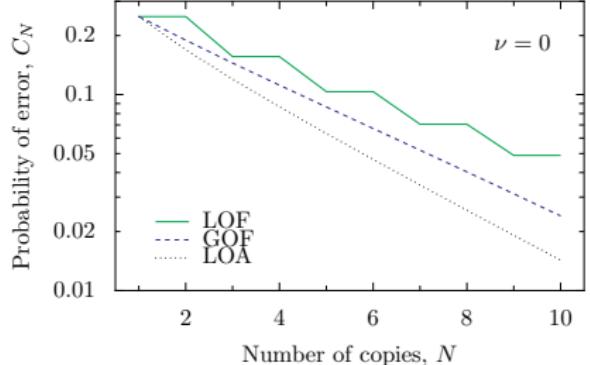
Locally optimal adaptive (LOA) scheme

$$\begin{array}{ccccccc} \hat{\rho}_{\pm} & & \hat{\rho}_{\pm} & & \hat{\rho}_{\pm} & & \dots \\ \phi_1 : \min_{\phi} C & \xrightarrow{} & \phi_2 : \min_{\phi} C & \xrightarrow{} & \phi_3 : \min_{\phi} C & \xrightarrow{} & \dots \end{array}$$

$$\phi_{n+1}(p_n) = \frac{1}{2} \operatorname{arccot}[(2p_n - 1) \cot \alpha]$$

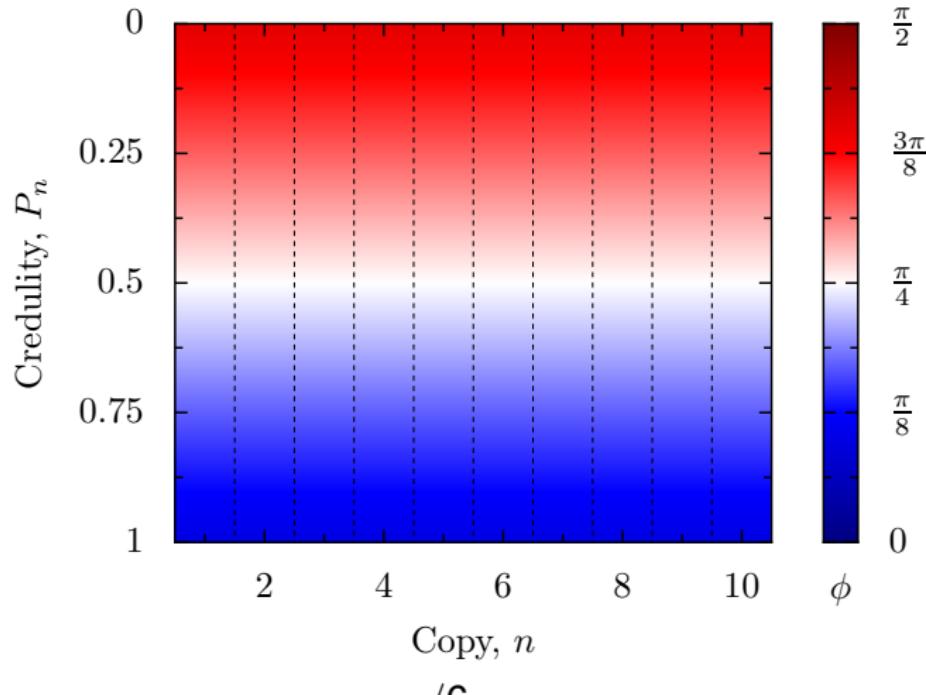
where p_n is an intermediate Bayes credulity.

