My Personal Adventure in Quantum Wonderland

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- *Quantum* is the minimum amount of any physical entity involved in an interaction

- *Quantum* is a misused word, as appeared in *Quantum* groups, *Quantum* fields and *Quantum* Computers.

- Sensible that *Quantum* Information was a sub-field of *Quantum* Mechanics.

- The first *Quantum* computer in real functioning was built in 1998.

- In 2006, there was a benchmarking of a 12-*qubit* computer.

My Personal Experience

- I am a pure mathematician with much interest in foundation of *Quantum* mechanics in the light of *Quantum* information.
- My old short paper of 1975 has been viewed as a pioneering paper about *Quantum* channels.
- Suddenly, I was awaken in the new era of *Quantum* computers to see frightening features of *Quantum* channels for communications of *Quantum* information.
- As the time runs backwards in an alternating world through the looking glass, I have to come back to the same old scene to release myself from *Quantum* entanglements.

Quantum Channels through my old Dream

UNITS OF INFORMATION-- Bits VS Qubits

- A ***bit*** (binary integer) is the base of conventional computer memory.

- 1-bit is read as either a zero or a one with probability in the real interval [0, 1].

- A 3-bit corresponds to an element in
  $\{0, 1\} \times \{0, 1\} \times \{0, 1\} = 2^3$, as vertices of a cube;

  but very WRONG to have a cube for probability!

- When n =30, we get 2^{30} = 1 giga

- To get a setting of a possible non-commutative generalization, we associate each 1-bit with a rank-1 diagonal 2 X 2 projection matrix, i.e.,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

- Each 3-bit corresponds to a rank-1 diagonal 8 X 8 projection matrix, as the tensor product of three 2 X 2 matrices where each is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Thus there are eight 3-bits located in an 8-dimensional space.

▪A ***qubit*** (quantum bit) is a unit of quantum computer memory.

▪Mathematically, each 1-qubit is regarded as an element in

$$S^2 \simeq \left\{ \frac{1}{2} \begin{pmatrix} 1-x & y+iz \\ y-iz & 1+x \end{pmatrix} \quad with\; x^2+y^2+z^2=1 \right\}$$

= {all 2 X 2 rank-1 projection matrices}

= {all vector states acting on $\mathbf{C}^2$ }

= {one-dimensional complex linear subspaces of $\mathbf{C}^2$ }

= {*special* two-dimensional real subspaces of $\mathbf{R}^4$}

Physically, a 1-qubit is a superposition of the spherical surface (called the Bloch sphere), because an electron can move freely to any direction from the origin of $\mathbf{R}^3$.

Def. An ***n-qubit*** (= a vector state) is regarded as a 1-dimensional complex linear subspace of the dim 2^n Hilbert space, which can also be identified as a rank-1 projection in the form as a $2^n \times 2^n$ complex matrix.

DENSITY MATRICES

- In the formal setting of non-commutative probability, the random position of an n-qubit can be regarded as a density matrix, to be defined as a convex combination of rank-1 projections in M_{2^n}.

Def: A *density matrix* is a positive semidefinite matrix of trace 1.

- Thus, each 2 x 2 density matrix is expressed as

$$\frac{1}{2}\begin{bmatrix} 1-x & y+iz \\ y-iz & 1+x \end{bmatrix}$$

with $x^2 + y^2 + z^2 \leq 1$; so all density matrices fill up the whole solid sphere with S^2 as boundary.

- Nevertheless, for the case n >2, there is no geometrical picture for the collection of all n x n density matrices.

Recap of the *simplest* quantum computer

- The setting of 1-qubit computer

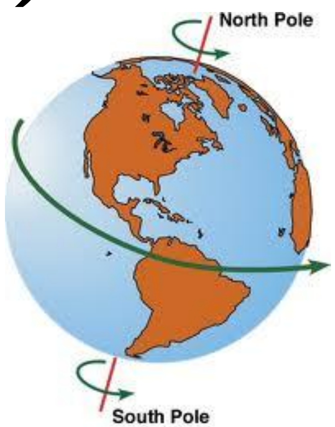
  is a solid sphere in R^3

--- just like the solid Earth in space.

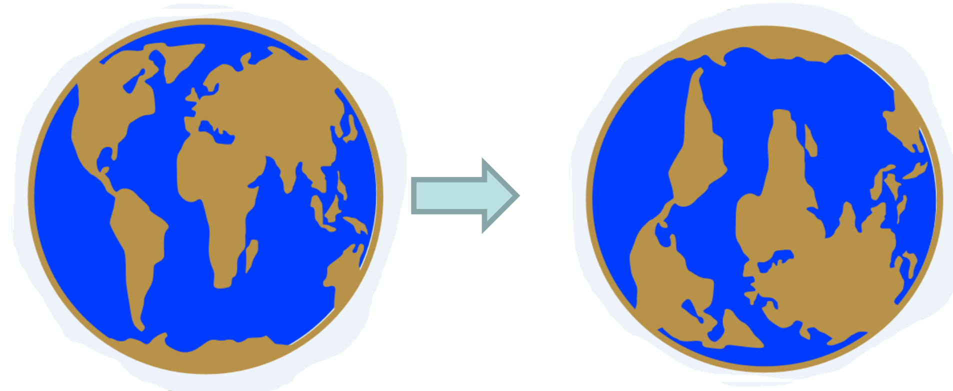
- To send out quantum information ----- to *communicate* between two 1-qubit computers, we consider a feasible *affine transform* (preserving the 3-dimensional convex structure) of the solid Earth.

