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On the Ambainis-Bach-Nayak-Vishwanath-Watrous Conjecture

by

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Introduction

Consider a one dimensional quantum walk determined by a 2×2 unitary matrix U on the state space $\{0,1,\cdots,N-1,N\}$ where N is finite or infinite

Let Φ below be the collection of initial quibit states: $\Phi = \left\{ \varphi = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in C^2 : |\alpha|^2 + |\beta|^2 = 1 \right\}$.

For $k \in \{0,1\cdots,N\}$ we are interested in the probabilites $P_k^\infty(\varphi)$ and $P_k^N(\varphi)$ for the walks.

Brief History of the Problem

Let $U = H(\rho) = \begin{bmatrix} \sqrt{\rho} & \sqrt{1-\rho} \\ \sqrt{1-\rho} & -\sqrt{\rho} \end{bmatrix}$ **. Bach *et.al* showed:**

$$\lim_{k\to\infty} P_k^{\infty}\left({}^{T}[0,1]\right) = \frac{\rho}{1-\rho}\left(\frac{\cos^{-1}(1-2\rho)}{\pi} - 1\right) + \frac{2\sqrt{\rho}}{\pi\sqrt{1-\rho}}$$

Yamaski-Kobayashi-Imai Conjecture

$$\lim_{k\to\infty} P_k^{\infty}\left({}^{T}[0,1]\right) = \frac{\cos^{-1}(1-2\rho)}{\pi}$$

Brief History of the Problem

Let $U = H\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$**.**

Ambainis *et.al* showed: $P_{1}^{\infty}\left({}^{T}[0,1]\right) = P\left({}^{T}[1,0]\right) = \frac{2}{\pi}$

and $\lim_{N\to\infty} P_{1}^{N}\left({}^{T}[0,1]\right) = \frac{1}{\sqrt{2}}$

Bach *et.al* showed:

$$\lim_{k\to\infty} P_{k}^{\infty}(\varphi) = \frac{1}{2}|\alpha|^{2} + \left(\frac{2}{\pi} - \frac{1}{2}\right)|\beta|^{2} + \left(\frac{2}{\pi} - 1\right)\mathrm{Re}(\overline{\alpha}\beta)$$

The Conjecture

Let $U = H\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ **and** $\varphi = |R\rangle =^{T} [0,1]$

? Is it true that

$$P_1^{N+1}(|R\rangle) = \frac{2P_1^{N}(|R\rangle)+1}{2P_1^{N}(|R\rangle)+2} \quad ; \; N \geq 1$$

$$P_1^{1}(|R\rangle) = 0$$

The Konno-Namiki-Soshi-Sudbury Attack

Lemma 1: $P_k^N(\varphi) = \sum_{n=1}^{\infty} P_k^N(n;\varphi)$ **, where**

$$P_k^N(n;\varphi) = C_1(n)|\alpha|^2 + C_2(n)|\beta|^2 + 2\,\mathrm{Re}\left(C_3(n)\overline{\alpha}\beta\right),$$

$$C_1(n) = \left|ap_k^N(n) + cr_k^N(n)\right|^2, \quad C_2(n) = \left|bp_k^N(n) + dr_k^N(n)\right|^2,$$

$$C_3(n) = \overline{\left(ap_k^N(n) + cr_k^N(n)\right)}\left(bp_k^N(n) + dr_k^N(n)\right)$$

where the bar denotes complex conjugation, and $p_k^N(n)$ and $r_k^N(n)$ satisfy certain difference equations that contribute to the solution of $P_k^N(\varphi)$

The Konno-Namiki-Soshi-Sudbury Attack

Theorem 2: $P_k^N(\varphi) = C_1|\alpha|^2 + C_2|\beta|^2 + 2\,\mathrm{Re}(C_3\overline{\alpha}\beta)$ **,**

where

$$C_1 = \frac{1}{2\pi}\int_0^{2\pi}\left|ap_k^N(e^{i\theta}) + cr_k^N(e^{i\theta})\right|^2 d\theta$$

$$C_2 = \frac{1}{2\pi}\int_0^{2\pi}\left|bp_k^N(e^{i\theta}) + dr_k^N(e^{i\theta})\right|^2 d\theta$$

$$C_3 = \frac{1}{2\pi}\int_0^{2\pi}\left(\overline{ap_k^N(e^{i\theta}) + cr_k^N(e^{i\theta})}\right)\left(bp_k^N(e^{i\theta}) + dr_k^N(e^{i\theta})\right)d\theta$$