Locally optimal adaptive (LOA) scheme

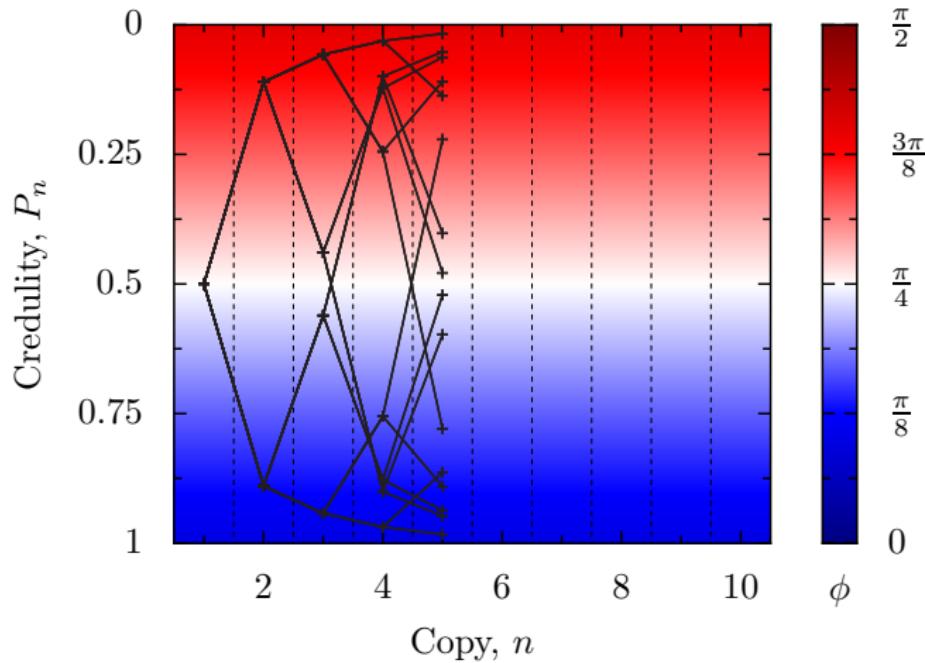


$$\alpha = \pi/6$$

Locally optimal adaptive (LOA) scheme

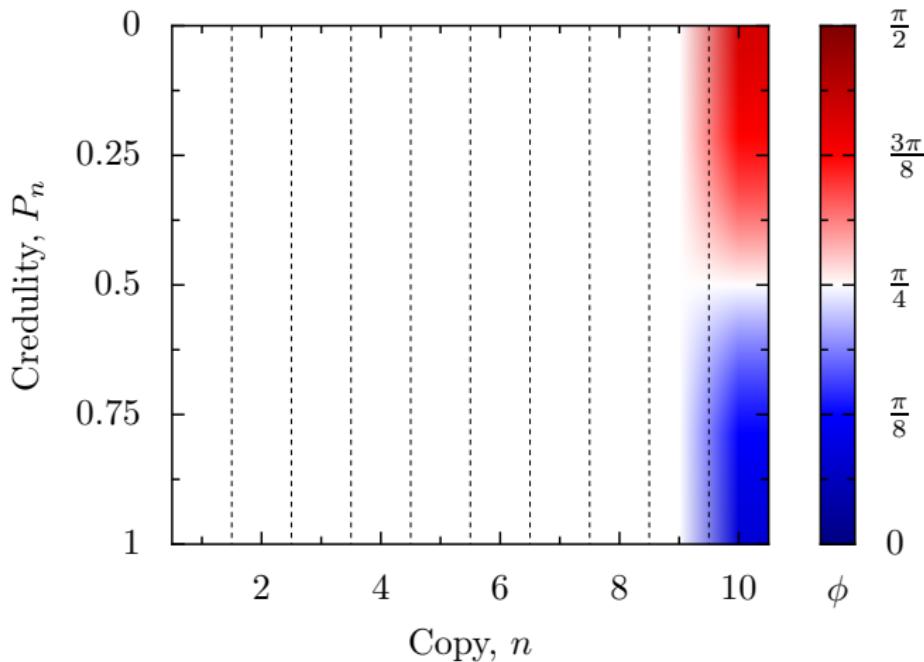


Locally optimal adaptive (LOA) scheme

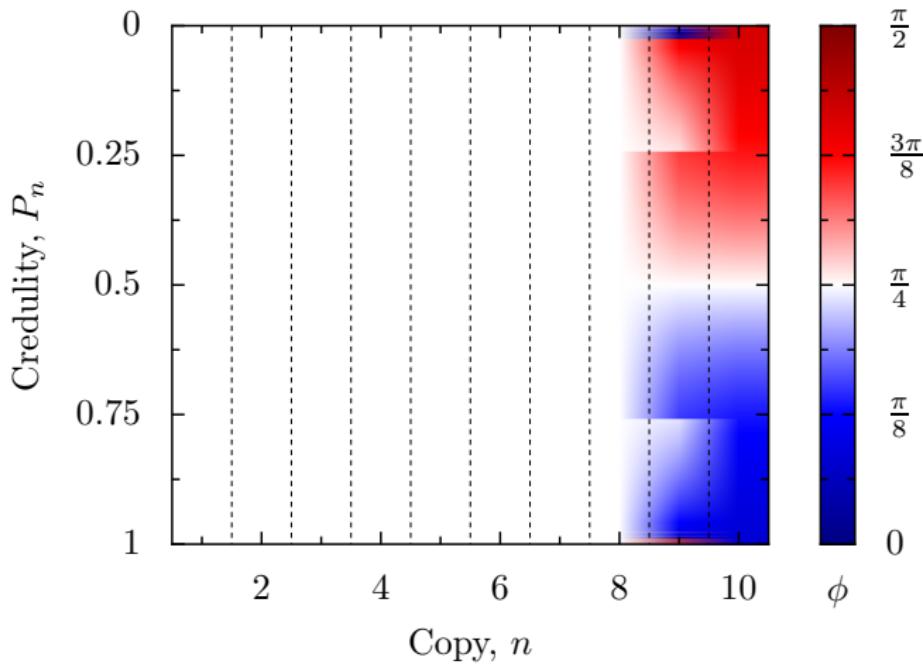