What on ETH does it mean?

(A) Rotation

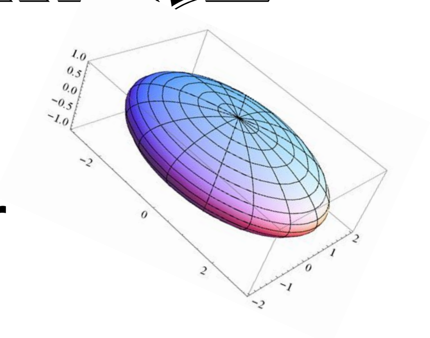


(B) *Reflection*



- Each 1-1 onto affine transform of the earth must be a *rotation* or *reflection* followed by a *rotation*.
- Note that *Reflection* is not feasible, in geography, or in physics, or by topology.

More examples of AFFINE TRANSFORMS OF THE SPHERE



- (C) *Shrinking* to a solid ellipsoid
- (D) Simple *translation* --- only after shrinking.

- The affine transform image must be a solid ellipsoid inside the original solid sphere.
- Each affine transform must be a composition of the four types above.
- An environmentally feasible affine transform is the composition of a *Rotation* and certain *Shrinking* followed by appropriate *Translation* (but no *Reflection*).

Notations

M_n = the algebra of all n x n complex matrices

M_n^+ = the cone of all n x n positive semidefinite matrices.

Def: A linear map $\Phi : M_n \rightarrow M_m$ is a *positive linear map* when $\Phi(M_n^+) \subseteq M_m^+$.

Φ is ***completely positive*** when Φ is of the form $\Phi(A) = \Sigma V_j^*AV_j$ for all A in M_n .

Completely positive linear maps in CIRCUIT THEORY

- Each transformer defines a positive linear map   A --> V*AV . Thus several transformers in series define a completely positive linear maps.

- Main question in circuit theory: Besides transformers, what else can make a positive linear map?

AFFINE TRANSFORMS induces LINEAR MAPS

- $M_n = M_n^+ - M_n^+ + i M_n^+ - i M_n^+$
- $M_n^+ = R^+ \times \{$ density matrices$\}$

➢ {affine transforms on density matrices}

≈ {trace-preserving positive linear maps}.

➢ {affine transforms on density matrices, fixing scalar matrices }

≈ {unital trace-preserving positive linear maps}.

➢ {feasible affine transforms on density matrices}

≈ {trace-preserving completely positive linear maps}.

FOR THE CASE n=2

- {density matrices} ≈ { the solid sphere in R^3}

---the setting of a 1-qubit computer.

- {trace-preserving positive linear maps}

  ≈ {Affine transforms of the solid sphere}

  = {Compositions of *rotations, reflection, shrinking,* & *translations*}

- {trace-preserving completely positive linear maps}

≈ {Feasible affine transforms of the solid sphere}

--- quantum channels between two 1-qubit computers

- The transpose map ≈ the reflection of the sphere.
- *-isomorphism ≈ rotation of the sphere.

Will need full information of special classes of positive linear maps.

- **Notation**: Each linear map $\Phi : M_n \to M_k$ can be extended to a linear map

$$\Phi \otimes id_p : M_n \otimes M_p \longrightarrow M_k \otimes M_p \; .$$

- **Def**: Φ is said to be *p-positive*

  when $\Phi \otimes id_p$ is a positive linear map.

- **Def**: Φ is said to be *completely positive* when Φ is a p-positive linear map for each positive integer p.

Full Structure Theory

Thm.(Choi, 1972) All p-positive linear maps from M_n to M_k are completely positive when $n \leq p$ or $k \leq p$.

- Nevertheless, various p provide distinct classes of p-positive linear maps as elaborated in the following:

- **Example:** (Choi, 1072)

  The linear map $\Phi: M_n \to M_n$ defined as

  $\Phi(A) = (n-1)(\text{trace } A)I_n - A$

  is (n-1)-positive but not n-positive.

- **Thm**: (Choi, 1975) A linear map

$\Phi: M_n \to M_k$ is ***completely positive***

iff $[\Phi(E_{ij})]$ is positive

iff when Φ is of the form

$$\Phi(A) = \Sigma V_j^*AV_j \text{ for all } A \text{ in } M_n .$$

- *More details for the structure theory.*

(This 1975 paper has been cited in more than 650 research papers, as of 2011 August Google Scholars.)

- Karl Kraus (1938-1988) was responsible for early development of quantum information.

Now, we are ready to carry out the full exploration of quantum information.

Let H_s be the system Hilbert space and let H_e be the environment Hilbert space. (Usually, $\dim H_e \geq \dim H_s < \infty$.)

Definition: A *quantum channel* (alias, *quantum operation*) is a linear map $\Phi : L(H_s) \longrightarrow L(H_s)$ satisfying the conditions:

(i) Φ is trace preserving

(ii) $\Phi \otimes id: L(H_s) \otimes L(H_e) \to L(H_s) \otimes L(H_e)$ is positive.

- From the structure theorem, it follows that Φ is ***completely positive*** and

$$\Phi(A) = \Sigma V_j^*AV_j \text{ for all } A \text{ in } M_n$$

with $\Sigma V_jV_j^* = I$.

Quantum Channels

as Trace-preserving completely positive linear maps

Open Question: Let T be an n x n matrix.

What are $\Phi(T)$ for all possible quantum channels Φ?

Partial answer: Suppose T is positive. Then all $\Phi(T)$ are precisely all positive n x n matrices of the same trace as T.