The Konno-Namiki-Soshi-Sudbury Attack

IDEOLOGY: **In the special case of the Hadamard walk, explicitly determine** $p_k^N\left(e^{i\theta}\right)$ **and** $r_k^N\left(e^{i\theta}\right)$ **in Theorem 2 , from the difference equation they satisfy and use it in the special case of** $\varphi = \left|R\right\rangle$ $k=1$ **to show the truth or falsity of the Conjecture**

Failure and the Ampadu Solution

Lemma 3: Let $\lambda_{\pm} = \dfrac{z^2 - 1 \pm \sqrt{z^4+1}}{\sqrt{2}z}$ **,** $p_k^N = (\lambda_+)^k$**,**

$r_k^N = (\lambda_-)^k$ **, then the solution to the difference equation** $p_k^N(z) = A_z p_{k-1}^N(z) + B_z r_{k-1}^N(z)$ **,**

$r_k^N(z) = A_z p_{k+1}^N(z) + B_Z r_{k+1}^N(z)$ **, with prescribed boundary conditions** $r_{N-1}^N(z) = 0$ **&** $p_1^N(z) = z$ **is**

$$p_k^N(z) = \frac{z(\lambda_-)^N}{(\lambda_-)^N - (\lambda_+)^N}\lambda_+^{k-1} + \frac{z(\lambda_+)^N}{(\lambda_+)^N - (\lambda_-)^N}\lambda_-^{k-1}$$

$$r_k^N(z) = \frac{z(\lambda_-)^N}{(\lambda_-)^N - (\lambda_+)^N}\lambda_+^{k+1} + \frac{z(\lambda_+)^N}{(\lambda_+)^N - (\lambda_-)^N}\lambda_-^{k+1}$$

Failure and the Ampadu Solution

Theorem 4: $P_k^N(\varphi) = C_1|\alpha|^2 + C_2|\beta|^2 + 2\mathrm{Re}(C_3\overline{\alpha}\beta)$

where

$$C_1 = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{\sqrt{2}}p_k^N(e^{i\theta}) + \frac{1}{\sqrt{2}}r_k^N(e^{i\theta})\right|^2 d\theta$$

$$C_2 = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{\sqrt{2}}p_k^N(e^{i\theta}) - \frac{1}{\sqrt{2}}r_k^N(e^{i\theta})\right|^2 d\theta$$

$$C_3 = \frac{1}{2\pi}\int_0^{2\pi}\left(\overline{\frac{1}{\sqrt{2}}p_k^N(e^{i\theta}) + \frac{1}{\sqrt{2}}r_k^N(e^{i\theta})}\right)\left(\frac{1}{\sqrt{2}}p_k^N(e^{i\theta}) - \frac{1}{\sqrt{2}}r_k^N(e^{i\theta})\right)d\theta$$

and $p_k^N(e^{i\theta})$**,** $r_k^N(e^{i\theta})$ **satisfy Lemma 3 with**

$$z = e^{i\theta};\theta \in [0,2\pi]$$

Failure and the Ampadu Solution

Note that $p_1^N(z)=z$ **for** $N\geq 2$**. Now let** $k=1$ **,** $\varphi=|R\rangle$ **in Theorem 4 then we have the following**

Corollary 5: For $N\geq 2$ **,** $P_1^N(|R\rangle)=\frac{1}{2}+\frac{1}{4\pi}\int_0^{2\pi}\left|r_1^N\left(e^{i\theta}\right)\right|^2 d\theta$

where $r_1^N(z)$ **,** $z=e^{i\theta};\theta\in[0,2\pi]$ **satisfies**

$$r_1^N(z)=\frac{z(\lambda_-)^N}{(\lambda_-)^N-(\lambda_+)^N}\lambda_+^2+\frac{z(\lambda_-)^N}{(\lambda_+)^N-(\lambda_-)^N}\lambda_-^2$$

Resolution of the Conjecture

From Corollary 5, using facts about $\lambda_{\pm}$, $\lambda_{+}\lambda_{-}$, $\lambda_{+}+\lambda_{-}$

it can be deduced that $r_1^3(z)=\dfrac{z^3}{2z^4-3z^2+2}$ **is an**

odd complex function. In particular, direct computation shows that $\dfrac{1}{4\pi}\int_0^{2\pi}\left|r_1^3\left(e^{i\theta}\right)\right|^2 d\theta=0$ **,**

thus, $P_1^3\left(\left|R\right\rangle\right)=\dfrac{1}{2}\neq\dfrac{2}{3}$ **(as the Conjecture gives)**

Open Problem

For a 1D quantum walk determined by a 2×2 unitary matrix on a state space $\{0,1\cdots,N\}$, where N is finite or infinite, find general analytic formula for the probabilities $P_k^{\infty}(\varphi)$ and $P_k^{N}(\varphi)$ for any $k\in\{1,\cdots,N-1\}$ and $\varphi\in\Phi$

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