$$\alpha = \pi/6, \nu = 0.1$$

Globally optimal adaptive (GOA) scheme

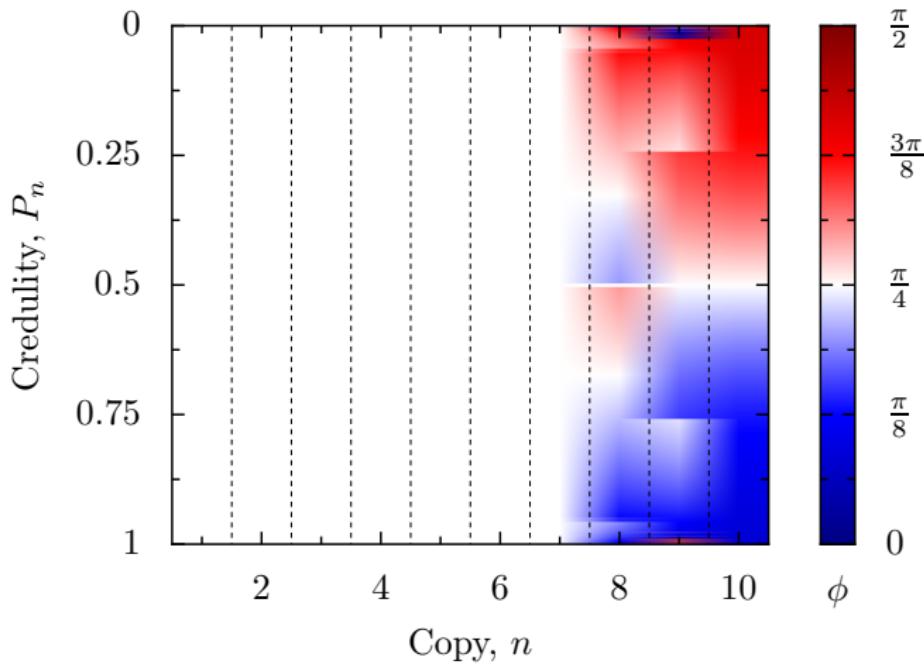


Globally optimal adaptive (GOA) scheme



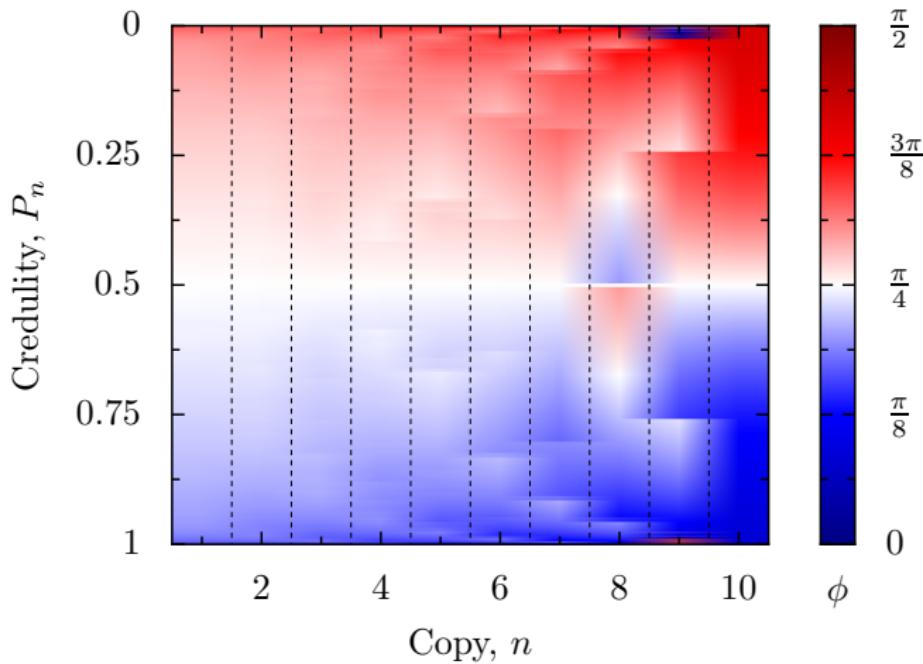
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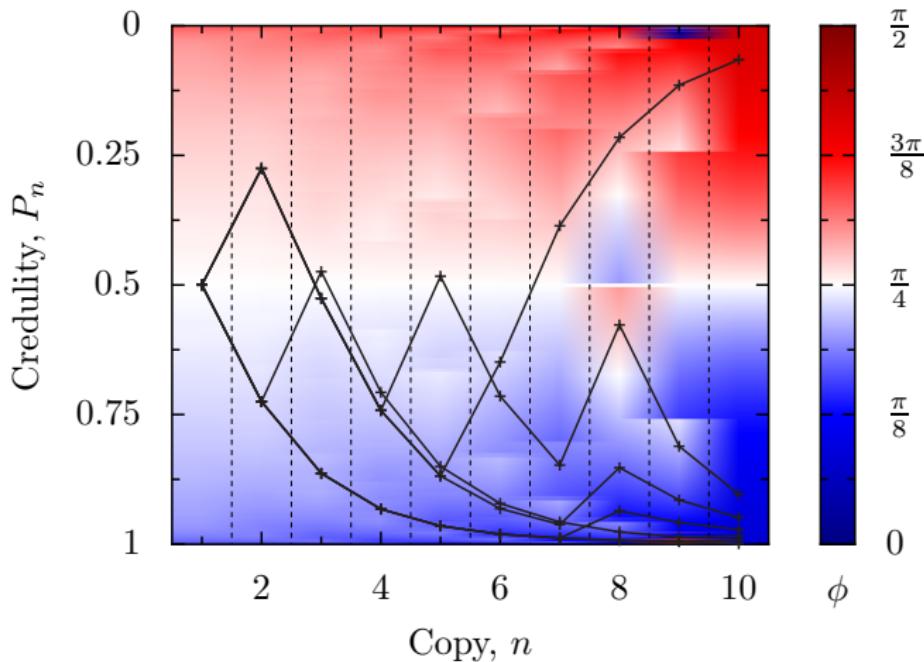
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Globally optimal adaptive (GOA) scheme

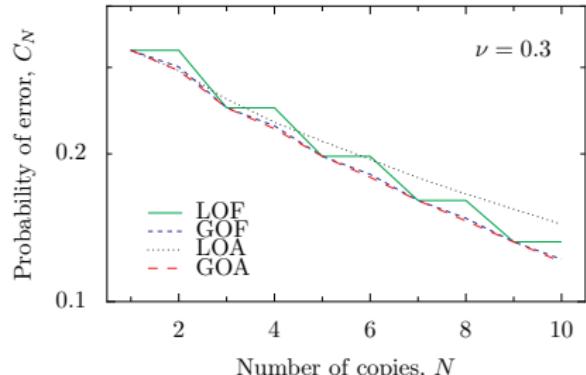
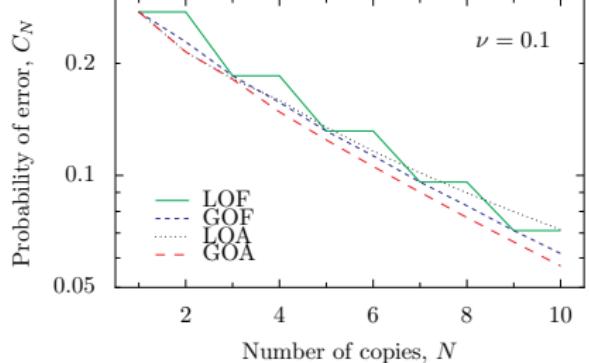
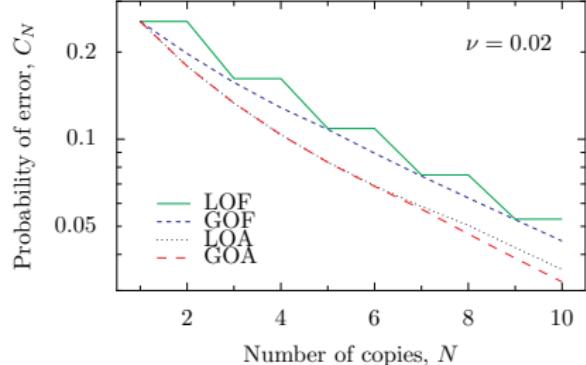
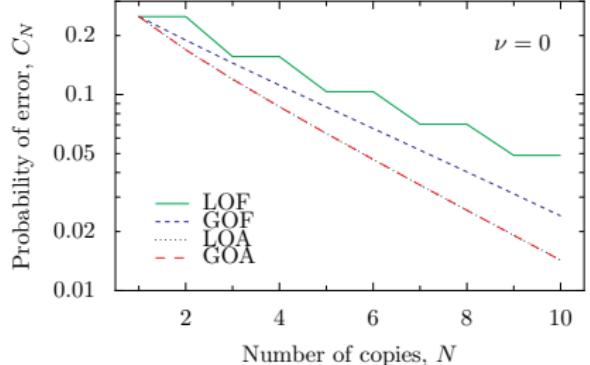


$$\alpha = \pi/6, \nu = 0.1$$

Globally optimal adaptive (GOA) scheme



Globally optimal adaptive (GOA) scheme



$$\alpha = \pi/6$$

The asymptotic limit

As $N \rightarrow \infty$:

$$C_N \sim e^{-\xi N}$$

Chernoff bound:

$$\xi = - \lim_{N \rightarrow \infty} \frac{\log C_N}{N}$$

Bounds on ξ : Hayashi¹; Calsamiglia, de Vicente, Muñoz-Tapia, Bagan²

¹IEEE Trans. Inf. Theory **55**, 3807 (2009)

²PRL **105**, 080504 (2010)

Locally optimal fixed (LOF)

$$C_N^{\text{LOF}} = \sum_{n=0}^{\lfloor N/2 \rfloor} \binom{N}{n} \left(1 - C^{\text{OSM}}\right)^n \left(C^{\text{OSM}}\right)^{N-n}$$

Locally optimal fixed (LOF)

$$C_N^{\text{LOF}} = \left(C^{\text{OSM}}\right)^N \sum_{n=0}^{\lfloor N/2 \rfloor} \binom{N}{n} \left(\frac{1 - C^{\text{OSM}}}{C^{\text{OSM}}}\right)^n$$

Locally optimal fixed (LOF)

$$C_N^{\text{LOF}} \sim \left(C^{\text{OSM}}\right)^N \int_0^{N/2} \binom{N}{n} \left(\frac{1 - C^{\text{OSM}}}{C^{\text{OSM}}}\right)^n dn$$

Locally optimal fixed (LOF)

$$C_N^{\text{LOF}} \sim \left(C^{\text{OSM}}\right)^N \binom{N}{N/2} \int_0^{N/2} \left(\frac{1 - C^{\text{OSM}}}{C^{\text{OSM}}}\right)^n dn$$

Locally optimal fixed (LOF)

$$C_N^{\text{LOF}} \sim \left(C^{\text{OSM}}\right)^N \frac{N!}{[(N/2)!]^2} \frac{[(1 - C^{\text{OSM}})/C^{\text{OSM}}]^{N/2}}{\log[(1 - C^{\text{OSM}})/C^{\text{OSM}}]}$$

Locally optimal fixed (LOF)

$$C_N^{\text{LOF}} \sim \left(C^{\text{OSM}}\right)^N \frac{N!}{[(N/2)!]^2} \frac{[(1 - C^{\text{OSM}})/C^{\text{OSM}}]^{N/2}}{\log[(1 - C^{\text{OSM}})/C^{\text{OSM}}]}$$

$$\begin{aligned}\log C_N^{\text{LOF}} \sim & (N/2) \log[(1 - C^{\text{OSM}})/C^{\text{OSM}}] + N \log C^{\text{OSM}} \\ & + \log(N!) - 2 \log[(N/2)!] + \text{const.}\end{aligned}$$

Locally optimal fixed (LOF)

$$C_N^{\text{LOF}} \sim \left(C^{\text{OSM}}\right)^N \frac{N!}{[(N/2)!]^2} \frac{[(1 - C^{\text{OSM}})/C^{\text{OSM}}]^{N/2}}{\log[(1 - C^{\text{OSM}})/C^{\text{OSM}}]}$$

$$\begin{aligned}\log C_N^{\text{LOF}} &\sim N \log \left[2\sqrt{(1 - C^{\text{OSM}})C^{\text{OSM}}} \right] \\ &\quad - (1/2) \log N + \text{const.}\end{aligned}$$

Locally optimal fixed (LOF)

$$C_N^{\text{LOF}} \sim (C^{\text{OSM}})^N \frac{N!}{[(N/2)!]^2} \frac{[(1 - C^{\text{OSM}})/C^{\text{OSM}}]^{N/2}}{\log[(1 - C^{\text{OSM}})/C^{\text{OSM}}]}$$

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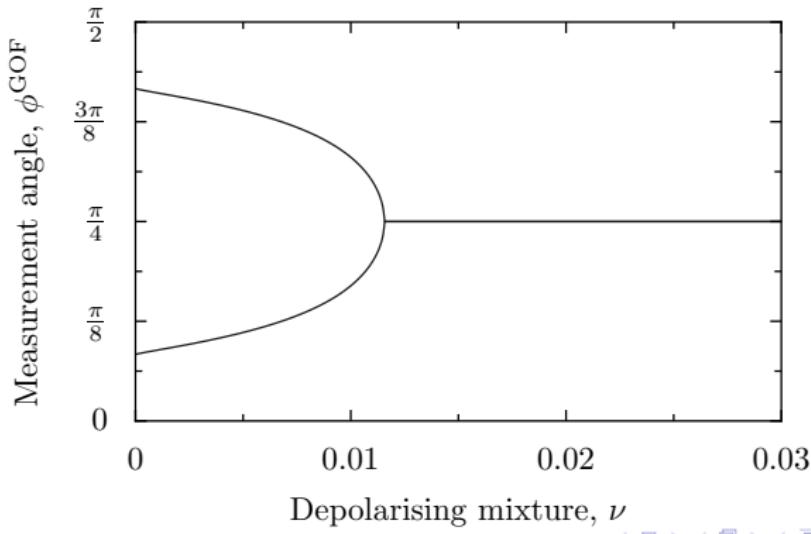
Taking $\xi^{\text{LOF}} = \lim_{N \rightarrow \infty} \log C^{\text{LOF}}/N$:

$$\begin{aligned}\xi^{\text{LOF}} &= -\log \left[2\sqrt{(1 - C^{\text{OSM}})C^{\text{OSM}}} \right] \\ &= -(1/2) \log \left[1 - (1 - \nu)^2(1 - c^2) \right]\end{aligned}$$

For pure states, $\xi^{\text{LOF}} = -\log c$, $\therefore C_N^{\text{LOF}} \sim c^N$, as expected.

Globally optimal fixed (GOF)

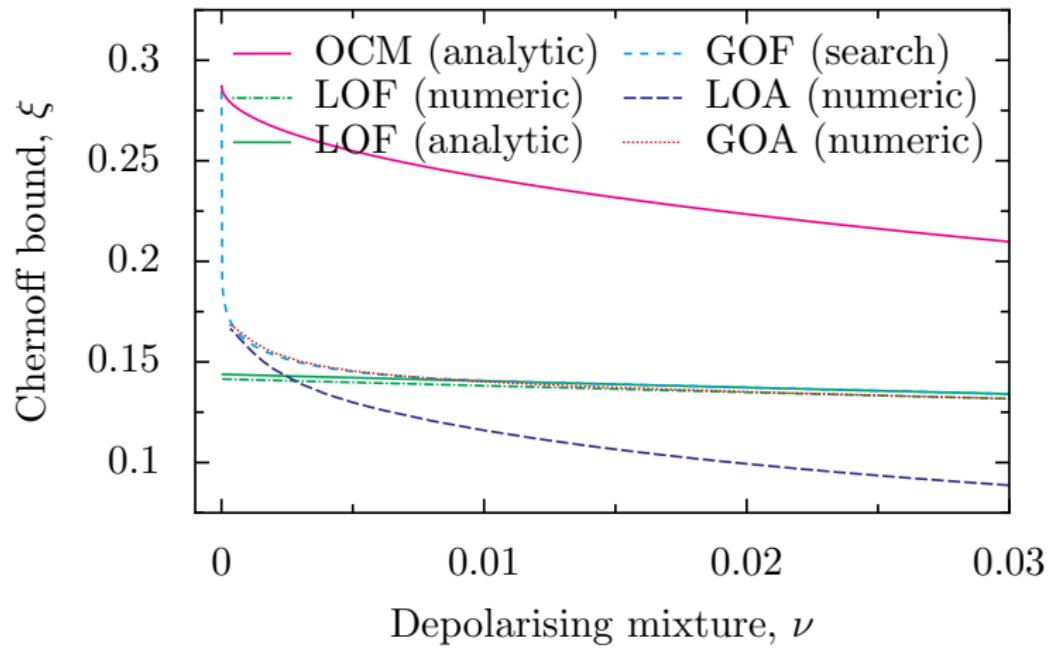
$$\xi^{\text{GOF}} = -\log \min_{0 \leq a \leq 1} \min_{0 \leq \phi \leq \pi/2} \left[\left(\text{Tr}[\hat{\phi} \hat{\rho}_+] \right)^a \left(\text{Tr}[\hat{\phi} \hat{\rho}_-] \right)^{1-a} + \left(\text{Tr}[\hat{\phi}^\perp \hat{\rho}_+] \right)^a \left(\text{Tr}[\hat{\phi}^\perp \hat{\rho}_-] \right)^{1-a} \right]$$



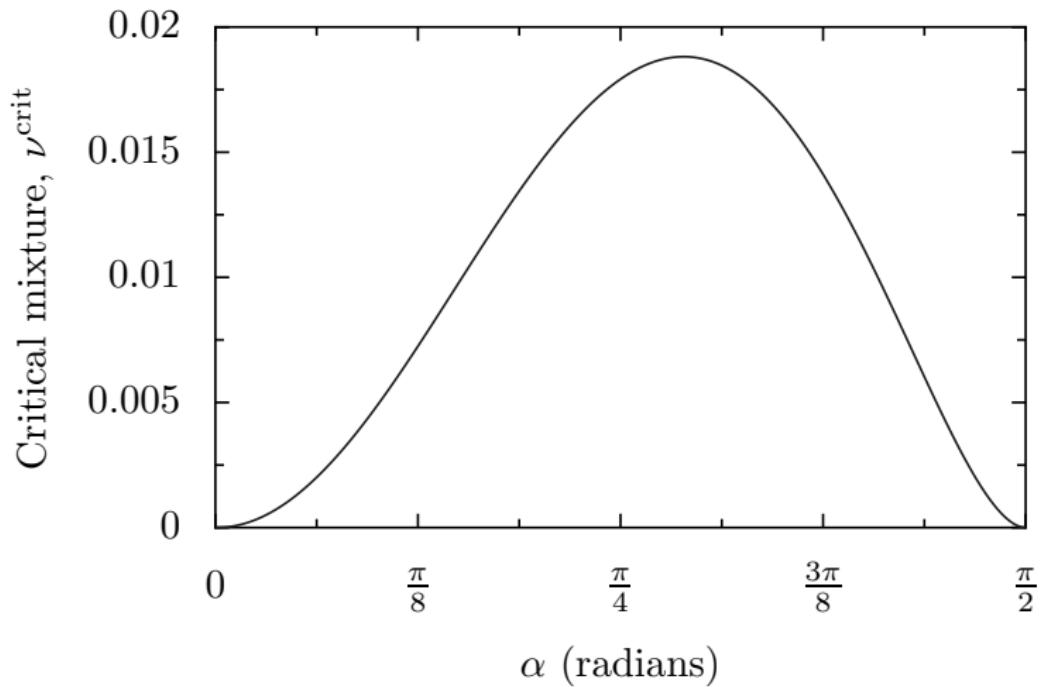
Adaptive schemes

- Calculate N to the asymptotic regime
- Large measurement trees
- Tabular sampling with cubic interpolation
- Estimation of infinite samples³

Chernoff bounds



Critical mixture



Conclusions

- Multi-copy state discrimination schemes using local and global optimization strategies
- Locally optimal adaptive measurements are only good enough for pure states (unrealistic)
- If you only care about scaling, adaptivity is unnecessary
- If mixture is $\gtrapprox 2\%$, being clever doesn't help scaling, regardless of α